MATH 230-1: Multivariable Differential Calculus
Northwestern University, Lecture Notes

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These are notes which provide a basic summary of each lecture for MATH 230-1, "Multivariable Differential Calculus", taught by the author at Northwestern University. The book used as a reference is the 14th edition of *Thomas' Calculus* by Hass, Heil, and Weir. Watch out for typos! Comments and suggestions are welcome.

Contents

Lecture 1: Introduction

This is a course in *multivariable differential calculus*. Our basic goal is extend the concepts you saw before in single-variable calculus, such as the notions of derivative, linear approximations, and optimization, to the setting of functions of more than one variable. Why do we care about functions of more than one variable, meaning functions which take more than one input? The answer is simply because most real-world phenomena do indeed depend on more than one thing. Whether it be the price of a share of stock in some company, or the windchill one experiences standing on a mountain, various quantities we care about cannot be accurately described by one variable alone.

We will see that much of multivariable calculus closely mimics single-variable calculus, only with more variables of which to keep track. However, we also see new phenomena arising in the multivariable setting, which somehow get at the *core* of what calculus is all about, which were hidden under the rug in the single-variable setting. The interpretations and uses of calculus cannot be truly appreciated in all their glory until we see what they look like in higher-dimensions.

Topics for the quarter. Briefly, here are some basic notions we will eventually come to study:

- *limits*: multivariable limits share some similarities with the single-variable limits you've seen before, but also some important differences, owing to the fact that moving one dimension higher (from the two dimensions of the xy-plane to the three dimensions of the world in which we live) gives more directions of which to keep track;
- *derivatives*: multivariable derivatives are computed using precisely the same tools as for computing single-variable derivatives, and their basic geometric interpretations will be the same, but moving into higher dimensions gives a whole range of new uses that were not present before;
- *gradients*: the notion of a "gradient vector" is one of the most important ones in multivariable calculus, and has no real analog in the single-variable calculus; more precisely, "one-dimensional gradients" are too simplistic and uninteresting to introduce at all; and
- *optimization*: finding extrema (i.e., maximums and minimums) of multivariable functions is one of the most important applications of multivariable calculus, just as it was for single-variable calculus. In addition to techniques that look familiar, we will see new techniques that, again, only really appear in the multivariable setting.

In fact, the topics listed above will take up the *second* half of this course. So, what, exactly, takes up the first half? The answer is *geometry*, which plays a much more important role now than it did in single-variable calculus. But before elaborating, let us consider a first basic example.

Partial derivatives. In a previous course you would have seen a function such as $f(x) = x^2$, which takes in one input x and outputs one number x^2. The derivative, which gives the instantaneous rate of change of f at any given input, is $f'(x) = 2x$. But now, we will consider a function such as $f(x, y) = x^2y$, which is an example of a function of two variables since it takes *two* numbers x and y as inputs. Evaluating at say $x = 1, y = 2$ results in the output $f(1, 2) = 1^2 \cdot 2 = 2$, or evaluating at $x = -2, y = 3$ gives $f(-2, 3) = (-2)^2 \cdot 3 = 12$.

What should the "derivative" of $f(x, y) = x^2y$ be? The basic fact is that in this case we now have *two* derivatives we can compute: one with respect to x and one with respect to y. These derivatives are called *partial derivatives* and are denoted by

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

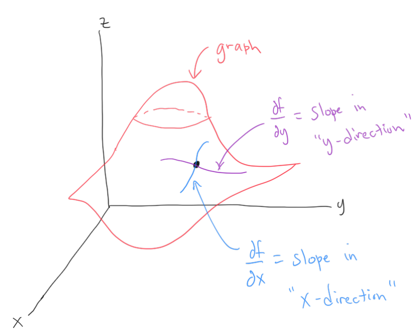
respectively. (Note the similarity with the notion $\frac{df}{dx}$ for single-variable derivatives. The ∂ in the notion instead of d is used to signify that this is a "partial" derivative taken with respect to only one variable or the other.) To compute such a partial derivative, we do the same thing as what we did with single-variable functions, only that we think of the *other* variable as being a constant. For example, to compute the "partial derivative of f with respect to x", we think of y as being a constant, so that x^2y is a x^2 times a "constant", and we then take the derivative of x^2 as your normally would; we get $2xy$. Similarly, to compute the "partial derivative of f with respect to y, we think of x as being constant—which in turns means that x^2 is a "constant", so that x^2y is a "constant" times y—and take a usual derivative with respect to y, to get x^2. To summarize, the partial derivatives of $f(x, y) = x^2y$ are

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2.$$

And thus, we will see that computing such partial derivatives uses just the same differentiation techniques—power rule, product rule, quotient rule, chain rule—that you saw for single-variable functions; nothing more, nothing less. The *new* concepts for us will come from using partial derivatives to compute other quantities of interest, and from understanding their interpretations.

Higher-dimensional slopes. Geometrically, in the single-variable case, derivatives computed slopes of tangent lines to graphs of curves. The graph of a function $y = f(x)$ of one variable is a curve in the xy-plane, and the derivative $f'(a)$ at a point $x = a$ gives the slope of the graph (or tangent line) at that point. To begin to get a sense of what *partial derivatives* mean geometrically, we first have to understand what the graph of a function of, say, two variables is. In the single-variable case we deal with the xy-plane because we need two coordinates of everything: one for the input x, and one for the output $y = f(x)$. But for something like $f(x, y) = x^2y$, we need three coordinates to keep track of everything: two for the inputs x and y, and one for the output $z = f(x, y)$. This forces us to work in three-dimensional space.

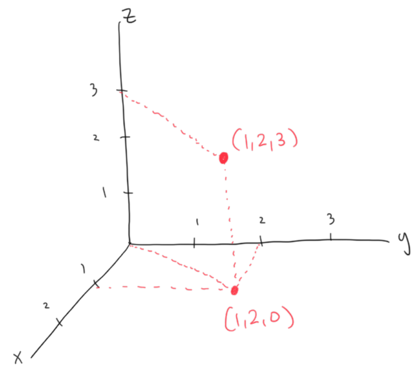
We will talk about three-dimensional space in a bit, but for now we point out the *graph* a function of two variables $f(x, y)$ is actually a surface, analogously to how in one dimensional lower the graph of a function of one variable is a curve. This graph might look something like



It turns out that the partial derivatives of $f(x, y)$ can indeed be interpreted geometrically as slopes, only that we now have to specify the *direction* in which we consider the slope. In the picture above, we might imagine varying only the x coordinate of our input (x, y), and we will see the partial derivative $\frac{\partial f}{\partial x}$ computes the slope in this "x-direction. Similarly, for the partial derivative $\frac{\partial f}{\partial y}$, we imagine changing only the value of y in the input (x, y), and this partial derivative computes the slope of the graph in the "y-direction". The upshot is that derivatives in the higher-dimensional setting do compute the same types of geometric information as in the single-variable case, with the added wrinkle that *direction* matters.

Geometry of space. The discussion above was only meant to be a brief introduction to the calculus topic we will eventually study, so it is not expected that you be familiar with the ideas (such as partial derivatives and "slopes in a certain direction") we mentioned. The key takeaway is that this discussion should make clear the point that multivariable derivatives are intimately connected with geometry, so that is why we will spend roughly the first half of this course studying three-dimensional geometry.

To begin with, we describe three-dimensional space using points with three coordinates (x, y, z). We draw the three corresponding axes as in the picture above. We should view the x-axis as coming "out" from the page towards us. Thus, the x-coordinate of a point (x, y, z) tells us how far we move in this direction away from the page (negative values of x means that we move away through back side *away* from us), the y-coordinate tells us how far we move horizontally (left is negative, right is positive), and the z-coordinate tells us the "height" at which we are, where positive z means we move "up" and negative z means "down". A point like $(1, 2, 3)$ would then be drawn as

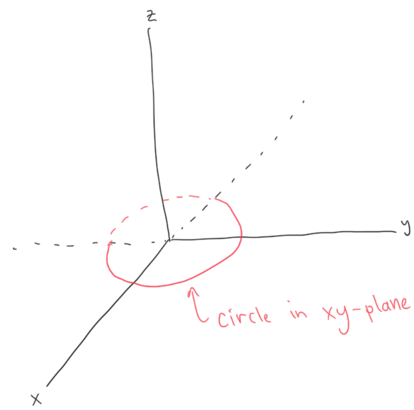


Notice here that we draw this in a way which maintains *perspective*. Importantly, we draw the right edge moving away from the y-axis so that is parallel to the x-axis, and the edge on top moving from $(1, 2, 3)$ back directly towards the z-axis (maintaining height) as being parallel to the segment connecting the origin $(0, 0, 0)$ to the point $(x, y, 0)$ in the xy-plane. (The xy-plane is the plane containing the x- and y-axes, or in other words the collection of points whose z-coordinate is exactly zero. This visually is the horizontal plane at the "bottom" of three-dimensional space. The yz-plane would be the plane on the "back" where the x-coordinate of points is zero, and the xz-plane is the plane on the "left" where the y-coordinate is zero.)

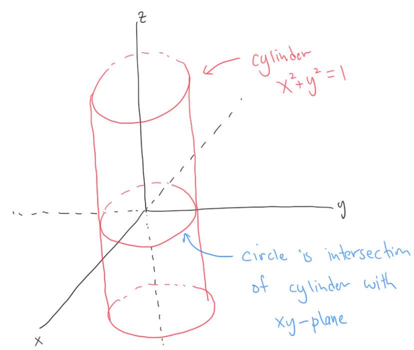
We use the notation $\mathbb{R}^3$ to denote three-dimensional space, where $\mathbb{R}$ denotes the real number line and the superscript 3 corresponds to the fact that we are considering three axes/coordinates. (Analogously, the xy-plane from single-variable calculus consists of the points that make up what we would call $\mathbb{R}^2$.) So, we would say that $(1, 2, 3)$, for example, is a point in $\mathbb{R}^3$.

Equations in space. Now let us consider regions in $\mathbb{R}^3$ defined by different equations. For example, we first want to describe (or better yet, draw!) the collection of points (x, y, z) satisfying $x^2 + y^2 = 1$. In two dimensions, this describes a circle of radius 1 centered at $(0, 0)$, but what does this same equation describe in three-dimensional space?

Well, if nothing else we still have the same unit circle as before, draw in the xy-plane, since a point $(x, y, 0)$ in this xy-plane that satisfied the unit circle equation still satisfies this new equation:

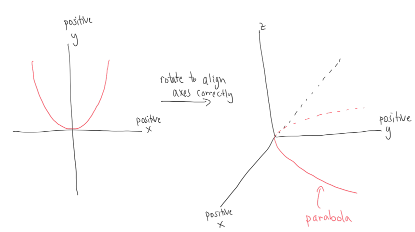


(The equation itself is the same, after all, and all that is different is the introduction of a new variable.) But, this alone does not describe *all* points satisfying $x^2 + y^2 = 1$. Indeed, if we take any point in this circle and *change* its z-coordinate—or visually, if we "move" it up or down—the points we get we still satisfy $x^2 + y^2 = 1$ since we are not changing the values of x or y at all. We thus have a "copy" of this same circle occurring at height 1, one at height 2, one at height -1 and so on. Thus $x^2 + y^2 = 1$ actually describes a *surface*, namely the surface obtained by taking the circle in the xy-plane as a starting point and then sliding it up and down in the z-direction:

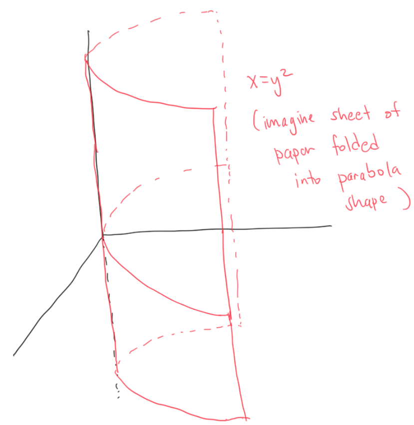


The upshot is that $x^2 + y^2 = 1$ describes a *cylinder* of radius 1 in $\mathbb{R}^3$.

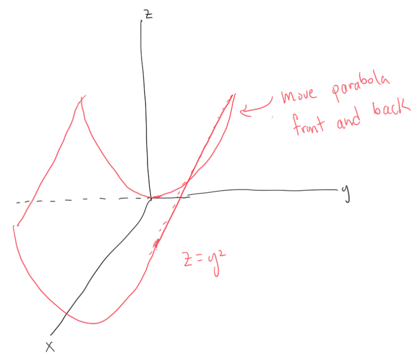
Next, what about the region characterized by the equation $y = x^2$ in $\mathbb{R}^3$? Again, what we mean here is the collection of points (x, y, z) whose coordinate satisfy $y = x^2$. In two-dimensions, $y = x^2$ describes a parabola in the xy-plane, so that is our starting point here as well:



But just as before, $y = x^2$ places no restriction on what the value of z can be, so taking any point on this parabola and changing its z-coordinate still results in points satisfying $y = x^2$. Thus, $y = x^2$ again describes a surface, namely the surface obtained by taking the parabola $y = x^2$ and sliding it up and down in the z-direction:



Now consider the surface defined by $z = y^2$. This does not as written describe something in the xy-plane, but by analogy we should really first consider what this describes in the yz-plane instead. Here, it again gives a parabola. (The point is that whether we consider $y = x^2$ in the xy-plane, or $z = y^2$ in the yz-plane, does not matter: the basic *shape* of the resulting curve is the same, and all that changes is the names we are giving to the variables.) In this case, modifying the x-coordinate of a point on this parabola still gives a valid point satisfying $z = y^2$. So, if we take the parabola and move it "back and forth" along the x-direction coming "out" of the page, we sweep out the full surface described by $z = y^2$, with no restriction on x:



The general idea that these examples demonstrate is the same, at least for equations that omit one of the variables: the corresponding collection of points satisfying that equation is a surface, which we can visualize by first taking the *curve* described by that equation in the appropriate plane, and then moving this curve along the direction of the missing variable to get the entire surface.

Spheres. Visualizing and understanding surfaces is something we will continue to get much more experience with as we go, but for now let us mention one standard examples. To start, we first note the formula for three-dimensional distance: the distance from (x, y, z) to (a, b, c) in $\mathbb{R}^3$ is

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

This formula is exactly analogous to the formula for two dimensional distance:

$$\text{distance from } (x, y) \text{ to } (a, b) \text{ is } \sqrt{(x-a)^2 + (y-b)^2},$$

only with one more term under the square root making use of the third coordinate. In the end, these formulas are consequences of the Pythagorean theorem.

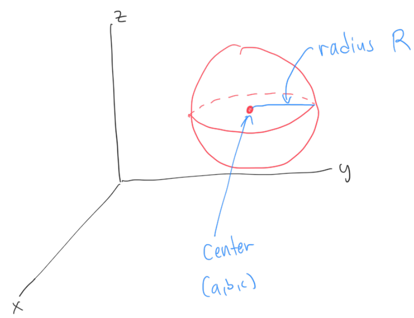
With this, we can then derive the formula of a sphere. The *sphere* of radius R centered at (a, b, c) in $\mathbb{R}^3$ is the collection of points (x, y, z) whose distance to (a, b, c) is exactly R. Using our distance formula, this mean that

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = R.$$

A points (x, y, z) is on this sphere when its coordinates satisfy this equation. But, you ordinarily would not see the equation of a sphere written like this. Instead, we can rewrite this equation by squaring both sides to get

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2,$$

which is the standard form of the equation of a sphere:



Note that if we instead consider the *inequality*

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \leq R^2,$$

we are then considering points whose distance to (a, b, c) is *at most* R, but possibly less than R. This gives the region *enclosed* by the sphere, which looks like a solid ball. (To be clear, a sphere is not solid and only consists of the outer "shell" of the solid ball.)

Lecture 2: Vectors

Warm-Up 1. We find an equation that characterizes points (x, y, z) whose distance to $(1, 2, 1)$ is the same as its distance to $(2, 1, 1)$. The distance from (x, y, z) to $(1, 2, 1)$ is given by

$$\sqrt{(x-1)^2 + (y-2)^2 + (z-1)^2}$$

and the distance from (x, y, z) to $(1, 2, 1)$ is

$$\sqrt{(x-2)^2 + (y-1)^2 + (z-1)^2}.$$

The equation we want says that this two distances should be the same, so (x, y, z) should satisfy

$$\sqrt{(x-1)^2 + (y-2)^2 + (z-1)^2} = \sqrt{(x-2)^2 + (y-1)^2 + (z-1)^2}.$$

This is one form of the desired equation, but actually we can simplify this a lot by doing some algebra. Our real goal is to visualize precisely what points (x, y, z) we're looking at, or in other words see what region (surface, in fact) this equation describes. First we can square both sides to get rid of the square root:

$$(x-1)^2 + (y-2)^2 + (z-1)^2 = (x-2)^2 + (y-1)^2 + (z-1)^2.$$

The $(z-1)^2$ term common to both sides cancels out, and then we can expand all the squares to get

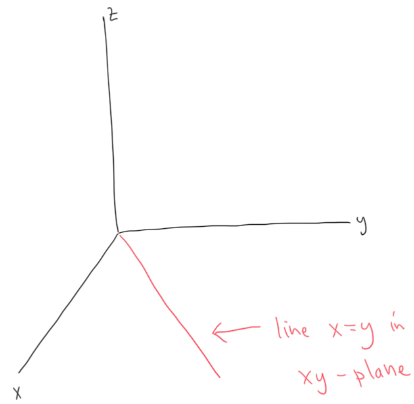
$$x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 - 4x + 4 + y^2 - 2y + 1.$$

More algebra gives

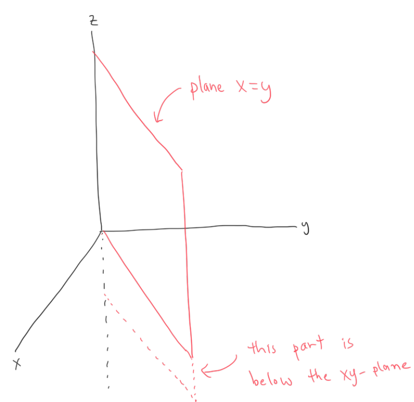
$$2x = 2y, \text{ and finally } x = y.$$

The conclusion is that the points (x, y, z) with distances to $(1, 2, 1)$ and to $(2, 1, 1)$ are the same are precisely those which satisfy $x = y$.

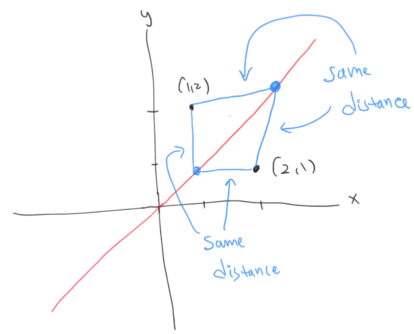
To visualize what the equation $x = y$ describes, we begin by first drawing the line we know it describes in the xy-plane:



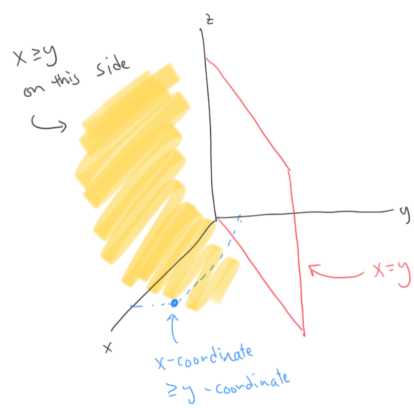
(Note how the xy-plane is oriented here: the positive x-direction is coming out towards us, and the positive y-direction moves towards the right. The line we drew above is in the first quadrant of the usual xy-plane.) But now we recognize that $x = y$ places no restriction on z, so taking this line and moving it up and down in the z-direction traces out the surface in question, which is a plane:



Here we are only drawing the portion of this plane coming out towards us; there is another half coming out the back side as well. The points on this plane are then the ones whose distances to $(1, 2, 1)$ and $(2, 1, 1)$ are equal. Indeed, this is the three-dimensional version of the two-dimensional observation that the points (x, y) in the xy-plane whose distances to $(1, 2)$ and $(2, 1)$ are the same is the line $y = x$:



As a follow-up, we can ask what the inequality $x \geq y$ describes. Now we want points whose x-coordinate is *at least* as large as its y-coordinate. Visually, this corresponds to taking the region on one side of the plane above, and we can determine which region we need by simply testing a point: the point



has x-coordinate larger than its y-coordinate, so $x \geq y$ describes the region to the "left" (as viewed from the perspective drawn above) of the plane $x = y$.

Warm-Up 2. We find the point satisfying the equation

$$x^2 - 4x + y^2 - 6y + z^2 = -12$$

that is *closest* to the xz-plane. The point here is to first determine what type of surface this equation describes, and then to use a picture of this surface to find the desired closest point. In fact, this is the equation of a sphere. To make it look like the standard sphere equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2,$$

we must "complete the square", which is the technique of writing something like $x^2 - 4x$ in terms of a single quantity square plus or minus a constant. In particular, we need $x^2 - 4x$ to arise from the $(x-a)^2$ term:

$$x^2 - 4x = (x-a)^2 \pm \text{something},$$

and for this we need to use $(x-2)^2$ since this is what will give the $-4x$ term. But $(x-2)^2$ gives a 4 that does not show up in our original equation, so we must compensate for this by subtracting 4:

$$x^2 - 4x = (x-2)^2 - 4.$$

We do the same thing for the y terms, and in theory the z terms, although in this case z^2 has already had its square completed since we can write it as $(z-0)^2$; we would only need to complete the square for z if we had an additional z to the first power term occurring in our equation.

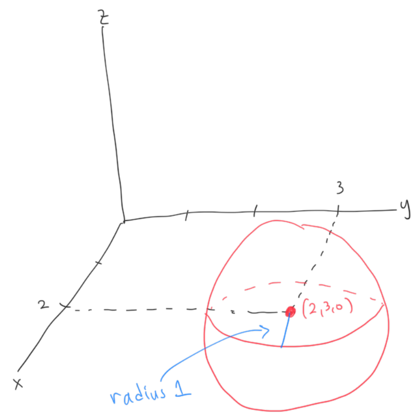
After completing the square, we get that our original equation can be written as

$$(x-2)^2 - 4 + (y-3)^2 - 9 + z^2 = -12,$$

which in turn we can write as

$$(x-2)^2 + (y-3)^2 + z^2 = 1.$$

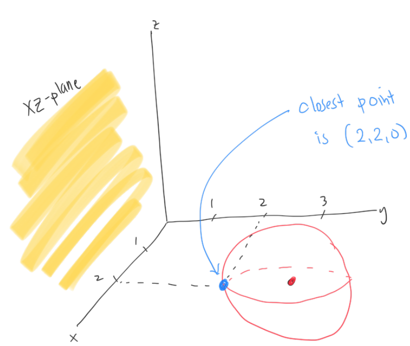
Thus, the given equation describes a sphere of radius 1 centered at $(2, 3, 0)$. We draw this sphere as follows:



Recall that our goal is to find the point on this sphere that is closest to the xz-plane. The xz-plane is here drawn on the left (it is the plane containing the x- and z-axes), which we imagine as being strictly vertical and coming out directly at us. Visually then, we can see that the point on the sphere closest to the this plane should be the "leftmost" point on the sphere. This point occurs on the equator of the sphere, which is the circle in the xy-plane given by $(x-2)^2+(y-3)^2 = 1$. (This comes from setting $z = 0$ in the sphere equation, since $z = 0$ is what characterizes the xy-plane.) The point on this circle closest to the xz-plane has x-coordinate, and thus has y-coordinate 1. (This y-coordinate comes from setting $z = 0$ and $x = 2$ in the sphere equation to get $(y-3)^2 = 1$; this gives $y = 4$ or $y = 2$, and $y = 2$ is the leftmost point.) Thus the point satisfying

$$x^2 - 4x + y^2 - 6y + z^2 = -12$$

that is closest to the xz-plane is the point $(2, 2, 0)$:



As a follow-up, we can ask about the points satisfying the inequality

$$x^2 - 4x + y^2 - 6y + z^2 \leq -12.$$

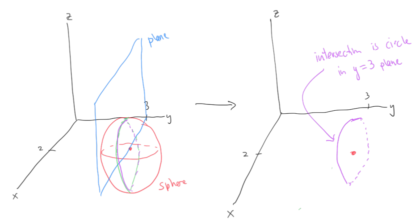
After completing the square as above, this becomes

$$(x-2)^2 + (y-3)^2 + z^2 \leq 1,$$

which thus describes the solid ball enclosed by the sphere from before. As another example, we can ask about the points satisfying

$$x^2 - 4x + y^2 - 6y + z^2 = -12 \quad \text{and} \quad y = 3.$$

Here we are looking at points on the sphere that have in particular y-coordinate 3. This is the intersection of the sphere with the vertical plane at $y = 3$ (this plane is parallel to the xz-plane), and so this intersection looks like a circle:



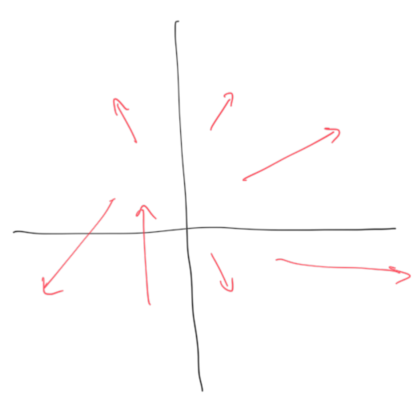
(This is a "line of longitude" on the sphere.) If instead we asked about the points satisfying

$$x^2 - 4x + y^2 - 6y + z^2 \leq -12 \quad \text{and} \quad y = 3,$$

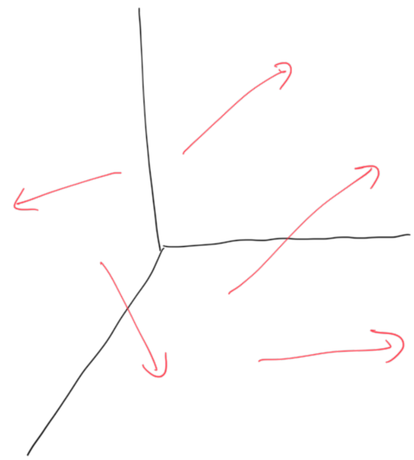
we would get the *disk* enclosed by this circle.

Vectors. As we further develop our understanding of three-dimensional space and how to visualize objects within it, it will be important to be able to describe *directions* of interest. In the two-dimensional case, we can often use slopes to specify directions, but the notion of "slope" is more subtle in three-dimensions since it depends on the way in which we are drawing space, and it depends on the pair of coordinates we pick: do we mean "slope" with respect to y and x, or "slope" with respect to z and x, etc.? Instead, we will indicate directions in general (even in two-dimensions) using the notion of a vector.

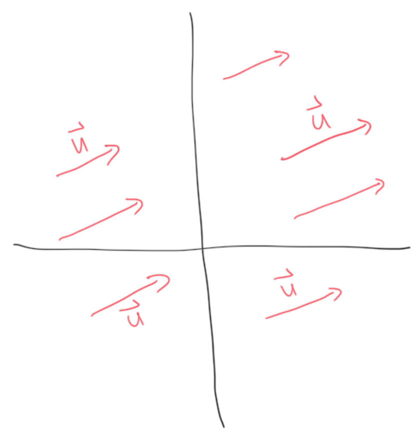
A *vector* is simply something that has a specified direction and a specified magnitude/length. We visualize vectors as arrows:



Apart from being used to indicate directions, practically vectors are often use to indicate some type of "force", or perhaps "velocity". Imagine, for example, that the xy-plane above describes the surface of a lake, and each vector drawn indicates the direction in which water flows at that location, and where we interpret the length of the vector as the "strength" (or magnitude) of the flow of water; a longer vector means a stronger flow. The same concepts work in three-dimensions, where we interpret three-dimensional vectors again as arrows:

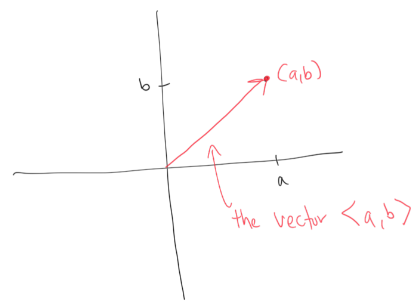


Now, one basic point we keep in mind when working with vectors is that all that matters when describing a given vector is its direction and its magnitude, and *not* the starting point of the arrow we draw. For example, all vectors drawn below are meant to the *same*



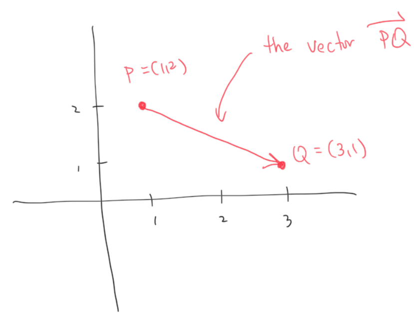
since each arrow drawn has the same length and the same "direction". (Notation wise, we typically indicate vectors using arrows above whatever letter or symbol we are using, as in $\vec{u}$. In typed writing, it is also common to use bold letters **u** to indicate vectors.) We can a certain arrow and translate it around so that the point at which it begins changes, but if maintain the same direction and magnitude throughout, we are not changing the vector itself.

Algebraic description. Even we can draw vectors as arrows wherever, it will still be important to be able to describe vectors using *coordinates* so that we can later use them in equations. We use the notation (say in two-dimensions for now, but the analogous story applies in three-dimensions as well) $\langle a, b\rangle$ to denote a vector, namely the vector drawn to start at the origin $(0,0)$ and *end* at the point (a,b):



To be clear, whenever we use coordinates (or "components") to describe a vector, we always we mean the vector beginning at the origin and ending at the point whose coordinates are given. Of course we can draw this same vector elsewhere by moving it around to change the point at which it begins, but we should always imagine it drawn at the origin when wanting to describes its coordinates. The reason for using angled brackets as in $\langle a, b\rangle$ as opposed to parentheses as in (a, b) is to emphasize the fact that we are talking about a *vector*, and not simply a point.

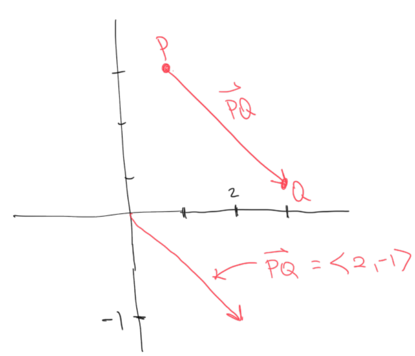
For example, we consider the vector which starts at $P = (1, 2)$ and ends at $Q = (3, 1)$, which we would denote as $\overrightarrow{PQ}$:



To describe this vector algebraically in terms of components, we must shift it so that it begins at the origin instead. To do this we should subtract the coordinates of P from all points involved: subtracting 1 from the x-coordinate of P and 2 from the y-coordinate of P moves P to $O = (0, 0)$ (O denotes the origin), so we should do the same to the coordinates of Q. We get

$$\overrightarrow{PQ} = \langle 3 - 1, 1 - 2\rangle = \langle 2, -1\rangle .$$

The vector beginning at the origin and *ending* at the point $(2, -1)$ is indeed the same as the vector going from P to Q, again only shifted so that it begins at $(0, 0)$ instead:



Scalar multiplication. Now, say we want to describe the vector beginning at P and pointing towards Q above, only that we want it to have length 10 overall? The vector we computed above

$\overrightarrow{PQ} = \langle 2, -1 \rangle$ does point in the direction we want, but it has length

$$|\overrightarrow{PQ}| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$$

instead of 10. (We use vertical bars $|\mathbf{u}|$ to denote the magnitude/length of a vector $\mathbf{u}$. The length just comes from the usual distance formula, in this example the distance from $(2, -1)$ to $(0, 0)$). So, the goal is to somehow modify this vector so that it has the correct length, without changing its direction.

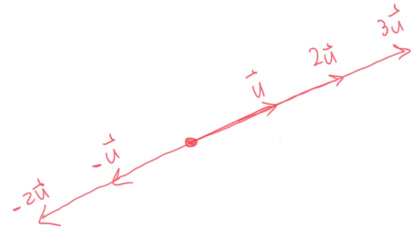
To do this we use the vector operation known as *scalar multiplication*. (The word "scalar" just means the same thing as "number". The name comes from using numbers to "scale" the length of vectors.) Scalar multiplication takes a vector, say algebraically written as $\mathbf{u} = \langle a, b \rangle$, and multiplies it by a number c, by which we mean to multiply each coordinate by that number:

$$c\mathbf{u} = c \langle a, b \rangle = \langle ca, cb \rangle .$$

Geometrically, this has the effect of scaling the length of $\mathbf{u}$, at least if c is positive:

the length of $c\mathbf{u}$ is c times the times of $\mathbf{u}$.

Thus, for example, $2\mathbf{u}$ is the vector in the same direction as $\mathbf{u}$ only with twice the length, and $3\mathbf{u}$ points in the same direction as $\mathbf{u}$ but has three times the length. Multiplying by a negative number has the same effect, only with the added effect of "flipping" the direction of the vector around:



So, $-\mathbf{u}$ has the same length as $\mathbf{u}$ but points directly in the opposite direction, $-2\mathbf{u}$ has twice the length and points in this opposite direction, and so on.

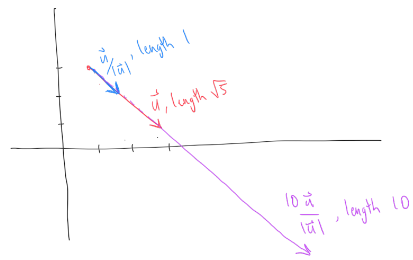
Going back to our previous example, our goal is to find a vector pointing in the direction from $P = (1, 2)$ towards $Q = (3, 1)$, and of length 10. The first step is to find the vector of length 1 pointing in this direction, or what we call the *unit vector* in this direction. For this we simply scale $\mathbf{u} = \overrightarrow{PQ} = \langle 2, -1 \rangle$ (I'm using $\mathbf{u}$ to denote this vector instead of $\overrightarrow{PQ}$ simply for the sake of simpler notation) by the reciprocal of its length, since multiplying the length $|\mathbf{u}| = \sqrt{5}$ by $\mathbf{u}$ by $1/|\mathbf{u}| = 1/\sqrt{5}$ will result in something of length $\sqrt{5}/\sqrt{5} = 1$ overall. (In other words, *dividing* a nonzero vector by its length always results in a vector of length 1.) So, in our case,

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{5}} \langle 2, -1 \rangle = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

is the unit vector pointing in the direction from P towards Q. Finally, to get something of length 10, we can simply scale by 10 to get

$$10\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{10}{\sqrt{5}} \langle 2, -1 \rangle = \left\langle \frac{20}{\sqrt{5}}, -\frac{10}{\sqrt{5}} \right\rangle$$

as our desired vector:

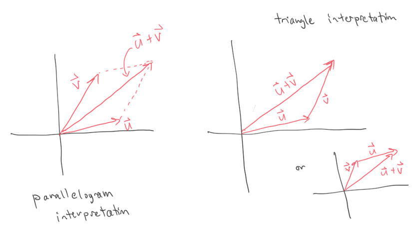


Vector addition. In addition to scalar multiplication of a vector by a number, there is another basic algebraic operation we will use, that of *addition* of vectors. Algebraically, this is straightforward: given, for example, $\mathbf{u} = \langle 3, 1\rangle$ and $\mathbf{v} = \langle 1, 2\rangle$, the sum $\mathbf{u} + \mathbf{v}$ is obtained by adding corresponding coordinates, which gives

$$\mathbf{u} + \mathbf{v} = \langle 3, 1\rangle + \langle 1, 2\rangle = \langle 3 + 1, 1 + 2\rangle = \langle 4, 3\rangle .$$

The same applies to vectors in three dimensions.

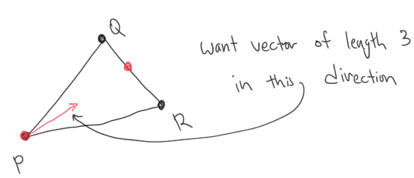
But, the great thing is that this also has a straightforward geometric interpretation. If we draw $\mathbf{u} = \langle 3, 1\rangle$ and $\mathbf{v} = \langle 1, 2\rangle$ as arrows, and visualize them as the sides of a parallelogram, then $\mathbf{u} + \mathbf{v}$ is precisely the vector that describes the main *diagonal* this parallelogram! Alternatively, we can shift $\mathbf{v}$ so that it begins at the point where $\mathbf{u}$ ends, so that $\mathbf{u}$ and $\mathbf{v}$ are then two sides of a triangle, and $\mathbf{u} + \mathbf{v}$ is the vector that completes this triangle



These geometric interpretations of vector addition will be crucial to understand many of their uses.

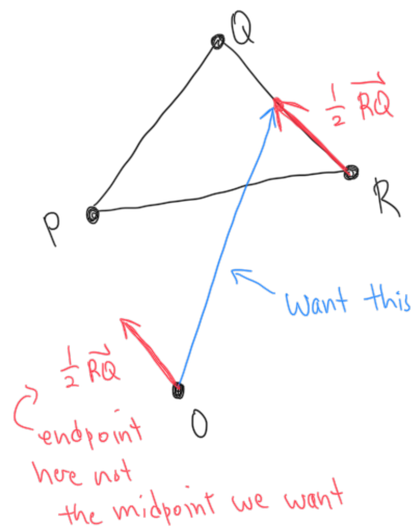
Lecture 3: Dot Product

Warm-Up. Given the triangle with vertices $P = (1, 1, 1), Q = (2, -1, 2)$, and $R = (-1, 3, 4)$, we find—meaning describe algebraically—the vector of length 3 pointing from P towards the midpoint of the segment QR:



The goal here is to understand how to use vector algebra to first describe the desired midpoint explicitly, and then to modify the length of the vector from P to this midpoint in order to have the desired length of 3. Spoiler alert: the coordinates of the midpoint are simply the averages of the coordinates of the points in question, but we seek to understand *why* this is true using vectors.

The desired midpoint lies on the line segment between Q and R, and this line segment (with starting point R and final point Q) is given by the vector $\overrightarrow{RQ}$. We want to move halfway along this vector to reach the midpoint, so we want to move along the vector $\frac{1}{2}\overrightarrow{RQ}$ *starting* at R. However, if we want to determine the coordinates of this midpoint, we need to describe the vector which begins at the origin and ends at this point:



The point is that simply taking $\frac{1}{2}\overrightarrow{RQ}$ does not work, since the coordinates of $\frac{1}{2}\overrightarrow{RQ}$ correspond to the endpoint of this vector as it were drawn starting at the origin instead of starting at R. To get the correct vector, we must take the sum

$$\overrightarrow{OR} + \frac{1}{2}\overrightarrow{RQ}$$

since this sum is what describes the vector obtained by first moving along $\overrightarrow{OR}$ to get from O to R, and then moving halfway along $\overrightarrow{RQ}$ to get to the desired midpoint. We have

$$\overrightarrow{RQ} = \langle 2-(-1), -1-3, 2-4\rangle = \langle 3, -4, -2\rangle\,,$$

so

$$\overrightarrow{OR} + \frac{1}{2}\overrightarrow{RQ} = \langle -1, 3, 4\rangle + \frac{1}{2}\langle 3, -4, -2\rangle = \left\langle \tfrac{1}{2}, 1, 3\right\rangle.$$

The midpoint of segment RQ is thus $(\frac{1}{2}, 1, 3)$.

Now we can write down the vector that goes from P to this midpoint: it is

$$\mathbf{u} = \left\langle \tfrac{1}{2}, 1, 3\right\rangle - \langle 1, 1, 1\rangle = \left\langle -\tfrac{1}{2}, 0, 2\right\rangle.$$

This vector has length $|\mathbf{u}| = \sqrt{\frac{1}{4}+0+4} = \sqrt{17/4} = \sqrt{17}/2$, which is larger than 3, so it is not the vector we want. To modify the length, we first divide $\mathbf{u}$ by its length to get a unit vector in the desired direction:

$$\frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{2}{\sqrt{17}}\left\langle -\tfrac{1}{2}, 0, 2\right\rangle,$$

and then we scale by 3 to make the length 3 instead of 1:

$$\frac{3}{|\mathbf{u}|}\mathbf{u} = \frac{6}{\sqrt{17}}\left\langle -\tfrac{1}{2}, 0, 2\right\rangle = \left\langle -\frac{3}{\sqrt{17}}, 0, \frac{12}{\sqrt{17}}\right\rangle.$$

This is hence the vector of length 3 pointing from P towards the midpoint of segment RQ.

$\mathbf{i},\mathbf{j},\mathbf{k}$ notation. Let us take a brief aside to introduce another common notation for vectors, namely $\langle a, b, c\rangle = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$. Here, $\mathbf{i}$ denotes $\langle 1, 0, 0\rangle$, $\mathbf{j}$ denotes $\langle 0, 1, 0\rangle$, and $\mathbf{k}$ denotes $\langle 0, 0, 1\rangle$, and writing a vector as a sum of multiples just emphasizes the geometry: for $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$, we move

a in the "**i**-drection", b in the "**j**-direction, and c in the "**k**-direction". Similarly, we can express two-dimensional vectors in this way using $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.

The dot product. The notion of the dot product of two vectors will be an essential tool going forward. Given vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$, their *dot product*, denoted $\mathbf{u}\cdot\mathbf{v}$ (pronounced "**u** dot **v**"), is the number given by taking the sum of products of their corresponding components:

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

In two dimensions the analogous definition applies, only with no u_3v_3 term. For a first basic example, we have

$$\langle 1, 2, 3\rangle \cdot \langle -2, 4, 1\rangle = 1(-2) + 2(4) + 3(1) = -2 + 8 + 3 = 9.$$

The dot products of a vectors is a type of "multiplication" of vectors, and it is important to note that the result of the dot product is always a *number*, and not a vector.

Now, why do we care about the number given by the dot product? What geometric information is it giving? The answer is the following:

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ drawn as arrows starting at the same point. (We always take the angle to be between 0 and π.) The upshot is that the value of the dot product is directly related geometrically to the position of the vectors in relation to one another, as measured by the angle between them, and to their lengths. We will not justify this geometric formula in this course, but if you're interested in learning more, it comes from the "law of cosines" in trigonometry.

For the example above, with $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle -2, 4, 1\rangle$, we have

$$9 = |\mathbf{u}||\mathbf{v}|\cos\theta = \sqrt{14}\sqrt{21}\cos\theta, \text{ so } \cos\theta = \frac{9}{\sqrt{14}\sqrt{21}}.$$

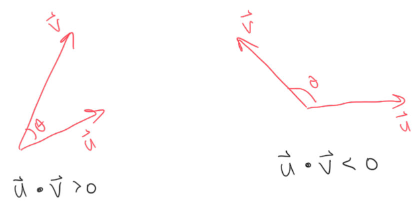
From this, using the arccos function and a calculator, we can directly determine the angle between $\mathbf{u}$ and $\mathbf{v}$, which in this case is $\arccos(9/\sqrt{14\cdot 21}) \approx 1.0182$ radians, or about 58.34°.

Positive vs negative. In the example above we saw that the angle between $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle -2, 4, 1\rangle$ was between 0 and $\frac{\pi}{2}$ radians ($\frac{\pi}{2}$ is larger than 1.0182), so that this is an acute angle. But actually, this is something we can determine without computing the angle explicitly, solely from the fact that the dot product of $\mathbf{u}$ and $\mathbf{v}$ is positive. Indeed, in

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

the two length terms on the right are never negative, so the sign of the number $\mathbf{u}\cdot\mathbf{v}$ is the same as the sign of $\cos\theta$; that is, $\mathbf{u}\cdot\mathbf{v}$ is positive if and only if $\cos\theta$ is positive, and $\mathbf{u}\cdot\mathbf{v}$ is negative if and only if $\cos\theta$ is negative. But $\cos\theta$ is positive when $0 \le \theta < \frac{\pi}{2}$, and $\cos\theta$ is negative when $\frac{\pi}{2} < \theta \le \pi$, so the upshot is that

> Dot products are positive when the two vectors make up an acute angle (less than 90°), and dot products are negative when the two vectors make up an obtuse (greater than 90° angle).



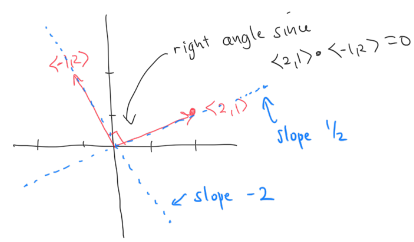
Thus, the sign of the dot product completely tells us whether the two vectors point in a "similar" direction or in somewhat "opposite" directions. (We will see this come up when discussing "projections" in a bit.)

Orthogonality. What about when the dot product between two vectors is zero? In this case the geometric formula for the dot product reads

$$0 = |\mathbf{u}||\mathbf{v}| \cos\theta.$$

For nonzero vectors the two length terms on the right are nonzero, so the dot product is zero if and only if $\cos\theta = 0$, which happens only when $\theta = \frac{\pi}{2}$. The upshot is that the dot product between two vectors is zero precisely when they are *orthogonal* to one another, which is just another way of saying *perpendicular* to one another. Thus, the dot product gives a purely algebraic way to determine whether or not vectors are orthogonal.

For example, $\langle 2, 1\rangle \cdot \langle -1, 2\rangle = -2+2 = 0$, so $\langle 2, 1\rangle$ and $\langle -1, 2\rangle$ are indeed perpendicular to each other. This makes sense visually:



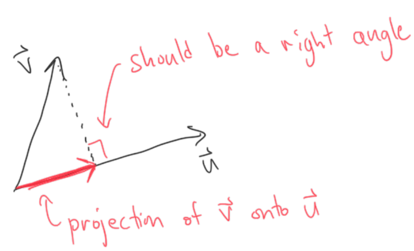
The "slope" of the vector $\langle 2, 1\rangle$ is $\frac{1}{2}$, and the "slope" of $\langle -1, 2\rangle$ is $\frac{2}{-1} = -2$; since these slopes are negative reciprocals of each other, $\langle 2, 1\rangle$ and $\langle -1, 2\rangle$ are indeed perpendicular/orthogonal.

This observation about zero dot product is more useful in three dimensions when whether or not things are orthogonal is tougher to determine based on a picture alone. For example, the dot product of $\langle 1, 2, 1\rangle$ and $\langle -1, 2, -3\rangle$ is

$$\langle 1, 2, 1\rangle \cdot \langle -1, 2, -3\rangle = -1 + 4 - 3 = 0,$$

so $\langle 1, 2, 1\rangle$ and $\langle -1, 2, -3\rangle$ are orthogonal, which is not easy to see based on a three-dimensional picture alone.

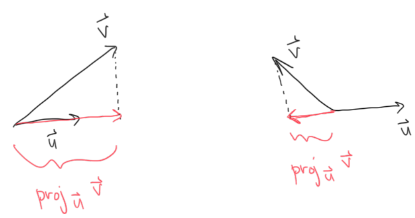
Vector projections. One important use of dot products comes in computing *vector projections.* (Also called *orthogonal projections.*) Given vectors $\mathbf{u}$ and $\mathbf{v}$, the goal is to describe the multiple of $\mathbf{u}$ that is "closest" to $\mathbf{v}$ in the sense of the following picture:



The vector we want is thus the one with the property that the line segment connecting its endpoint to the endpoint of $\mathbf{v}$ is *orthogonal* to $\mathbf{u}$ itself. This vector is called the vector (or orthogonal) projection of $\mathbf{v}$ onto $\mathbf{u}$; we denote it by $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ and is explicitly given by

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}.$$

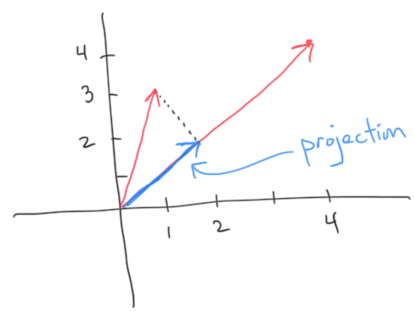
To be clear here, dot products give numbers, so the fraction $\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}$ is a number, and it is this particular multiple of $\mathbf{u}$ which gives the projection. We will not derive the formula for the projection here, but it comes from determine exactly which multiple $a\mathbf{u}$ of $\mathbf{u}$ results in $\mathbf{v} - a\mathbf{u}$ being orthogonal to $\mathbf{u}$. (The vector $\mathbf{v} - a\mathbf{u}$ is the one pointing from the endpoint of $a\mathbf{u}$ to the endpoint of $\mathbf{v}$. It completes the right triangle in the first projection picture we gave above.) Note that it does not matter if $\mathbf{u}$ is shorter or points in the opposite direction, the projection is perfectly well-defined either way:



Let us compute, for example, the projection of $\mathbf{v} = \langle 1, 3 \rangle$ onto $\mathbf{u} = \langle 4, 4 \rangle$. This is

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \left(\frac{4 + 12}{16 + 16} \right) \langle 4, 4 \rangle = \frac{16}{32} \langle 4, 4 \rangle = \langle 2, 2 \rangle .$$

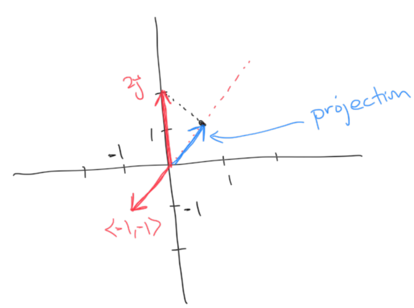
Visually we have



As another example, the projection of $2\mathbf{j}$ onto $\mathbf{v} = \langle -1, -1 \rangle$ is

$$\operatorname{proj}_{\mathbf{v}} 2\mathbf{j} = \left(\frac{\langle 0, 2 \rangle \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \left(\frac{-2}{2} \right) \langle -1, -1 \rangle = \langle 1, 1 \rangle .$$

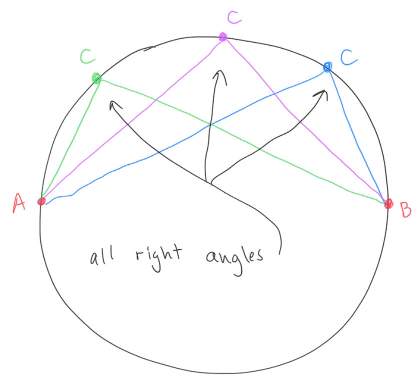
In this case, note that the vector $\langle -1, -1 \rangle$ we projected onto is in the opposite direction of the projection $\langle 1, 1 \rangle$, which reflects the fact that $2\mathbf{j}$ and $\langle -1, -1 \rangle$ meet an angle greater than $\frac{\pi}{2}$ from each other:



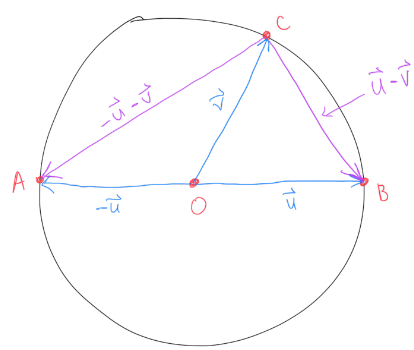
The fact that $\mathbf{v}$ got flipped around comes from the fact that the dot product $2hj \cdot \mathbf{v} = -2$ in the numerator of the projection formula is negative, which, from earlier, also reflects the fact that the angle between these vectors is greater than $\frac{\pi}{2}$.

Lecture 4: Cross Product

Warm-Up 1. Take a circle, and two points A and B on opposite ends of the diameter. Now take any other point on C on the circle, and consider the angle at C formed by the segments AC and BC. We use properties of vectors to verify that this angle is always a right angle, regardless of which point C we pick:



This a more conceptual problem than previous ones we've seen, and is meant to highlight algebraic properties of the dot product that mimic properties with which we're already familiar when it comes to ordinary multiplication of numbers. We include the center O of the circle, and label vectors as follows:



To be clear: the vector $\overrightarrow{OA}$ is $-\mathbf{u}$ because it has the same length as $\mathbf{u} = \overrightarrow{OB}$ (both of these lengths correspond to the radius of the circle) but points in opposite direction; the vector connecting the endpoint of $\mathbf{v}$ to the endpoint of $\mathbf{u}$ is $\mathbf{u} - \mathbf{v}$ (start minus end); and the vector connecting the endpoint of $\mathbf{v}$ to the endpoint of $-\mathbf{u}$ is $-\mathbf{u} - \mathbf{v}$.

In this notation, what we need to verify is that $\mathbf{u} - \mathbf{v}$ is orthogonal to $-\mathbf{u} - \mathbf{v}$, since the angle between these vectors is precisely the angle at C we want. To verify this, we must compute the dot product

$$(\mathbf{u} - \mathbf{v}) \cdot (-\mathbf{u} - \mathbf{v})$$

of these vectors and see that it is zero. First, we use the fact that dot product satisfies a "distributive" property, meaning that in the expression above we can simply "multiply everything out" as you normally would for something like $(a + b)(c + d)$. We get four terms:

$$(\mathbf{u} - \mathbf{v}) \cdot (-\mathbf{u} - \mathbf{v}) = -\mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}.$$

The second term came from $\mathbf{u}$ in the first parentheses dot $-\mathbf{v}$ in the second, and the third from $-\mathbf{v}$ in the first parenthses dot $-\mathbf{u}$ in the second. (The two negatives here combine to be $\mathbf{v} \cdot \mathbf{u}$ instead of $-\mathbf{v} \cdot \mathbf{u}$. The same is true for $-\mathbf{v}$ from the first parentheses dot $-\mathbf{v}$ from the second to get $\mathbf{v}\dot{\mathbf{v}}$ as the fourth term.) Next, we use the fact that dot product is "commutative", which means that order in which dot two vectors does not matter: $\mathbf{u} \cdot \mathbf{v}$ is the same as $\mathbf{v} \cdot \mathbf{u}$. This results in the second and third terms above, $-\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u}$, cancelling each other out. So, we are left with

$$(\mathbf{u} - \mathbf{v}) \cdot (-\mathbf{u} - \mathbf{v}) = -\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}.$$

The final property we use is that taking a vector dot *itself* has a nice geometry interpretation: apply $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ when $\mathbf{v}$ is the same as $\mathbf{u}$ gives $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}||\mathbf{u}|cos0$, since the angle between $\mathbf{u}$ and $\mathbf{u}$ itself is zero. But $\cos 0 = 1$, so

$$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2.$$

Thus, the dot product of a vector with itself always gives the length of that vector squared, so

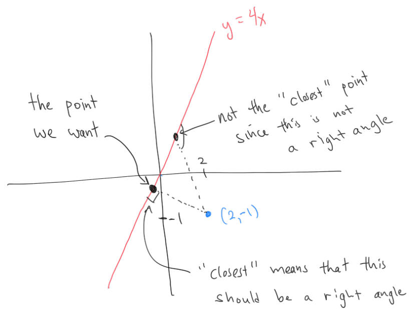
$$(\mathbf{u} - \mathbf{v}) \cdot (-\mathbf{u} - \mathbf{v}) = -|\mathbf{u}|^2 + |\mathbf{v}|^2.$$

However, $|\mathbf{u}|$ and $|\mathbf{v}|$ are the same since both of these lengths are the radius of the circle we are considering, so $|\mathbf{u}|^2$ is the same as $|\mathbf{v}|^2$, and hence

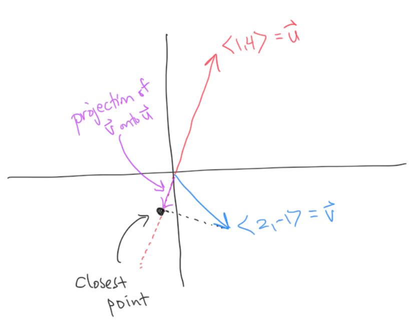
$$(\mathbf{u} - \mathbf{v}) \cdot (-\mathbf{u} - \mathbf{v}) = -|\mathbf{u}|^2 + |\mathbf{v}|^2 = 0.$$

Since the dot product of these vectors is zero, $\mathbf{u} - \mathbf{v}$ and $-\mathbf{u} - \mathbf{v}$ are orthogonal to each other, so the angle at C is a right angle as we claimed.

Warm-Up 2. We find the point on the line $y = 4x$ in the xy-plane that is closest to $(2, -1)$. In fact, this is simply a problem about computing a vector projection. Indeed, here is the picture of what we want:



The closest point we want is precisely the *endpoint* of the projection of $\langle 2, -1 \rangle$ onto *any* vector which lies on the line $y = 4x$. For example, the vector $\langle 1, 4 \rangle$ lies on this line (since its y-coordinate is indeed 4 times its x-coordinate), so the projection of $\langle 2, -1 \rangle$ onto $\langle 1, 4 \rangle$ should give us the point we need:



Note that this projection points in the direction opposite $\langle 1,4\rangle$, so we expect to get a negative coefficient in the vector projection formula, which, as stated last time, reflects the fact that the angle between $\langle 2,-1\rangle$ and $\langle 1,4\rangle$ is greater than $\frac{\pi}{2}$.

The projection of $\mathbf{v} = \langle 2,-1\rangle$ onto $\mathbf{u} = \langle 1,4\rangle$ is

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \left(\frac{2-4}{1+16}\right)\langle 1,4\rangle = -\frac{2}{17}\langle 1,4\rangle = \left\langle -\tfrac{2}{17}, -\tfrac{8}{17}\right\rangle.$$

The point on the line $y = 4x$ that is closest to $(2,-1)$ is thus $(-2/17, -8/17)$. Note that picking a different vector on $y = 4x$ instead of $\langle 1,4\rangle$ leads to the same projection: if we use $\mathbf{u} = \langle -2,-8\rangle$ as the vector on the line instead (it's coordinates still satisfy $y = 4x$), the projection of $\mathbf{v} = \langle 2,-1\rangle$ onto $\mathbf{u}$ is

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \left(\frac{-4+8}{4+64}\right)\langle -2,-8\rangle = \frac{4}{68}\langle -2,-8\rangle = \left\langle -\tfrac{2}{17}, -\tfrac{8}{17}\right\rangle,$$

which gives the same answer as before.

The cross product. Taking the dot product of two vectors is in some sense a way to "multiply" them together. But while this type of "multiplication" results in a *number* as a result, we now look at one final vector operation, which is another type of "multiplication", only this time resulting in a *vector*. Rather than giving a general definition, it is simplest to jump straight into an example to illustrate the desired computation.

Take $\mathbf{u} = 2\mathbf{i} - 3\,\mathbf{j} - \mathbf{k} = \langle 2,-3,-1\rangle$ and $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k} = \langle 1,2,-2\rangle$. The *cross product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}\times\mathbf{v}$ (pronounced "$\mathbf{u}$ cross $\mathbf{v}$") is the vector computed in the following way. We setup of the following expression:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 1 & 2 & -2 \end{vmatrix}.$$

This is an example of what's called a "3 by 3 *determinant*", but it will not be important to know what "determinant" means in this course; we will simply view this notation as a way to help us compute cross product. (Determinants are things you would learn more about in a course in *linear algebra*, such as MATH 240.) The key point to remember is that the coefficients of $\mathbf{i},\mathbf{j},\mathbf{k}$ in the cross product come from the numbers in the array above that are *not* in the same column as that vector: the coefficient of $\mathbf{i}$ comes from the numbers in the second and third columns; the coefficient of $\mathbf{j}$ from the numbers in the first and third columns; and the coefficient of $\mathbf{k}$ from the numbers in the first and second columns. At first this computation will seem to come out of nowhere, but we will come to the geometric interpretation of all this afterwards. In the notation above, the second row is made up of the components of $\mathbf{u}$ (the first vector in our cross product $\mathbf{u}\times\mathbf{v}$ notation), and the third row comes from the components of the second vector $\mathbf{v}$. Also note that cross products only make sense for three-dimensional vectors, although given a two-dimensional vector like $\langle 2,3\rangle$, by introducing a z-coordinate of zero $\langle 2,3,0\rangle$, we can still compute cross products of vectors in the xy-plane viewed as sitting inside of $\mathbb{R}^3$.

For the coefficient of $\mathbf{i}$ we use the numbers in the second and third columns, second and third rows:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & \color{red}{-3} & \color{red}{-1} \\ 1 & \color{red}{2} & \color{red}{-2} \end{vmatrix}.$$

We multiply the "diagonal" entries -3 and -2, and subtract the product of the "off-diagonal" entries 2 and -1. The coefficient of $\mathbf{i}$ in the cross product is thus

$$(-3)(-2) - (2)(-1) = 6 + 2 = 8.$$

For the coefficient of $\mathbf{j}$ we do the same thing, only now using the numbers in the first and third columns, second and third rows:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \color{red}{2} & -3 & \color{red}{-1} \\ \color{red}{1} & 2 & \color{red}{-2} \end{vmatrix}.$$

We take the diagonal product $2(-2)$ and the off-diagonal product $1(-1)$, and subtract:

$$2(-2) - 1(-1) = -4 + 1 = -3.$$

This is almost the coefficient of $\mathbf{j}$, but for this coefficient (and this coefficient only) we change the sign, so that the coefficient of $\mathbf{j}$ in the cross product is 3 and not -3. (The reason why the same changes has to do with how determinants work in general, but this is not something we will focus on. For our purposes, just have it ingrained in your minds that the $\mathbf{j}$-coefficient changes sign.

Finally, the coefficient of $\mathbf{k}$ uses the numbers in the first and second columns, second and third rows:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \color{red}{2} & \color{red}{-3} & -1 \\ \color{red}{1} & \color{red}{2} & -2 \end{vmatrix}.$$

We get $2(2) - 1(-3) = 4 + 3 = 7$ (no sign change here) as the coefficient of $\mathbf{k}$. Altogether then, the cross product of $\mathbf{u} = \langle 2, -3, -1 \rangle$ and $\mathbf{v} = \langle 1, 2, -3 \rangle$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 1 & 2 & -2 \end{vmatrix} = (6 - (-2))\,\mathbf{i} \underbrace{-}_{\text{sign change}} (-4 - (-1))\,\mathbf{j} + (4 - (-3))\,\mathbf{k} = \langle 8, 3, 7 \rangle.$$

After trying this in a few examples, this computation should become fairly quick to carry out.

Geometric meaning. The question remains: Why would anyone think of doing such a random-looking computation? What is the point behind it? The answer comes the following observation: in the example with

$$\mathbf{u} = \langle 2, -3, -1 \rangle, \ \mathbf{v} = \langle 1, 2, -2 \rangle, \ \mathbf{u} \times \mathbf{v} = \langle 8, 3, 7 \rangle$$

above, we note that

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= \langle 8, 3, 7 \rangle \cdot \langle 2, -3, -1 \rangle = 16 - 9 - 7 = 0, \text{ and} \\ (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} &= \langle 8, 3, 7 \rangle \cdot \langle 1, 2, -2 \rangle = 8 + 6 - 14 = 0. \end{aligned}$$

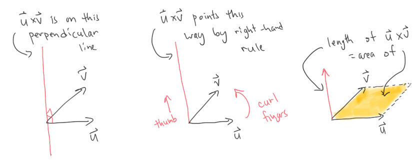
From this we thus conclude that the cross product $\mathbf{u} \times \mathbf{v}$ is in fact perpendicular to both $\mathbf{u}$ and $\mathbf{v}$! For our purposes, *this* is the main reason why we care about the cross product: it always gives a vector perpendicular to both vectors with which we began.

In fact, the cross product of two vectors encodes much geometric information, which we summarize here:

- $\mathbf{u} \times \mathbf{v}$ is always orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (as we said above);

- the direction in which $\mathbf{u} \times \mathbf{v}$ points is determined by the "right-hand rule": curling the fingers of your right hand from $\mathbf{u}$ towards $\mathbf{v}$ results in your thumb pointing in the direction of $\mathbf{u} \times \mathbf{v}$;
- the length of $\mathbf{u} \times \mathbf{v}$ is $|\mathbf{u}||\mathbf{v}| \sin \theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, which is precisely the area of the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as edges.

These properties fully characterize the cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$, and it is somewhat a miracle of nature that the crazy-looking computation we described before results in something that has all of these properties. The reasons for why this is again have to do with the notion of a "determinant", but we will say no more about this in this course. Visually, given $\mathbf{u}$ and $\mathbf{v}$, the first property tells us the "line" along which $\mathbf{u} \times \mathbf{v}$ will point since there is only one line that will be orthogonal to both $\mathbf{u}$ and $\mathbf{v}$; then, the direction we want along this line is determined by the right-hand rule; and finally, the length that we should go in this direction is determined by the area of the appropriate parallelogram:



Note that flipping the order in which we compute the cross product, say $\mathbf{v} \times \mathbf{u}$ instead of $\mathbf{u} \times \mathbf{v}$, simply has the effect of multiplying the cross product by -1:

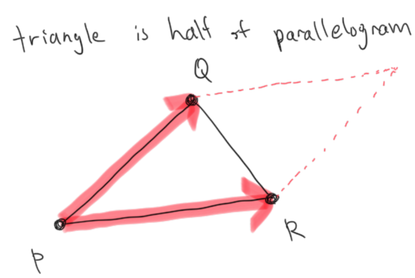
$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v},$$

thereby changing its direction. This comes from the right-hand rule, where your thumb points in the opposite direction if you curl your fingers from $\mathbf{v}$ towards $\mathbf{u}$ instead of from $\mathbf{u}$ towards $\mathbf{v}$.

Example. For our purposes, the most important use of the cross product will be to produce a vector that is orthogonal to two given ones. But the other geometric properties can be useful too. For example, let us compute the area of the triangle in $\mathbb{R}^3$ with vertices

$$P = (1, 1, 1), \; Q = (2, -1, 0), \; R = (-1, 3, 2).$$

The point is that we can view this triangle as half of a parallelogram, and then use a cross product to compute the area of this parallelogram:



In this case, the edges of the parallelogram are given by

$$\overrightarrow{PQ} = \langle 2, -1, 0 \rangle - \langle 1, 1, 1 \rangle = \langle 1, -2, -1 \rangle \quad \text{and} \quad \overrightarrow{PR} = \langle -1, 3, 2 \rangle - \langle 1, 1, 1 \rangle = \langle -2, 2, 1 \rangle .$$

The cross product of these is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -1 \\ -2 & 2 & 1 \end{vmatrix} = (-2 - (-2))\,\mathbf{i} - (1 - (-2)(-1))\,\mathbf{j} + (2 - (-2)(-2))\,\mathbf{k} = \langle 0, 1, -2 \rangle .$$

(Note again the extra sign change in the $\mathbf{j}$ term.) The area of the parallelogram with edges $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle 0, 1, -2\rangle| = \sqrt{0^2 + 1^2 + (-2)^2} = \sqrt{5},$$

and so the area of the triangle in question is

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{\sqrt{5}}{2}.$$

In fact, instead of using $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as edges of the parallelogram, we could have used any pairs of vectors giving the sides of the triangle, say $\overrightarrow{QP}$ and $\overrightarrow{QR}$, or $\overrightarrow{RP}$ and $\overrightarrow{RQ}$ instead. You can check (which you should do for the sake of practice!) that

$$\frac{1}{2}|\overrightarrow{QP} \times \overrightarrow{QR}| \quad \text{and} \quad \frac{1}{2}|\overrightarrow{RP} \times \overrightarrow{RQ}|$$

both give $\frac{\sqrt{5}}{2}$ as well. (In other words, we can view the given triangle as half of three different possible parallelograms, but all should give the same area.)

Another example. Now we want to find a vector of length 2 that is orthogonal to both $\mathbf{u} = \langle 2, 1, 2\rangle$ and $\mathbf{v} = \langle -1, 1, 1\rangle$. The method is clear: first find *some* vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then modify its length as needed. A possible vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is given by the cross product of the two:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ -1 & 1 & 1 \end{vmatrix} = \langle 1 - 2, -(2 - (-2)), 2 - (-1)\rangle = \langle -1, -4, 3\rangle.$$

Then to modify the length, we first get a unit vector in this direction:

$$\frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \frac{1}{\sqrt{1 + 16 + 9}}\langle -1, -4, 3\rangle = \left\langle -\tfrac{1}{\sqrt{26}}, -\tfrac{4}{\sqrt{26}}, \tfrac{3}{\sqrt{26}}\right\rangle,$$

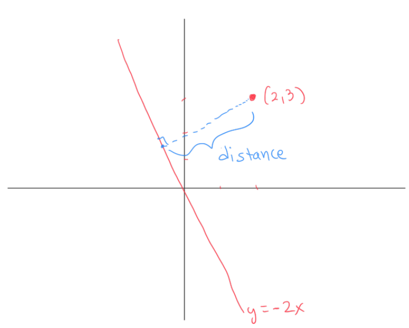
and then scale by 2:

$$2\frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = 2\left\langle -\tfrac{1}{\sqrt{26}}, -\tfrac{4}{\sqrt{26}}, \tfrac{3}{\sqrt{26}}\right\rangle = \left\langle -\tfrac{2}{\sqrt{26}}, -\tfrac{8}{\sqrt{26}}, \tfrac{6}{\sqrt{26}}\right\rangle.$$

You can verify if you'd like that this resulting vector does indeed have length 2 and is indeed orthogonal to both $\mathbf{u} = \langle 2, 1, 2\rangle$ and $\mathbf{v} = \langle -1, 1, 1\rangle$.

Lecture 5: Lines

Warm-Up. We use a cross product to find the distance from the point $(2, 3)$ to the line $y = -2x$ in the xy-plane. We have not seen this type of computation before, so the goal is to not only find the distance but also to understand exactly *how* we can approach something like this. To be clear distance is visually the following:



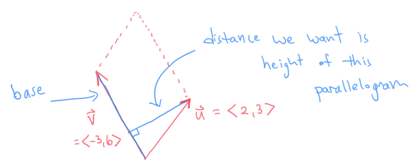
By definition, by the "distance" from a point to a line we meant the *shortest* distance we can obtain from that point to any point on the line, or in other words, the distance from the given point to the point on the line to which it is *closest*. In the picture above, the line segment whose length is the desired distance is a perpendicular to the line in question. This use of "closest" and "perpendicular" might remind you of the Warm-Up example from last time, where we used dot products and vector projections to find the point on a line closest to a given point, and indeed the approach from last time gives one way of approaching this new distance problem: we find the point on $y = -2x$ closest to $(2, 3)$ using a projection, and find the distance between the point we find and $(2, 3)$. In this approach, we take $\mathbf{v} = \langle 1, -2 \rangle$ as a vector lying along this line (y- coordinate is -2 times the x-coordinate), and project $\mathbf{u} = \langle 2, 3 \rangle$ onto it:

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \left(\frac{2-6}{1+4} \right) \langle 1, -2 \rangle = \left\langle -\tfrac{4}{5}, \tfrac{8}{5} \right\rangle.$$

Thus $(-4/5, 8/5)$ is the point on $y = -2x$ that it closest to $(2, 3)$, so the distance from $(2, 3)$ to this line is the distance between these two points:

$$\text{distance} = \sqrt{(2-(-4/5))^2 + (3-8/5)^2} = \sqrt{\tfrac{196}{25} + \tfrac{49}{25}} = \tfrac{\sqrt{245}}{5}.$$

But our aim in this Warm-Up is to find this distance using a cross product instead. The key is the interpretation of the length of the cross product in terms of area. Consider the following picture:



Here we have a parallelogram with edges $\mathbf{v} = \langle -3, 6 \rangle$ (lying along the line $y = -2x$) and $\mathbf{u} = \langle 2, 3 \rangle$. The desired distance, going perpendicularly from $(2, 3)$ to $\mathbf{v} = \langle -3, 6 \rangle$, is precisely the *height* of this parallelogram. (A "height" of a parallelogram is just the perpendicular distance from one corner to the line passing through the opposite edge.) The area of parallelogram is precisely such a height times the length of the "base" (the edge opposite the corder from which we compute the height), which in this case is the vector $\mathbf{v}$, so we can express the height as

$$\text{height} = \frac{\text{area}}{\text{length of base}}.$$

We can use a cross product to find the area, so thus this formula will give us a way to find the height, which is the desired distance.

The only wrinkle is that we are dealing with two-dimensional vectors $\mathbf{v} = \langle -3, 6 \rangle$ and $\mathbf{u} = \langle 2, 3 \rangle$, but the notion of "cross product" only applies to three=dimensional vectors. But this is simple to fix: we simply view our chosen vectors as vectors in the xy-plane in three-dimensions instead by including a z-component of zero; i.e. we use $\mathbf{v} = \langle -3, 6, 0 \rangle$ and $\mathbf{u} = \langle 2, 3, 0 \rangle$ instead. The cross product of these is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -3 & 6 & 0 \end{vmatrix} = \langle 0-0, -(0-0), 12-(-9) \rangle = \langle 0, 0, 21 \rangle.$$

Note that it makes sense that the cross-product has zero x and y components, so that it points only in the z-direction: $\mathbf{u} \times \mathbf{v}$ should be perpendicular to $\mathbf{u}$ and $\mathbf{v}$, and since $\mathbf{u}$ and $\mathbf{v}$ lie in the xy-plane, the perpendicular direction is indeed the z-direction. Moreover, the fact that we get a positive z-component and not negative is a reflection of the right-hand rule: curling $\mathbf{u}$ towards $\mathbf{v}$ with your right hand will have your thumb pointing in the positive z-direction, which is the direction coming out "at" us in the original two-dimensional picture.

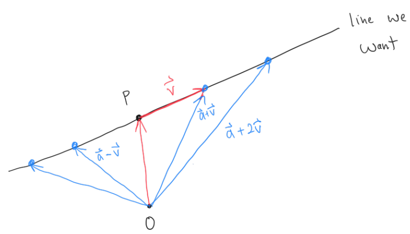
The distance from $(2, 3)$ to the line $y = -2x$, or in other words the height of the parallelogram with edges $\mathbf{u} = \langle 2, 3, 0\rangle$ and $\mathbf{v} = \langle -3, 6, 0\rangle$, is thus

$$\text{height} = \frac{\text{area}}{\text{length of base}} = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{0^2 + 0^2 + 21^2}}{\sqrt{9 + 36 + 0}} = \frac{21}{\sqrt{45}}.$$

In fact, this the same as the answer $\sqrt{245}/5$ we found using the projection approach, which you can see after you simplify the square roots a bit.

Lines. Our goal now is to use the material on vectors we've developed so far to describe certain geometric objects of interest, namely lines and planes. We start for now with lines. In two-dimensions, a line is given by an equation like $y = mx + b$. However, something like this cannot describe a line in three-dimensions, and indeed we've already seen examples (like $y = x$ on the second day of class) where equations of this form describe planes in $\mathbb{R}^3$: namely, $y = mx + b$ is the plane obtained by taking the line $y = mx + b$ in the xy-plane and sliding it in the z-direction to sweep out a plane.

So we need to look elsewhere if we want to describe lines. The answer comes from the following picture:



Here, we want to describe the line passing through a given point P and moving in the direction of a given vector $\mathbf{v}$. It turns out that this is the only data we need: to describe a line, we need a point on the line and a direction vector for the line. In the picture above, $\mathbf{a} = \overrightarrow{OP}$ is the position vector of the point P (i.e., the vector drawn at the origin to end at P), and the point is that we can obtain other points on this line by adding to $\mathbf{a}$ multiples of $\mathbf{v}$; for example, $\mathbf{a} + \mathbf{v}$ as drawn ends at a point on the line, so does $\mathbf{a} + 2\mathbf{v}$, and so does $\mathbf{a} - \mathbf{v}$. (Here we are using the triangle interpretation of vector addition.) In general, the endpoints of the vectors $\mathbf{a} + t\mathbf{v}$, where t is a parameter that varies, trace out the desired line. We call

$$\mathbf{r}(t) = \mathbf{a} + t\mathbf{v}$$

the *vector* equation of the line in the direction of $\mathbf{v}$ and passing through P, the endpoint of $\mathbf{a} = \overrightarrow{OP}$.

Here is a first example. We want to describe the line pass through the points $(1, 2, 3)$ and $(0, 2, -4)$, and to do so we need a point on the line and vector which gives the direction of the desired line. We are given two points, so we pick one to use as P in the notation above, say $P = (1, 2, 3)$. For a direction vector, we take the vector going from one given point to the next, so

$$\mathbf{v} = \langle 0, 2, -4\rangle - \langle 1, 2, 3\rangle = \langle -1, 0, -7\rangle\,.$$

With $\mathbf{a} = \overrightarrow{OP} = \langle 1, 2, 3 \rangle$, we get that the line passing through $(1, 2, 3)$ and $(0, 2, -4)$ has vector equation

$$\mathbf{r}(t) = \mathbf{a} + t\mathbf{v} = \langle 1, 2, 3 \rangle + t \langle -1, 0, -7 \rangle = \langle 1 - t, 2, 3 - 7t \rangle .$$

Again, the interpretation of this is that as t varies, the endpoints of this vector $\mathbf{r}(t)$ trace out the desired line. Note that other valid vector equations are possible: perhaps we might have used $P = (0, 2, -4)$ as our chosen point on the line, or $\mathbf{v} = -3 \langle -1, 0, -7 \rangle = \langle 3, 0, 21 \rangle$ as a direction vector. Such choices will give a different equation, but will nonetheless describe the same line.

Parametric equations. Vector equations give a first way of describing lines, but in practice it is often useful to have *parametric equations* instead. Parametric equations are equations for the x, y, and z coordinates of points on the line, that depend on a parameter: as the parameter varies, the point changes and the line is traced out. We extract these from the x, y, and z components of the vector equation. For example, for the passing through $(1, 2, 3)$ and $(0, 2, -4)$, which has vector equation

$$\mathbf{r}(t) = \langle 1 - t, 2, 3 - 7t \rangle ,$$

we get the parametric equations

$$x = 1 - t, \ y = 2, \ z = 3 - 7t.$$

Again, these equations give valid x, y, z coordinates for all points on the line, with different points occurring for different values of the parameter t.

To get the entire line we must allow all values of t, so we might right

$$x = 1 - t, \ y = 2, \ z = 3 - 7t, \quad -\infty < t < \infty$$

to indicate this. If we place restrictions on what values of t we allow, we only describe certain portions of the line. For example,

$$x = 1 - t, \ y = 2, \ z = 3 - 7t, \quad t > 0$$

gives only half of the line, or in other words a *ray*, namely the one that starts at $(1, 2, 3)$ and passes through $(0, 2, -4)$ but includes no points on the "other side" of $(1, 2, 3)$. The parametric equations

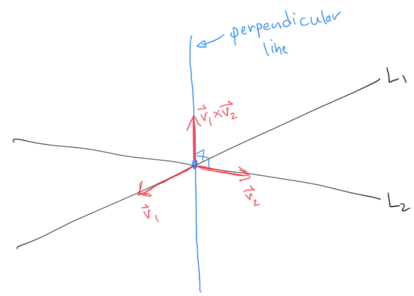
$$x = 1 - t, \ y = 2, \ z = 3 - 7t, \quad 0 \leq t \leq 1$$

describe only a line *segment*, namely the one going between $(1, 2, 3)$ and $(0, 2, -4)$ and no further. (Note that at $t = 0$ we are at the point $(x, y, z) = (1 - 0, 2, 3 - 7(0)) = (1, 2, 3)$, while at $t = 1$ we are at $(x, y, z) = (1 - 1, 2, 3 - 7(1)) = (0, 2, -4)$.)

Another example. Suppose we are given lines with parametric equations

$$\begin{cases} x = 1 + 4t \\ y = 2 - t \\ z = 1 + t \end{cases} \qquad \text{and} \qquad \begin{cases} x = 1 - 2t \\ y = 2 + 3t \\ z = 1 - t. \end{cases}$$

To be clear, we are considering one line L_1 given by the first set of parametric equations, and a second line L_2 given by the second set. We want to find (by which we mean find parametric equations for) the line that is perpendicular to both L_1 and L_2 and which passes through their point of intersection:



As always, to find these parametric equations we need a point on the desired line and a vector that gives the direction of the desired line. The point we need is the point where L_1 and L_2 intersect, and in this case this is simple to find: $(1, 2, 1)$ is the point of intersection. Indeed, note that $(1, 2, 1)$ is the point on L_1 occurring when $t = 0$, and that it is also the point on L_2 given by $t = 0$. (Note that the constant terms in the parametric equations always give a point on the line.) Since $(1, 2, 1)$ lies on both lines, it must be the point where they intersect. (Finding points of intersection will not always be this simple; we will see a more general approach to finding such points as a Warm-Up next time.)

We have our point so now we need our direction vector. This direction vector should be perpendicular to both lines, so in particular it should be perpendicular to the direction vectors of these two lines. This says that the cross product of these two direction vectors should thus point in the perpendicular direction we want. (See the picture above.) Finding direction vectors for a line when given its parametric equations is also simple: we simply use the coefficients of the parameter. So, since L_1 has parametric equations

$$x = 1 + 4t,\ y = 2 - t,\ z = 1 + t,$$

it has direction vector $\mathbf{v}_1 = \langle 4, -1, 1\rangle$ (i.e., the coefficients of t), and L_2 has direction vector $\mathbf{v}_2 = \langle -2, 3, -1\rangle$. This works because of where these coefficients come from in our original vector equation: in $\mathbf{r}(t) = \mathbf{a} + t\mathbf{v}$, the numbers which show up as the coefficients of t are precisely the entries of the direction vector $\mathbf{v}$.

The line we want thus has direction vector

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ -2 & 3 & -1 \end{vmatrix} = \langle 1 - 3, -(-4 - (-2)), 12 - 2\rangle = \langle -2, 2, 10\rangle\,.$$

With these direction and point $(1, 2, 1)$ on the line we get

$$\mathbf{r}(t) = \langle 1, 2, 1\rangle + t\,\langle -2, 2, 10\rangle = \langle 1 - 2t, 2 + 2t, 1 + 10t\rangle$$

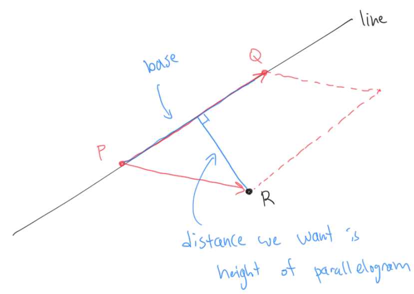
as a vector equation for the desired line, and parametric equations

$$x = 1 - 2t,\ y = 2 + 2t,\ z = 1 + 10t.$$

Distance from point to line. For a final computation, we find the distance from the point $(5, -1, 0)$ to the line we just found with parametric equations

$$x = 1 - 2t,\ y = 2 + 2t,\ z = 1 + 10t.$$

The "distance" we want is analogous to the one we described in the Warm-Up: it is the distance from $(5, -1, 0)$ to the point on the line to which it is closest, or equivalently the "perpendicular" distance from $(5, -1, 0)$ to the line:



This can be found using a vector projection approach as in the first approach to the Warm-Up, but here we'll instead use the cross product approach.

The idea is the same: view the desired distance as the height of an appropriate parallelogram. Take two points P and Q on the line, say for example $P = (1, 2, 1)$ for $t = 0$ and $Q = (3, 0, -9)$ for $t = -1$; it does not matter which points we pick, they will all give the same distance in the end. Set $R = (5, -1, 0)$, and use the parallelogram with edges $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ in the picture above. The distance we want is then

$$\text{distance/height} = \frac{\text{area}}{\text{length of base}} = \frac{|\overrightarrow{PR} \times \overrightarrow{PQ}|}{|\overrightarrow{PQ}|}.$$

In our case we have

$$\overrightarrow{PR} = \langle 5, -1, 0 \rangle - \langle 1, 2, 1 \rangle = \langle 4, -3, -1 \rangle$$

and

$$\overrightarrow{PQ} = \langle 3, 0, -9 \rangle - \langle 1, 2, 1 \rangle = \langle 2, -2, -10 \rangle,$$

so

$$\overrightarrow{PR} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & -1 \\ 2 & -2 & -10 \end{vmatrix} = \langle 30 - 2, -(-40 - (-2)), -8 - (-6) \rangle = \langle 28, 38, -2 \rangle.$$

Thus the distance from $(5, -1, 0)$ to the line with parametric equations

$$x = 1 - 2t, \ y = 2 + 2t, \ z = 1 + 10t$$

is

$$\frac{|\overrightarrow{PR} \times \overrightarrow{PQ}|}{|\overrightarrow{PQ}|} = \frac{\sqrt{28^2 + 38^2 + 4}}{\sqrt{4 + 4 + 100}} = \frac{\sqrt{2232}}{\sqrt{108}}.$$

Lecture 6: Planes

Warm-Up. We find the distance from the point $(5, 1, 1)$ to the line that is perpendicular to the lines with parametric equations

$$\begin{cases} x = 1 + 2t \\ y = 2 - t \\ z = 3 + t \end{cases} \qquad \text{and} \qquad \begin{cases} x = 5 - 3t \\ y = 5 - t \\ z = -2 + 2t \end{cases}$$

and passes through their point of intersection. The first step is to find the line that is perpendicular to the two given lines and passes through their point of intersection, since we will need to use points on *this* line in our distance formula. We did something like this last time, but in that case there was something that made this simpler: plugging in $t = 0$ into the given parametric equations immediately gave us the point at which the lines intersected. In our new example, for $t = 0$ we get the point $(1, 2, 3)$ on the first line and $(5, 5, -2)$ on the second, so we do not get the intersection point from this alone.

Instead, we must do some additional work. The point at which the given lines intersect will be for which the (x, y, z) coordinates along the first line agree with those along the second line. A first instinct might be to setup the equations

$$\begin{aligned} 1 + 2t &= 5 - 3t \\ 2 - t &= 5 - t \\ 3 + t &= -2 + 2t \end{aligned}$$

obtained by setting x coordinates equal to each other, y coordinates equal to each other, and z coordinates equal, but this does not work: there is no value for t that satisfies the equations above. The reason why this does not work is that this assumes the common intersection point occurs at the *same* value of the parameter t for both lines, but there is no reason why this should be true; perhaps there is some value t_1 of the parameter that gives the intersection point on the first line, but then a *different* value t_2 of the parameter that gives this *same* point along the second line. So instead we setup and solve the equations

$$\begin{aligned} 1 + 2t_1 &= 5 - 3t_2 \\ 2 - t_1 &= 5 - t_2 \\ 3 + t_1 &= -2 + 2t_2. \end{aligned}$$

Again, this says that we get the same (x, y, z) coordinates along both lines, but allows for the possibility that they occur at different t for each line.

From the second equation we can express t_1 in terms of t_2 as

$$t_1 = 2 - 5 + t_2 = -3 + t_2.$$

Substituting this into the first equation gives

$$1 + 2(-3 + t_2) = 5 - 3t_2, \text{ or } -5 + 2t_2 = 5 - 3t_2.$$

This gives $t_2 = 2$, and from this we get. $t_1 = -3 + t_2 = -3 + 2 = -1$. This says that $t_1 = -1$ and $t_2 = 2$ gives at least the same x and y coordinates (since these are the equations we used to derive these values) on the two lines. To verify that these give a point of intersection we must check that they also give the same z coordinate: we get

$$3 + t_1 = 3 + (-1) = 2 \quad \text{and} \quad -2 + 2t_2 = -2 + 2(2) = 2$$

on the first and second lines respectively, so since these agree we do indeed have a point of intersection occurring at $t_1 = -1$ on the first line and $t_2 = 2$ on the second. Plugging in either $t_1 = -1$ in the first line or $t_2 = 2$ into the second gives $(-1, 3, 2)$ as the point of intersection.

The direction vector of the perpendicular line is found just like last time, by taking the cross product of the direction vectors of the two lines. The first line has direction vector $\langle 2, -1, 1\rangle$ (coefficients of t), and the second has direction vector $\langle -3, -1, 2\rangle$, so the perpendicular line has direction vector

$$\langle 2, -1, 1\rangle \times \langle -3, -1, 2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -3 & -1 & 2 \end{vmatrix} = \langle -1, -7, -5\rangle .$$

The line which is perpendicular to the two given lines and passes through their point of intersection thus has vector equation

$$\mathbf{r}(t) = \langle -1, 3, 2 \rangle + t \langle -1, -7 - 5 \rangle = \langle -1 - t, 3 - 7t, 2 - 5t \rangle$$

and parametric equations

$$x = -1 - t, \; y = 3 - 7t, \; z = 2 - 5t.$$

Now we get to our original problem, which was to find the distance from $(5, 1, 1)$ to this line we just found. For this we take the points $P = (-1, 3, 2)$ for $t = 0$ and $Q = (-2, -4, -3)$ for $t = 1$ on the line, and set $R = (5, 1, 1)$. The distance from R to the line is the height of the parallelogram with edges $\overrightarrow{PR}$ and $\overrightarrow{PQ}$, with $\overrightarrow{PQ}$ as the base, so

$$\text{distance/height} = \frac{\text{area}}{\text{length of base}} = \frac{|\overrightarrow{PQ} \times \overrightarrow{PR}|}{|\overrightarrow{PQ}|}.$$

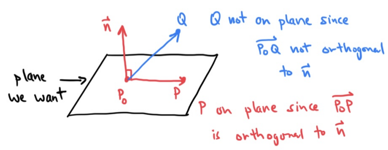
We have $\overrightarrow{PQ} = \langle -1, -7, -5 \rangle$ and $\overrightarrow{PR} = \langle 6, -2, -1 \rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -7 & -5 \\ 6 & -2 & -1 \end{vmatrix} = \langle -3, -31, 44 \rangle .$$

Thus the desired distance is

$$\frac{|\langle -3, -31, 44 \rangle|}{|\langle -1, -7, -5 \rangle|} = \frac{\sqrt{3^2 + 31^2 + 44^2}}{\sqrt{1 + 49 + 25}} = \sqrt{\frac{2906}{75}}.$$

Planes. Now that we've seen how to describes lines in $\mathbb{R}^3$, we move to the next simplest type of geometric objects, planes. To describe a plane we need two pieces of data: a point P_0 on the plane, and a vector $\mathbf{n}$ that is orthogonal to the plane we want. We call such a vector a *normal* vector to the plane. Given these, in order to characterize the other points P on this plane, we use the following picture:



In order for P to be on the plane, the vector $\overrightarrow{P_0P}$ should be on the plane as well. But this means that is would have to be orthogonal to the normal vector $\mathbf{n}$, so the conclusion is that P is on the plane we want precisely when

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

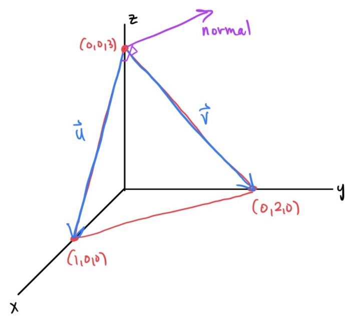
In the picture above, the point Q is *not* on the plane containing P_0 and normal to $\mathbf{n}$ precisely because the vector $\overrightarrow{P_0Q}$ is not orthogonal to $\mathbf{n}$.

If we denote by $\mathbf{r}_0 = \overrightarrow{OP_0}$ the position vector of the point P_0, and by $\mathbf{r} = \overrightarrow{OP} = \langle x, y, z \rangle$ the position the vector of the point $P = (x, y, z)$ we want to characterize as being on the plane, then $\overrightarrow{P_0P}$ is $\mathbf{r} - \mathbf{r}_0$, so (x, y, z) is on this plane when

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

We call this the *vector equation* of the desired plane, from which we can extract an equation in terms of x, y, z, as we'll see.

Example. We find an equation of the plane containing the points $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$. We need two things: a point on the plane, and a vector normal to the plane. For a point on the plane we simply pick one of the three we're already given; let' say $P_0 = (0,2,0)$. For a normal vector, we need something perpendicular to, say, the triangle with vertices the given points since this triangle itself should lie on our plane:



Thus, we can get a normal vector by taking the cross product of the vectors forming two of the edges of this triangle, since this cross product will indeed be perpendicular to the triangle. Let us use

$$\mathbf{u} = \langle 1,0,0\rangle - \langle 0,0,3\rangle = \langle 1,0,-3\rangle \text{ and } \mathbf{v} = \langle 0,2,0\rangle - \langle 0,0,3\rangle = \langle 0,2,-3\rangle$$

as edges, and so

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ 0 & 2 & -3 \end{vmatrix} = \langle 6,3,2\rangle$$

as a normal vector.

With $\mathbf{r}_0 = \langle 0,2,0\rangle$ and $\mathbf{r} = \langle x,y,z\rangle$, where (x,y,z) is the arbitrary point on the plane we seek to characterize, the vector equation of this plane is

$$\underbrace{\langle 6,3,2\rangle}_{\mathbf{n}} \cdot \underbrace{\langle x-0, y-2, z-0\rangle}_{\mathbf{r}-\mathbf{r}_0} = 0.$$

If we compute this dot product, we get

$$6(x-0) + 3(y-2) + 2(z-0) = 0, \text{ or more simply } 6x + 3(y-2) + 2z = 0.$$

This is the *standard* equation of the plane, which can be in turn simplified to $6x + 3y + 2z = 6$. One takeaway is that planes are, as we've alluded to before, describes by linear equations, meaning equations where x, y, z occur to at most a first power only.

Reading off the normal. We finish by noting the following. Say we had the plane with equation

$$4x - 6y + 3z = 11.$$

Then we can immediately write down a normal vector to this plane, namely $\mathbf{n} = \langle 4,-6,3\rangle$. The point is that the coefficients of the variables x, y, z are precisely the entires in this normal vector. This works because, if we go back to the vector equation of a plane

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) = 0$$

with $\mathbf{r} = \langle x, y, z\rangle$, it is indeed the entries of $\mathbf{n}$ that appears as coefficients of x, y, z in the standard equation. Namely, if $\mathbf{n} = \langle a, b, c\rangle$ is the normal vector and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ describes point $P_0 = (x_0, y_0, z_0)$ we already know to be on the plane, then the vector equation above becomes

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0,$$

which is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

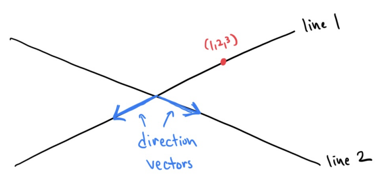
After simplifying further the coefficients of x, y, z are a, b, c, the entries of the normal vector.

Lecture 7: Quadric Surfaces

Warm-Up 1. We find the distance from the point $(-3, 1, 5)$ to the plane containing the lines with parametric equations

$$\begin{cases} x = 1 + 2t \\ y = 2 - t \\ z = 3 + t \end{cases} \quad \text{and} \quad \begin{cases} x = 5 - 3t \\ y = 5 + t \\ z = -2 + 2t \end{cases}.$$

We have not spoken about distances to planes yet, but before we get to that we have to know what plane we're dealing with exactly. Here is the picture we keep in mind:



These lines intersect, and indeed we found their point of intersection last time.

To find the plane containing these lines, we need two things: a point on the plane, and a normal vector. For a point on the plane we can take the point of intersection we found last time, but actually this is overkill: all we need is *some* point on the plane, and there are simpler to points to find than this intersection point. (Recall that finding this intersection point last time involved a fair amount of algebra.) Instead, we simply note that $(1, 2, 3)$ is on the first line in question (attained at $t = 0$), and hence is also on the plane containing this line and the second line. To be clear, there are many points on the plane we can use, but we just use the one we can find with as minimal work possible.

For a normal vector, we note that any normal vector must be perpendicular to both given lines on the plane, and so must be perpendicular to their direction vectors as in the picture above. Thus we use the cross product of these direction vectors as a normal vector:

$$\mathbf{n} = \langle 2, -1, 1\rangle \times \langle -3, 1, 2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -3 & 1 & 2 \end{vmatrix} = \langle -3, -7, -1\rangle.$$

The vector equation of the plane containing $(1, 2, 3)$ with normal vector $\langle -3, -7 - 1\rangle$ is then

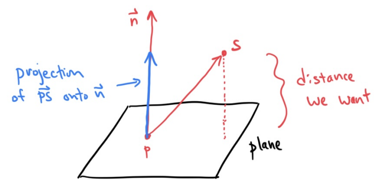
$$\langle -3, -7, -1\rangle \cdot \langle x - 1, y - 2, z - 3\rangle = 0,$$

which gives

$$-3(x - 1) - 7(y - 2) - (z - 3) = 0$$

as the standard equation. (This could, of course, be simplified further if desired.)

Now, to find the distance from $S = (-3, 1, 5)$ to this plane, we use the following picture:



The distance we want is obtained from dropping a perpendicular segment from S to the plane. We pick any point P on the plane, say $P = (1, 2, 3)$, and consider the projection of $\overrightarrow{PS}$ onto the normal vector $\mathbf{n}$; the upshot is that the length of this projection is *precisely* the distance we want from S to the plane. In our case, we have

$$\overrightarrow{PS} = \langle -3, 1, 5 \rangle - \langle 1, 2, 3 \rangle = \langle -4, -1, 2 \rangle \text{ and } \mathbf{n} = \langle -3, -7, -1 \rangle,$$

so the projection is

$$\operatorname{proj}_{\mathbf{n}} \overrightarrow{PS} = \left(\frac{\overrightarrow{PS} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} = \frac{12 + 7 - 2}{9 + 49 + 1} \langle -3, -7, -1 \rangle = \frac{17}{59} \langle -3, -7, -1 \rangle.$$

The distance from $S = (-3, 1, 5)$ to our plan is thus

$$\left| \operatorname{proj}_{\mathbf{n}} \overrightarrow{PS} \right| = \left| \frac{17}{59} \langle -3, -7, -1 \rangle \right| = \frac{17}{59} | \langle -3, -7 - 1 \rangle | = \frac{17}{59} \sqrt{9 + 49 + 1} = \frac{17}{\sqrt{59}}.$$

(The second-to-last expression is perfectly fine as an answer; it does not have to be simplified.)

We should note that the book gives the distance from S to the plane as the formula

$$\frac{\left| \overrightarrow{PS} \cdot \mathbf{n} \right|}{|\mathbf{n}|}.$$

This is precisely the same thing we computed since it is the length of the projection of $\overrightarrow{PS}$ onto $\mathbf{n}$:

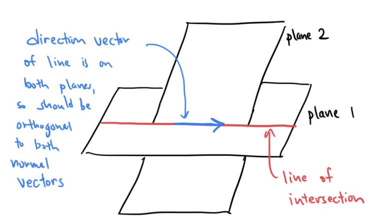
$$\left| \operatorname{proj}_{\mathbf{n}} \overrightarrow{PS} \right| = \left| \left(\frac{\overrightarrow{PS} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right| = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{|\mathbf{n}|^2} |\mathbf{n}| = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}.$$

For me, it is simpler to compute the projection as a vector and then its length, as opposed to having to memorize yet another formula as in the book.

Warm-Up 2. We find parametric equations for the line in which the planes with equations

$$-3x + 2y - z = 1 \quad \text{and} \quad 2x - y - 2z = -8$$

intersect. Here is a picture:



As always, to describe a line we need a point on the line and a direction vector for the line. A point on the line of intersection will be some (x, y, z) that satisfies the equations of both planes simultaneously. Since we only need one such point, let us look for one that happens to have y-coordinate 0. The point is that this simplifies our plane equations so that we are left only needing to find x, z satisfying

$$-3x - z = 1 \quad \text{and} \quad 2x - 2z = -8.$$

(Of course, taking $y = 0$ is not the only option, and we could have just as easily set $x = 0$ and solved for y, z, or set $z = 0$ and solved for x, y instead. Again, all we need is *some* point on the line of intersection.) We the simplified equations above, the first gives $z = -3x - 1$, and then the second gives

$$2x - 2(-3x - 1) = -8, \text{ or } 8x + 2 = -8.$$

Hence $x = -10/8 = -5/4$. This in turn gives $z = -3(-5/4) - 1 = 11/4$, so $(-5/4, 0, 11/4)$ is a point on the line of intersection between the given planes.

Now we need a direction vector for this line. Here the point is that since this direction vector in particular lies on the first plane, it must be orthogonal to the normal vector $\mathbf{n}_1 = \langle -3, 2, -1 \rangle$ to this plane, and since the desired direction vector lies on the second plane, it must also be orthogonal to the normal vector $\mathbf{n}_2 = \langle 2, -1, -2 \rangle$ to this plane. Hence our direction vector must be orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$, so we can take $\mathbf{n}_1 \times \mathbf{n}_2$ as a valid direction vector:

$$\langle -3, 2, -1 \rangle \times \langle 2, -1, -2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -1 \\ 2 & -1 & -2 \end{vmatrix} = \langle -5, -8, -1 \rangle.$$

Thus we get

$$x = -\frac{5}{4} - 5t, \; y = -8t, \; z = \frac{11}{4} - t$$

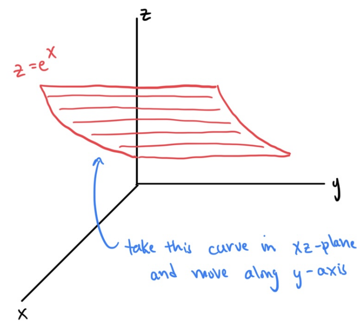
as parametric equations for the line of intersection.

Cylinders. We've seen how to describe planes, and now we move to describing other types of surfaces. We start with the types of surfaces we saw back on the first day, namely ones described by equations which only involve two variables. For example, we saw back then that

$$x^2 + y^2 = 1$$

is the equation of a standard cylinder of radius 1 centered along the z-axis. To visualize this cylinder, we start with the *curve* described by $x^2 + y^2 = 1$ in the xy-plane, which is a circle, and then slide it in the z-direction to sweep out the cylinder. We also saw the surface $z = y^2$ on the first day, where we take start with the curve $z = y^2$ (a parabola) in the yz-plane, and slide it in the x-direction to sweep out the desired surface.

This same idea applies to all surfaces with equations which omit a variable. For example, the surface with equation $z = e^x$ is visualized by taking the curve $z = e^x$ in the xz-plane, and then sliding it in the y-direction:



We will use the term *cylinder* to refer to all of these types of surfaces, even though they do not necessarily look like "cylinders" in the usual sense of the word. So, cylinders are described by equations with one variable missing, and are visualized/drawn by taking the curve the given equation describes in an appropriate plane, and then sliding that curve in the direction of the missing variable.

Quadric surfaces. After planes and cylinders, the next simplest types of surfaces to consider are known as *quadric surfaces*. Planes are described with equations with x, y, z occurring to only a first power, and quadric surfaces are where we now allow second powers. As a first example, we consider the quadric surface with equation

$$x^2 + y^2 = z^2.$$

Our goal is to understand what this surface looks like. To be clear, the surface consists of all points (x, y, z) in $\mathbb{R}^3$ whose coordinates satisfy the given equation.

To visualize the 3-dimensional surface, we first consider certain 2-dimensional portions of it, namely the 2-dimensional curves obtained by intersecting the surface with some plane. These are called *cross-sections* of the surface. So, to start, we first consider the cross-section of this surface at $z = 1$, which is the curve obtained by intersecting $x^2 + y^2 = z^2$ with the horizontal plane $z = 1$. The explicit equation for this cross-section is found by setting $z = 1$ in the surface equation, which gives

$$x^2 + y^2 = 1.$$

This is the equation of a circle of radius 1 centered at $(0, 0)$ in the xy-plane. This is the portion of surface $x^2 + y^2 = z^2$ we see occurring at a "height" of $z = 1$, and the idea is that if we can determine enough of these cross-sections at different "heights", we can use them to piece the entire surface together.

The cross-section of this surface at $z = 2$ has equation

$$x^2 + y^2 = 2^2 = 4.$$

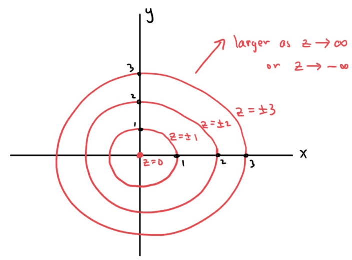
This is a circle of radius 2 centered as $(0, 0)$, so it is larger than the circle we had at $z = 1$. The cross section at $z = -1$ has equation

$$x^2 + y^2 = (-1)^2 = 1,$$

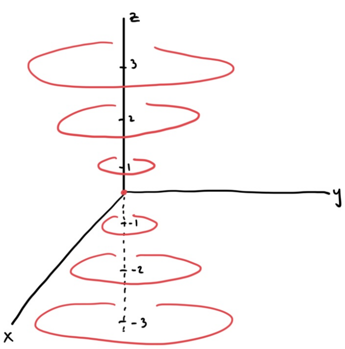
so it too is a unit circle just like the cross section at $z = 1$. The cross-section at $z = -2$ is the same as that at $z = 2$, and in general the cross-section at $z = k$ is

$$x^2 + y^2 = k^2.$$

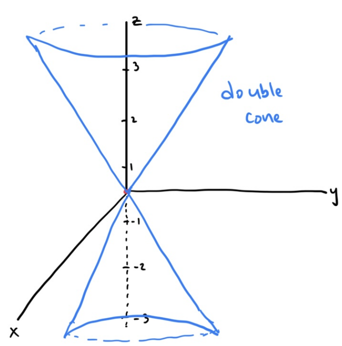
For $k \neq 0$, this is a circle centered at $(0,0)$, which gets larger as k increases in the positive direction or decreases in the negative direction. For $k = 0$, we have $x^2 + y^2 = 0$, which is only satisfied by $x = 0, y = 0$, so this cross-section is a single point. Altogether then, we get the following picture of cross-sections:



To visualize the full surface from these 2-dimensional cross-sections, we draw these cross-sections in $\mathbb{R}^3$ at the appropriate heights z, so we get the origin at $z = 0$, a unit circle at $z = \pm 1$, a circle of radius 2 at $z = \pm 2$, and so on:



These curves are the intersections of our desired surface with different horizontal planes, and by imagine what the "sweep" out as the value of z changes, we get the following picture:

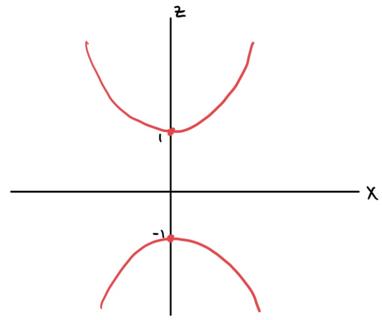


Thus, $x^2+y^2 = z^2$ is the equation of what's called a *double cone*, where "double" means that we get a cone in both the positive direction of z in the negative direction as well. The important takeaway is knowing the cross-sections alone is enough to get a pretty good picture of the full surface.

Conic sections. There are other cross-sections we can consider for the double cone $x^2 + y^2 = z^2$ apart from those occurring at a constant value of z. Instead, for example, we can consider cross-sections at $y = k$, which give the intersections of the double cone with vertical planes at a point along the y-axis. For $y = 1$, for instance, we get a cross-section with equation

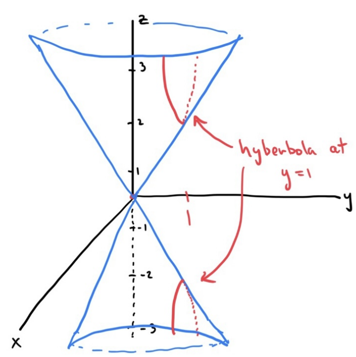
$$x^2 + 1 = z^2, \text{ which can be written as } z^2 - x^2 = 1.$$

This is the equation of a *hyperbola*:



The fact that this hyperbola opens up vertically instead of horizontally comes from determining which axis it can and cannot cross: here, for $x = 0$ we get $z^2 = 1$ and hence $z = \pm 1$, which give intersections with the z-axis, but for $z = 0$ we get $-x^2 = 1$, which has no solutions, meaning that this hyperbola will not cross the x-axis.

We can visualize this specific cross-section on the actual double cone by cutting through it with the plane at $y = 1$:



In general, the types of curves we get as cross-sections of quadric surfaces will be ellipses (of which circles are examples), hyperbolas, and parabolas. These are known as *conic sections*, but a full discussion of conic sections will not be so important for us. We will recall the basic facts we need, such as how to find where they intersect an axis, as we work through more examples of quadric surfaces next time.

Lecture 8: More on Surfaces

Warm-Up. We sketch the cross-sections of the quadric surface

$$z = 2x^2 + 3y^2$$

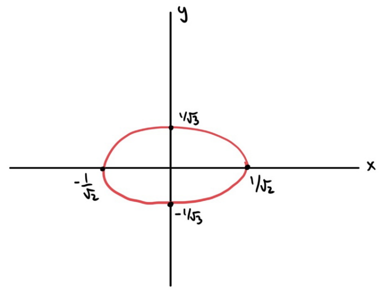
at $z = 0, 1, 2, -1$, and then use them to sketch the full surface. The cross-section at $z = 0$ has equation

$$0 = 2x^2 + 3y^2,$$

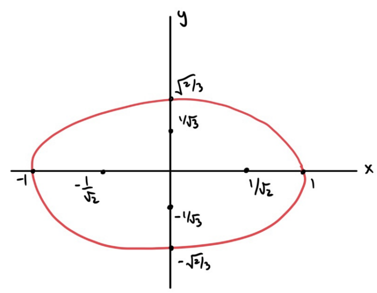
and the only point satisfying this is $x = 0, y = 0$, so this cross-section is a single point. The cross-section at $z = 1$ has equation

$$1 = 2x^2 + 3y^2.$$

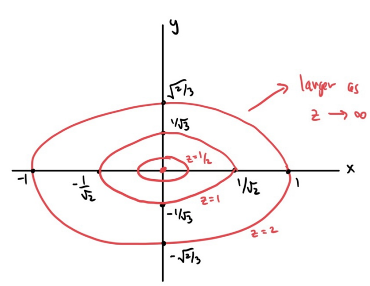
This is the equation of an ellipse, namely one that crosses the x-axis at $x = \pm 1\sqrt{2}$ (found by setting $y = 0$), and the y-axis at $y = \pm 1/\sqrt{3}$, which is found by setting $x = 0$. This cross-section thus looks like



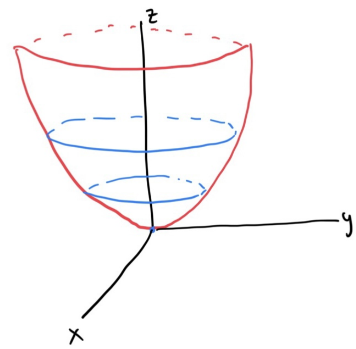
The cross-section at $z = 2$ has equation $2 = 2x^2 + 3y^2$, which is also an ellipse, this time with x-intercepts at $x = \pm 1$ and y-intercepts at $y = \pm\sqrt{2/3}$:



The cross-section at $z = -1$ has equation $-1 = 2x^2 + 3y^2$, and this has no solutions since the right side cannot be negative. We thus say that this cross-section is *empty*, which means that there is no portion of the 3-dimensional surface $z = 2x^2 + 3y^2$ occurring at a "height" of $z = -1$, or in other words this surface does not intersect the plane $z = -1$. In general, the quadric surface $z = -2x^2 + 3y^2$ always has empty cross-sections at negative values of $z = k$, so no portion of the surface is below the xy-plane in $\mathbb{R}^3$. (This is a key difference between this surface and the double cone $z^2 = x^2 + y^2$ from last time.) For positive $z = k$, we get ellipses $k = 2x^2 + 3y^2$ which become larger as k increases:

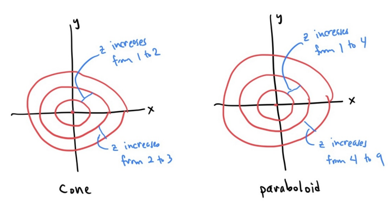


To visualize the full surface we then place these cross-sections at the appropriate values of z in $\mathbb{R}^3$, and imagine the surface they trace out:

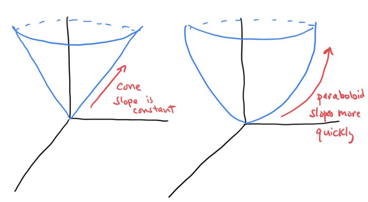


This surface is called a *paraboloid*, and is a 3-dimensional analog of a parabola. (More formally, it is an *elliptic paraboloid*, where "elliptic" emphasizes the ellipses that occur as cross-sections.)

Now, we should highlight the following, which points out a key difference between paraboloids and cones. Consider instead the quadric surface $z = x^2 + y^2$, which is still a paraboloid, but only with z cross-sections as circles. (At least, circles for positive z.) But we saw last time that the double cone $z^2 = x^2 + y^2$ also has cross-sections $z = k$ as circles, so how exactly do we distinguish between cones and paraboloids if they have similar cross-sections? (Let's ignore the behavior for negative z here.) The point is that, even though cross-sections at $z = k > 0$ for both are circles, the way in which the circles get larger as $z = k$ increases differs:



For the cone, the rate at which the radius of a cross-section increases is the same as the rate at which the $z = k$ value increases, so that moving from the cross-section at $z = 1$ to $z = 2$ to $z = 3$ results in the same change in radius. For the paraboloid, however, height $z = k$ changes more quickly as the radius changes: to move from radius 1 to 2 we jump from $z = 1$ to $z = 4$, and then to move to radius 3 causes as jump to $z = 9$. This is what causes the shape of the paraboloid to "slope upwards" more quickly than for the cone:

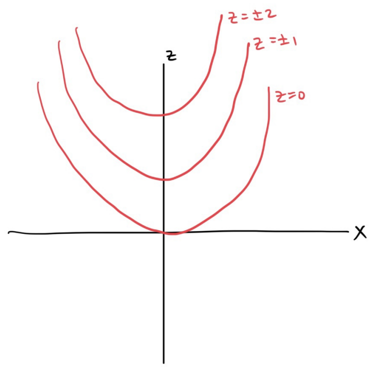


So, it is not just the fact that cross-sections are circles that is important in these examples, the way in which these circles change is also important.

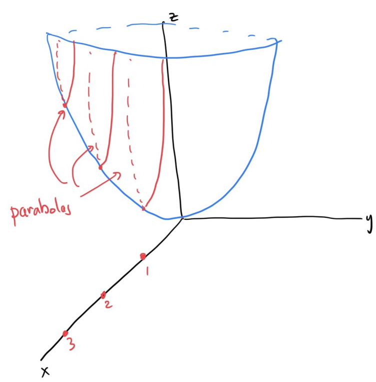
Going back to the elliptic paraboloid $z = 2x^2 + 3y^2$, what about the cross-sections at $x = k$ instead? Here we imagine intersecting the surface with a vertical plane $x = k$, and the resulting cross-section has equation

$$z = 2k^2 + 3y^2.$$

The only variables here are z and y (k is a constant), and in the yz-plane this is the equation of a *parabola* with z-intercept at $z = 2k^2$ when $y = 0$:



These parabolas thus move upwards as $x = k$ gets either more positive or more negative. On the full paraboloid in $\mathbb{R}^3$, we can visualize these x cross-sections as follows:



Indeed, as x moves in the positive direction, say, the parabola cross-section moves up.

Saddles. Now we look at the quadric surface with equation

$$z = 2x^2 - 3y^2.$$

This is a similar equation to the paraboloid we just considered, only with the sign of the y^2 term changed. We take some cross-sections. At $z = 1$. we get

$$1 = 2x^2 - 3y^2,$$

which is a hyperbola crossing the x-axis (since y can be zero) at $x = \pm 1\sqrt{2}$, but not the y-axis since x cannot be zero. At $z = 2$ we again get a hyperbola

$$2 = 2x^2 - 3y^2,$$

only now crossing the x-axis at $x = \pm 1$.

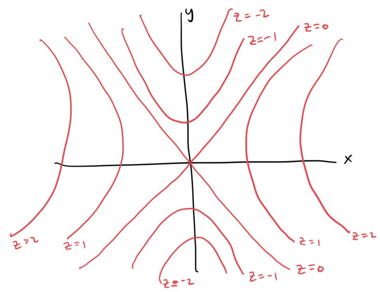
At $z = 0$ we get $0 = 2x^2 - 3y^2$, which can be written as

$$2x^2 = 3y^2.$$

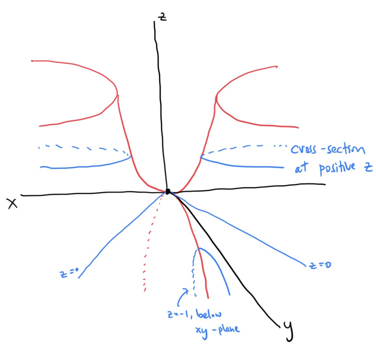
Taking square roots gives $2x = \pm 3y$, so this cross-section consists of the pair of lines $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$. For $z = -1$ we get

$$-1 = 2x^2 - 3y^2, \text{ which we rewrite as } 1 = -2x^2 + 3y^2.$$

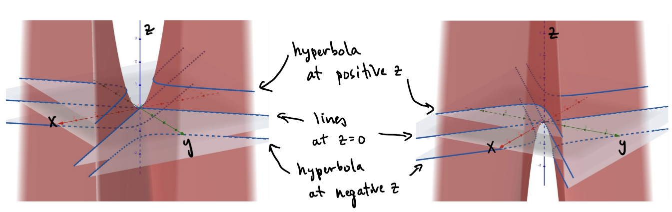
This too is a hyperbola, only now crossing the y-axis (at $y = \pm 1/\sqrt{3}$) but not the x-axis since now x can be zero but y cannot. For other negative z we also get hyperbolas that cross the y-axis and not the x-axis. These different cross-sections thus look like



The resulting surface is not easy to draw at all, and takes some practice to get it looking somewhat right. It is called a *hyperbolic paraboloid*, and resembles the surface of a saddle:



The name comes from getting hyperbolas for cross-sections in one direction (or a pair of lines), but parabolas for cross-sections in other directions; in this case for cross-sections at $y = k$, for example, we get $z = 2x^2 - 3k^2$, which is a parabola opening in the positive z-direction, while for cross-sections at $x = k$ we get $z = 2k^2 - y^2$, which is a parabola opening in the negative z-direction. With the aid of a computer we get a better picture:



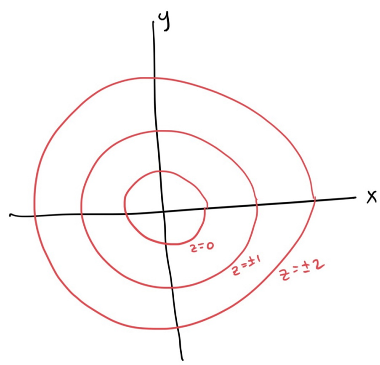
where we have drawn the hyperbolas (and pair of lines) occurring as cross-sections at certain fixed z. The picture on the right is a slightly rotated version of the first picture, which views the surface from a slightly different perspective. Note that the origin is at the bottom of a "valley" (first picture) in the x-direction, specifically at the bottom of the parabola $z = 2x^2$ at the cross-section

$y = 0$, but at the top of a "peak" (second picture) in the y-direction, specifically at the top of the parabola $z = -3y^2$ at the cross-section $x = 0$.

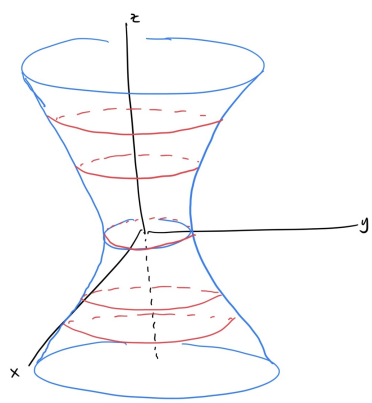
Hyperboloids. Consider now the quadric surface with equation

$$x^2 + y^2 - z^2 = 1.$$

The cross-section at $z = 0$ is a circle $x^2+y^2 = 1$; the cross-sections at $z = \pm 1$ are circles $x^2+y^2 = 2$; the cross-sections at $z = \pm 2$ are circles $x^2 + y^2 = 4$; and in general the cross-section at $z = k$ is always a circle $x^2 + y^2 = 1 + k^2$. (This is always a circle since the right side $1 + k^2$ is always positive.) We thus have the following cross-sections:

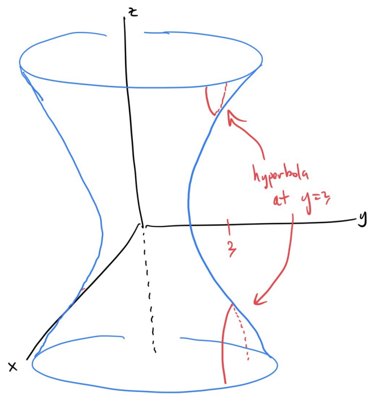


The smallest circle occurs at $z = 0$, so the resulting surface is thinnest on the xy-plane. With circles cross-sections increasing in size $z = k$ gets more positive or more negative, we have a picture like:



This surface is called a *hyperboloid of one-sheet*.

To see an example of a cross-section for a different variable, we take the cross-section at $y = 3$, which has equation $x^2 + 9 - z^2 = 1$, or $x^2 - z^2 = -8$, or $-x^2 + z^2 = 8$. This is a hyperbola opening in the z-direction, which can visualize on the one-sheeted hyperboloid as follows:



More hyperboloids. The term hyperboloid of "one sheet" above suggests there might be hyperboloids with more "sheets". Indeed, an example is given by the quadric surface with equation

$$-x^2 - y^2 + z^2 = 1.$$

At $z = 2$ is the cross section is the circle $-x^2 - y^2 + 4 = 1$, or $3 = x^2 + y^2$, but at $z = 1$ the cross section is $-x^2 - y^2 = 0$, which describes a single point $(0, 0)$. In general, the cross-section at $z = k$ has equation

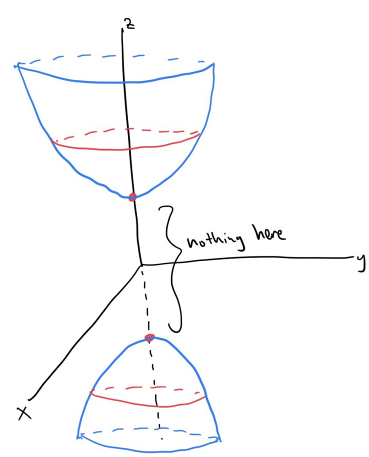
$$-x^2 - y^2 + k^2 = 1, \text{ or } k^2 - 1 = x^2 + y^2.$$

This *almost* looks like the equation of a circle, except for the fact that the left side $k^2 - 1$ can be negative! (This was avoided in the one-sheeted hyperboloid example where we ended up with $k^2 + 1 = x^2 + y^2$.) If $k^2 - 1$ is positive, we definitely get a circle, but when $k^2 - 1 = 0$, so for $k = \pm 1$, we get single points.

Moreover, when $k^2 - 1 < 0$, there are no points satisfying

$$k^2 - 1 = x^2 + y^2$$

since the left side is negative but the right side is non-negative. Thus in this case we have an empty cross-section; in particular, $k^2 - 1 < 0$ when $-1 < k < 1$, so the upshot is that no portion of our surface in $\mathbb{R}^3$ occurs strictly between $k = -1$ and $k = 1$. The 3-dimensional picture these give is



and the resulting surface is called a *hyperboloid of two sheets*

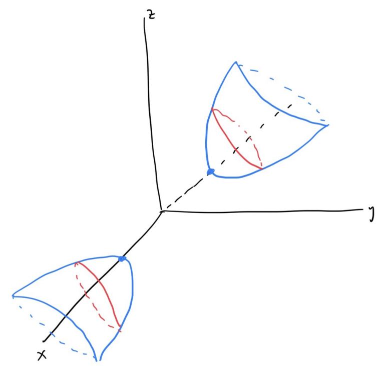
Modifications. The equation

$$x^2 - y^2 - z^2 = 1$$

is still a two-sheeted hyperboloid, only now centered along the x-axis. Indeed, the circles/points/empty cross-sections that characterized the previous example, which was centered along the z-axis, now occur at different values of $x = k$:

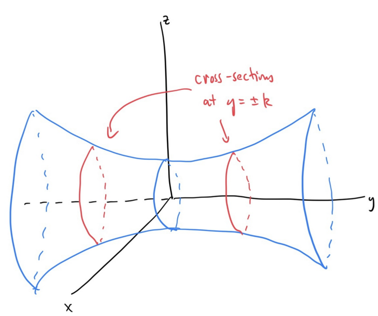
$$k^2 - y^2 - z^2 = 1, \text{ or equivalently } k^2 - 1 = y^2 + z^2.$$

This gives circles for $k < -1$ or $k > 1$ (i.e. when $k^2 - 1 > 0$); single points at $k = \pm 1$; and nothing (i.e. empty cross-sections) for $-1 < k < 1$, so the picture is



The point is that once we have one basic shape down, can get similar things centered along different axes by switching some of the variables.

Similarly, $x^2 - y^2 + z^2 = 1$ is a one-sheeted hyperboloid centered along the y-axis, since $y = k$ gives $x^2 + z^2 = 1 + k^2$, which is always a circle:



Compare this equation to the previous one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ we saw: here y and z are switched, which is why what was centered along the z-axis before is now centered along the y-axis.

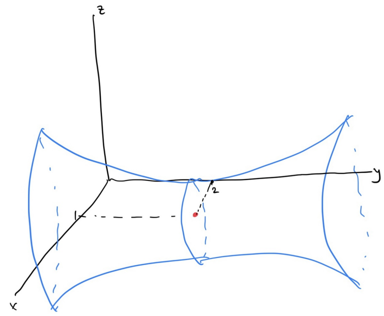
Finally we consider the quadric surface with equation

$$x^2 - 2x - y^2 + 4y + z^2 = 4.$$

This is still one of our basic shapes, only with the "origin" shifted. To put this into a more recognizable form, we complete the square in the x and y terms (z is fine) to get

$$(x-1)^2 - 1 - (y-2)^2 + 4 + z^2 = 4, \text{ or } (x-1)^2 - (y-2)^4 + z^2 = 1.$$

Whereas $x^2 - y^2 + z^2 = 1$ before was centered at $(0, 0, 0)$, this new surface retains the same basic shape but it is centered at $(1, 2, 0)$:



Lecture 9: Polar Coordinates

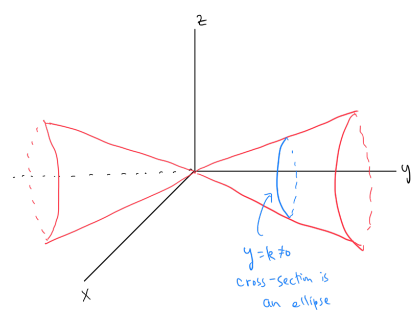
Warm-Up. Given a number k, we identify and sketch the surface with equation

$$x^2 - 2y^2 + 3z^2 = k.$$

The type of surface we get will depend on whether k is positive, negative, or zero. If $k = 0$, our equation is

$$x^2 - 2y^2 + 3z^2 = 0, \text{ which can be written as } x^2 + 3z^2 = 2y^2.$$

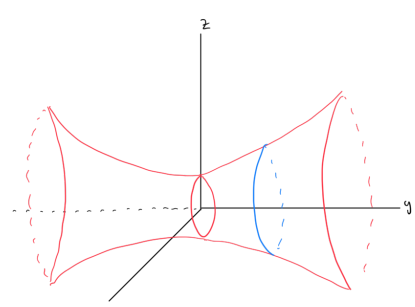
This is the equation of a double-cone centered along the y-axis. Indeed, the cross sections at $y = 0$ is a single point, while other cross sections at $y \neq 0$ are ellipses, so our surface looks like



For $k \neq 0$, let us rewrite our surface equation as

$$x^2 + 3z^2 = k + 2y^2.$$

If $k > 0$, the right side is always positive, so this gives ellipses as cross-sections for any k. The smallest ellipse is at $y = 0$, and the ellipses get larger as $y \to +\infty$ or $y \to -\infty$, so this gives a hyperboloid of one sheet centered along the y-axis:



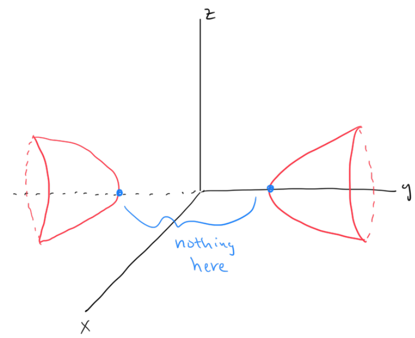
If $k < 0$, the right side of

$$x^2 + 3z^2 = k + 2y^2$$

can be positive, negative, or zero depending on which cross-section at y we take. When $k + 2y^2 < 0$ we get empty cross-sections since no points satisfy

$$x^2 + 3z^2 = \text{negative};$$

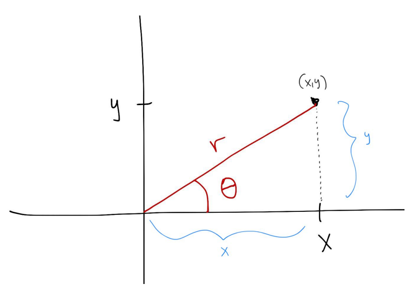
when $k + 2y^2 = 0$ we get single points as cross-sections; and for $k + 2y^2 > 0$ we get ellipses as cross-sections. This gives a hyperboloid of two sheets centered along the y-axis as our surface:



The intercepts with the y-axis here come from the values of y at which $k + 2y^2 = 0$, so $y = \pm\sqrt{-k/2}$. (Recall that in this case $k < 0$, so $-k/2$ is indeed a positive number so that we can take its square root.)

Polar coordinates. All of our lives up until this point, when dealing with the xy-plane we have used *rectangular* (also called *Cartesian*) coordinates. These are just your standard (x, y) coordinates, where the name comes from viewing (x, y) as the corner of a rectangle whose other vertices are $(0, 0), (x, 0)$, and $(0, y)$. But depending solely on Cartesian coordinates is often too restrictive, and many geometric objects of interest can be more easily understood if we modify the coordinates we use.

One common choice of alternative coordinates are *polar coordinates*, defined by the following picture:



So, r is the distance from the point (x, y) to the origin and θ is the angle you have to tilt away from the positive x-axis in order to point in the direction of (x, y), where positive angles correspond to counterclockwise rotations and negative angles to clockwise ones. A negative value of r is interpreted as describing a point in the direction *opposite* to θ; for instance, $\theta = \frac{\pi}{2}$ points us in the positive y-direction and $r = -1$ then gives the point $(0, -1)$ on the negative y-axis.

By looking at the right triangle given in the picture, we get the following relations between polar coordinates (r, θ) and rectangular/Cartesian coordinates (x, y):

$$r^2 = x^2 + y^2, \ \tan\theta = \tfrac{y}{x}, \ x = r\cos\theta, \ y = r\sin\theta.$$

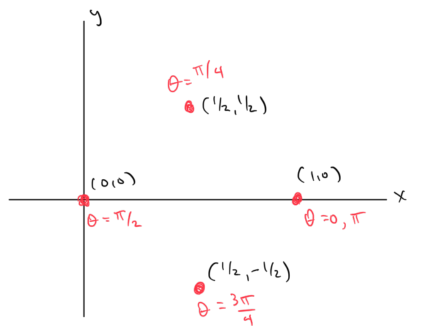
We can use these to convert from polar to Cartesian coordinates, or from Cartesian to polar.

Example 1. We sketch the curve with polar equation $r = \cos\theta$, meaning the curve consisting of all points in $\mathbb{R}^2$ whose polar coordinates satisfy $r = \cos\theta$. For instance, when $\theta = \frac{\pi}{2}$ we get $r = \cos\frac{\pi}{2} = 0$ so the origin (the only point with r value 0) is on this curve; when $\theta = 0$ we get $r = \cos 0 = 1$ so the point $(x, y) = (1, 0)$ (which is at a distance $r = 1$ in the positive x-axis $\theta = 0$ direction) is on this curve.

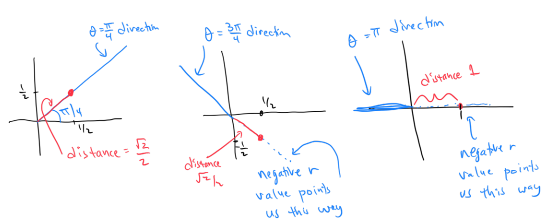
Let us create a table of a few points on this curve for some different values of θ:

θ	$r = \cos\theta$	$x = r\cos\theta$	$y = r\sin\theta$
0	1	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0	0	0
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
π	-1	1	0

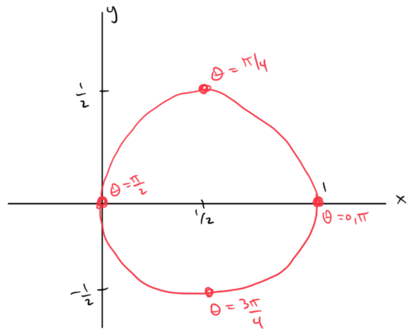
Plotting the resulting points (x, y) gives



We can also make sense of these values geometrically: in the $\theta = \pi/4$ direction we should be at a distance of $r = \sqrt{2}/2$ away from the origin; in the $\theta = 3\pi/4$ direction we should be at a distance of $r = -\sqrt{2}/2$ away from the origin, but since this is negative we actually move into the fourth quadrant opposite the $\theta = 3\pi/4$ direction; and in the $\theta = \pi$ direction we are at a distance of $r = -1$ away from the origin, which actually points us in the direction of the positive x-axis and puts us back at the point $(1, 0)$ we started at:



By connecting these points we can guess that our curve looks something like



It is no accident that this picture appears to be that of a circle, as we can confirm definitively by finding the Cartesian equation of the curve. From our conversions

$$x^2 + y^2 = r^2, \; \tan\theta = \tfrac{y}{x} \qquad \text{we get} \qquad \sqrt{x^2+y^2} = r, \; \theta = \arctan(\tfrac{y}{x})$$

so

$$r = \cos\theta \qquad \text{becomes} \qquad \sqrt{x^2+y^2} = \cos(\arctan \tfrac{y}{x}).$$

This is *a* Cartesian equation for this curve, but perhaps not a very helpful one since it is does not make clear why our curve is actually a circle. Instead, we find a better Cartesian equation by multiplying the polar equation by r to get

$$r^2 = r\cos\theta,$$

and then converting to rectangular coordinates to get

$$x^2 + y^2 = x.$$

After completing the square, this becomes $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$, which describes a circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$, precisely what our picture suggests. (There is one subtlety here, in that when we multiplied $r = \cos\theta$ by r to get $r^2 = r\cos\theta$ we made it so that the $r = 0$ would always satisfy the resulting equation, so that the origin would be on the resulting curve even if it was not on the original curve. For example, if instead we had the curve with polar equation $r = 10 + \cos\theta$, then the origin is not on this curve since r can never be zero here because $\cos\theta$ is always greater than or equal to -1. However, after multiplying by r we get $r^2 = 10r + r\cos\theta$, and $r = 0$ does satisfy this new equation. Essentially, multiplying by r gives rise a potentially extra point on the curve that might not have been present originally. The upshot is to only multiply by r in this way when trying to find Cartesian equations, but not for determining which points are actually on the curve.)

Example 2. We find a polar equation for the line $y = 2$. Since $y = r\cos\theta$ in polar coordinates, our line has equation $r\cos\theta = 2$, or

$$r = \frac{2}{\cos\theta}.$$

(Note that $\cos\theta = 0$ would give $y = r(0) = 0$, and so not the line $y = 2$, which is why can assume $\cos\theta \neq 0$ in this fraction.)

Lecture 10: Parametric Curves

Warm-Up. We find rectangular/Cartesian equations for the curves with the following polar equations and sketch the curves for $0 \leq \theta \leq \pi$:

$$(a)\; r = 2\sin\theta \qquad (b)\; r = 1 - 2\cos\theta.$$

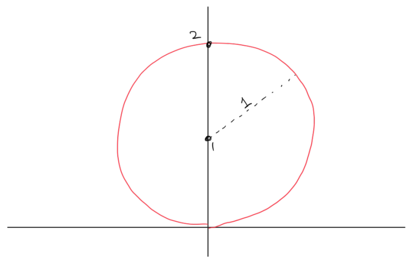
For (a), we find a Cartesian equation by first multiplying through by r to get $r^2 = 2r\sin\theta$, and then using $r^2 = x^2 + y^2$ and $y = r\sin\theta$ to get

$$r = 2\sin\theta \rightsquigarrow x^2 + y^2 = 2y.$$

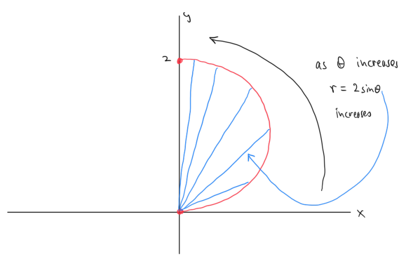
After rearranging and completing the square, this becomes

$$x^2 + y^2 - 2y = 0 \rightsquigarrow x^2 + (y-1)^2 = 1,$$

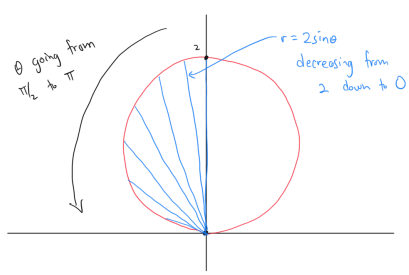
so $r = 2\sin\theta$ is a circle of radiu s 1 centered at $(0, 1)$ on the y-axis. Thus the curve looks like



We can obtain this picture even without converting to Cartesian coordinates by keeping track of the behavior of r as θ changes. At $\theta = 0$ we have $r = 2\sin(0) = 0$, which places us at the origin. Then, as θ increases from 0 to $\frac{\pi}{2}$ in the first quadrant, $r = 2\sin\theta$ increases from 0 to 2, so we get:



(Note that r is measuring distance to the origin, so these distances are the ones which are increasing from 0 when pointing in the positive x-direction to 2 when pointing in the positive y-direction.) Then as θ moves from $\frac{\pi}{2}$ to π in the second quadrant, $r = 2\sin\theta$ decreases from 2 back down to 0, so we get



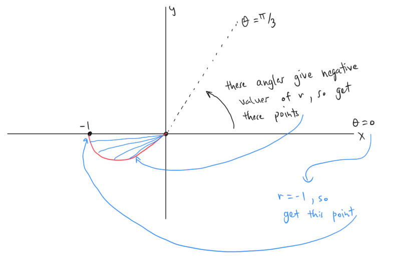
and we're back at the origin at the end.

For the curve in (b), we can find a Cartesian equation by again multiplying by r and using some conversions:

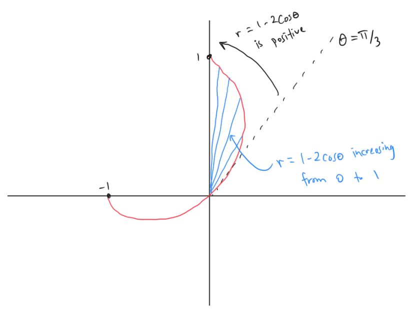
$$r = 1 - 2\cos\theta \rightsquigarrow r^2 = r - 2r\cos\theta \rightsquigarrow x^2 + y^2 = \sqrt{x^2 + y^2} - 2x.$$

This particular Cartesian equation however is not going to help in sketching the curve since it is not a "standard" equation like that of a circle. Instead, we will rely on determining the behavior of r

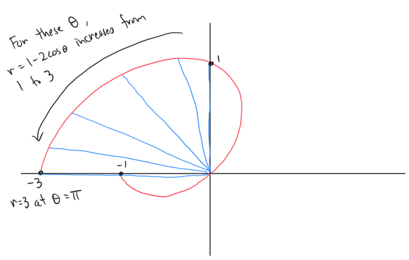
as θ varies directly from $r = 1 - 2\cos\theta$. At $\theta = 0$ we get $r = 1 - 2\cos(0) = 1 - 2 = -1$. Now, $\theta = 0$ points us towards the positive x-direction, but with the negative value of $r = -1$ the point we get occurs in the opposite direction, so at a distance of 1 from the origin in the negative x-direction. Hence we get the point $(-1, 0)$ on the x-axis as our starting point at $\theta = 0$. As θ increases from 0 to $\frac{\pi}{/}3$, r moves from $r = -1$ up to $r = 0$, so at $\theta = \pi/3$ we are at the origin. But these values of r between -1 and 0 are negative, which means that for $0 \le \theta \le \frac{\pi}{/}3$ in the first quadrant we actually get points in the third (i.e. opposite) quadrant, so the curve for $0 \le \theta \le \frac{\pi}{3}$ looks like



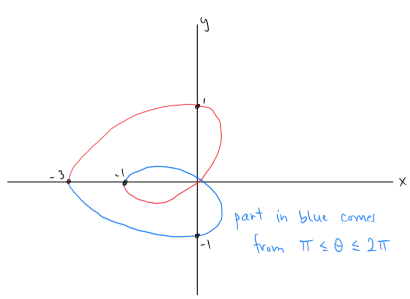
As θ increases from $\frac{\pi}{3}$ up to $\frac{\pi}{2}$ (sweeping out the rest of the first quadrant), $r = 1 - 2\cos\theta$ increases from 0 to 1, so we get



Finally, as θ moves from $\frac{\pi}{2}$ to π in the second quadrant, $r = 1 - 2\cos\theta$ increases from 1 to $1 - 2\cos(\pi) = 3$, so we get



and finish at the point $(-3, 0)$, which is at a distance of $r = 3$ from the origin in the negative x (i.e., $\theta = \pi$) direction. The setup only asked to sketch the curve for $0 \le \theta\pi$, but if you kept going for $\pi \le \theta \le 2\pi$ you would end up with



as the full curve.

Curves. We have seen how to describes lines in $\mathbb{R}^3$, and now we want to describe more general types of curves in 2- or 3-dimensions. Recall that for lines the end result was a set of *parametric equations*, where we specify different equations for the x, y, and z coordinates of points on the line all in terms of a common parameter. For example,

$$x = 1 + 2t, \ \ y = -2 - t, \ \ z = 3 + 4t$$

gives the line passing through $(1, -2, 3)$ and parallel to the vector $\langle 2, -1, 4 \rangle$. As t varies, the point (x, y, z) varies, tracing out the line in question.

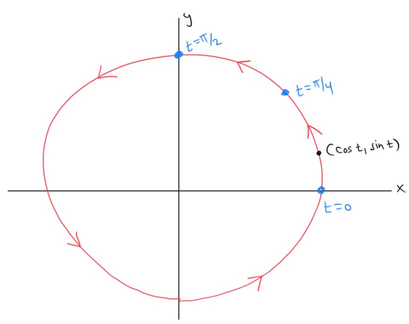
We use the same idea to describe other curves, only with parametric equations that can involve non-linear functions in terms of a parameter t. Take for example the curve with parametric equations

$$x = \cos t, \ \ y = \sin t, \ \ 0 \leq t \leq 2\pi.$$

We claim that these describe the unit circle in $\mathbb{R}^2$ centered at the origin. Indeed, first notice that the coordinates $x = \cos t$ and $y = \sin t$ satisfy

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

for all t, so the point $(x, y) = (\cos t, \sin t)$ is always on this unit circle. At $t = 0$, we are starting at the point $(x, y) = (\cos 0, \sin 0) = (1, 0)$, and at $t = \frac{\pi}{2}$ we are at $(x, y) = (\cos \frac{\pi}{2}, \sin \frac{\pi}{2}) = (0, 1)$. Thus the circle is being traced out by these changing (x, y) coordinates in a *counterclockwise* direction as t varies:



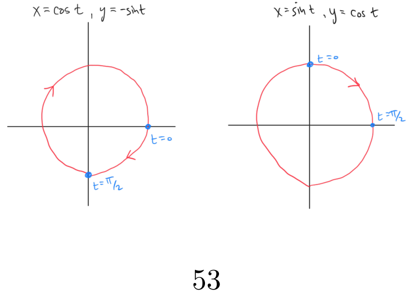
We can get other behaviors in terms of how the circle should be traced out by making modifications to the parametric equations above. For example, with

$$x = \cos t, \ \ y = -\sin t,$$

which still satisfy $x^2 + y^2 = 1$, we get the circle traced out *clockwise*: at $t = 0$ we start at $(x, y) = (1, 0)$, and at $t = \frac{\pi}{2}$ we are at $(x, y) = (0, -1)$ on the negative y-axis, so we move clockwise as t increases. With

$$x = \sin t, \ \ y = \cos t,$$

we still get a circle ($x^2 + y^2 = 1$ is still true), only we start tracing out at the circle at $(x, y) = (\sin 0, \cos 0) = (0, 1)$ at $t = 0$:



Example. We find parametric equations for the circle with equation

$$(x-1)^2 + (y-2)^2 = 4.$$

The equations $x = \cos t, y = \sin t$ we had before were for a circle centered at the origin of radius 1, so the goal is to modify these to make center be $(1, 2)$ and the radius 2 instead. The center can be shifted by adding 1 to the x-coordinate and 2 to the y-coordinate:

$$x = \cos t + 1, \;\; y = \sin t + 1.$$

But this does not have the right radius: with these equations we get $x - 1 = \cos t$ and $y - 2 = \sin t$, so

$$(x-1)^2 + (y-2)^2 = \cos^2 t + \sin^2 t = 1.$$

Instead, we must scale the cosine and sine terms by 2 in order to get the correct radius: with

$$x = 2\cos t + 1, \;\; y = 2\sin t + 2$$

we have

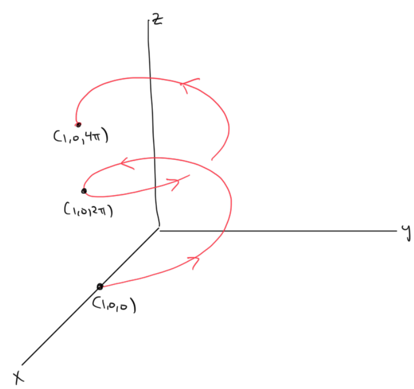
$$(x-1)^2 + (y-2)^2 = (2\cos t)^2 + (2\sin t)^2 = 4(\cos^2 t + \sin^2 t) = 4$$

as desired, so $x = 2\cos t + 1, y = 2\sin t + 2$ are valid parametric equations for this circle.

3D example. We sketch the curve in $\mathbb{R}^3$ with parametric equations

$$x = \cos t, \;\; y = \sin t, \;\; z = t \quad 0 \leq t \leq 4\pi.$$

First, note that the x and y equations are describing circular motion, just as before. But now we must incorporate the behavior of z as well. As x, y move in a circular pattern around the z-axis, the z-coordinate $z = t$ increases as we go, so (x, y, z) moves along a circular type of curve that moves up at the same time:

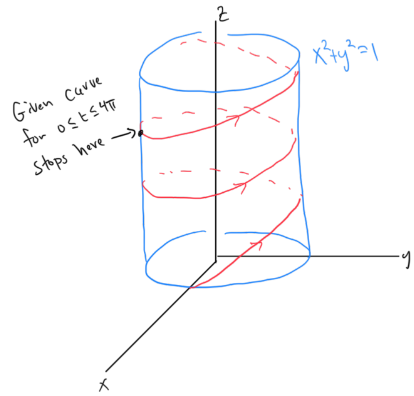


This is called a *helix*. The bounds $0 \leq t \leq 4\pi$ on the parameter t in this case describe two revolutions of the helix, since $x = \cos t, y = \sin t$ will complete two revolutions around the z-axis over this range of values of t.

To get a clearer picture, note that the x and y equations for points on this curve satisfy

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1,$$

so all points (x, y, z) on this curve satisfy the equation $x^2 + y^2 = 1$, which is the equation of a cylinder. Thus, our curve should be one that lies completely on this cylinder, moving up as it wraps around:



Other examples. Now we consider the curve with parametric equations

$$x = t\cos t,\ y = t\sin t,\ z = t \quad 0 \le t \le 4\pi.$$

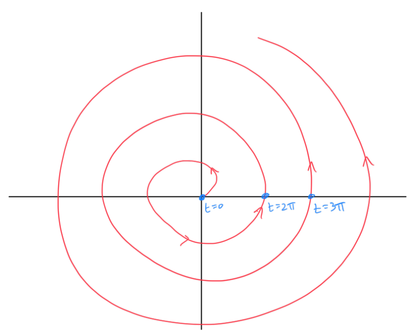
The presence of $\cos t$ and $\sin t$ again suggests some circular motion in the x- and y-directions, only the extra coefficient of t makes it so that the radius changes as we go. These x and y equations satisfy

$$x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2(\cos^2 t + \sin^2 t) = t^2,$$

so in the x- and y-directions we get "circular" motion which moves further away from the z-axis as we go. Another way of saying this is that in $\mathbb{R}^2$, the parametric equations

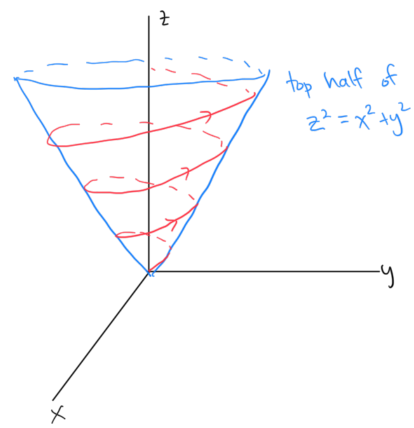
$$x = t\cos t,\ y = t\sin t$$

describe a *spiral*:



Indeed, as we spiral around the origin, we move further away from the origin at the same time.

Going back to our 3-dimensional curve, as the curve "spirals" around the z-axis the value of $z = t$ increases, so again we move upward as the spiraling occurs. To get a more concrete picture, we note that $x^2 + y^2 = t^2$, which is the same as $x^2 + y^2 = z^2$ in our case since $z = t$. Thus, our curve lies on the (double) cone with equation $x^2 + y^2 = z^2$:



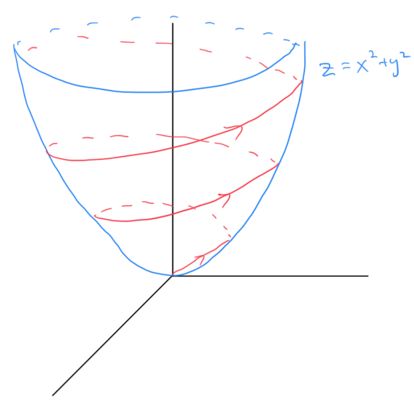
If instead we considered the curve with parametric equations

$$x = t\cos t, \;\; y = t\sin t, \;\; z = t^2,$$

we would have coordinates that satisfy

$$x^2 + y^2 = t^2 = z,$$

so that in this case our curve lies on the paraboloid $z = x^2 + y^2$:



Intersections. We find parametric equations for the intersection of the surface $x = y^2$ with the surface $z = y^3$. The key thing to note here is that once we know y, the values of x and z are completely determined since they must satisfy

$$x = y^2 \text{ and } z = y^3$$

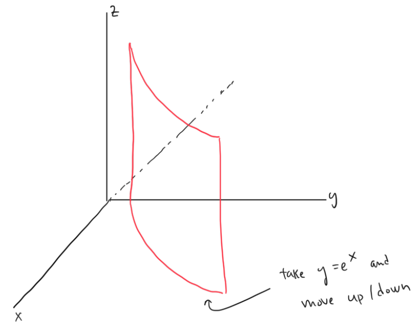
in order to have (x, y, z) lie on both given surfaces. Thus, if we take $y = t$ to be the parameter itself, we must have $x = y^2 = t^2$ and $z = y^3 = t^3$, so that

$$x = t^2, \;\; y = t, \;\; z = t^3$$

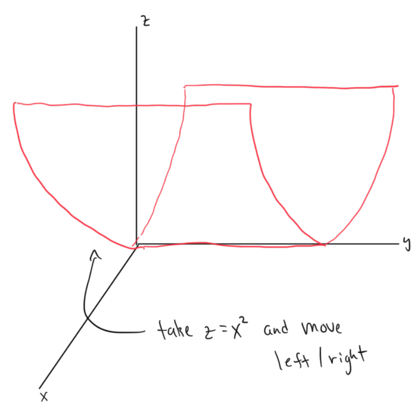
is a possible set of parametric equations for this intersection. If we want to get the full intersection, we should use $-\infty < t < \infty$ with no restrictions on the values of t.

Lecture 11: Tangent Vectors

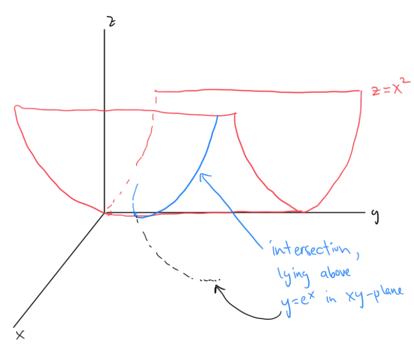
Warm-Up 1. We find parametric equations the curve where the surfaces with equations $y = e^x$ and $z = x^2$ intersect. First let us get a sense for what this intersection looks like. The surface $y = e^x$ is obtained by taking the curve $y = e^x$ in the xy-plane and sliding it up and down to change the value of z:



The surface $z = x^2$ is obtained by taking the parabola $z = x^2$ in the xz-plane and sliding it in the y-direction:



The curve in question is where these overlap, so it looks something like:



Now, once we specify the value x the values of y and z are completely determined by the requirements that x, y, z satisfy $y = e^x$ and $z = x^2$. Thus we can take $x = t$, and then $y = e^t$ and $z = t^2$, so that

$$x = t,\ y = e^t,\ z = t^2,\ -\infty < t < \infty$$

is a set of parametric equations for this intersection. This is not on the only set though (parametric equations are never unique), and with the initial choice of $x = t^3$ for example we would get

$$x = t^3,\ y = e^{t^3},\ z = (t^3)^2 = t^6,\ -\infty < t < \infty$$

as another valid set of parametric equations for this curve. However, note that with something like $x = \cos t$ and then

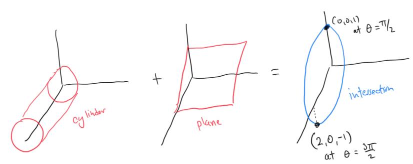
$$x = \cos t, \; y = e^{\cos t}, \; z = \cos^2 t,$$

we only get the portion of the intersection which consists of points with x-coordinate between -1 and 1 since $x = \cos t$ is restricted to these values alone.

Warm-Up 2. We describe the curve in $\mathbb{R}^3$ with parametric equations

$$x = 1 - \sin\theta, \; y = \cos\theta, \; z = \sin\theta \quad \text{for} \quad 0 \le \theta \le 2\pi.$$

First we note that the y- and z-coordinates satisfy $y^2 + z^2 = 1$, which means that our curve lies on the cylinder $y^2 + z^2 = 1$. Moreover, the x- and z-coordinates satisfy $x = 1 - z$, so our curve also lies on the plane $x = 1 - z$. Thus this curve is precisely the intersection of these two surfaces:



Note if nothing else that $x = 1 - \sin\theta$ only takes values between 0 and 2 (because sine only takes values between -1 and 1), so the entire curve should lie between these values of x, which the picture above confirms.

Vector functions. We can encode the parametric equations of a curve, such as the

$$x = 1 - \sin\theta, \; y = \cos\theta, \; z = \sin\theta$$

example above, as the components of a vector

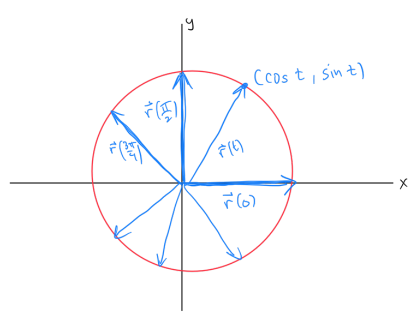
$$\mathbf{r}(\theta) = \langle 1 - \sin\theta, \cos\theta, \sin\theta \rangle$$

depending on θ, or whatever we happen to call the parameter in other examples. This is an example of what's called a *vector-valued function* since its values (i.e., outputs) are vectors. As the parameter varies, the vectors we get vary, and if we drawn them starting at the origin, their *endpoints* are the points that trace out the curve in question.

For example, take the unit circle with parametric equations $x = \cos t, \sin t$, or equivalently parametrized by the vector-valued function

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle .$$

At different t we get



We call $\mathbf{r}(t)$ the *position vector* of a point on this curve because it literally point (when drawn to start at the origin) at the position of a point on the curve.

Limits of vector functions. We can take limits of vector-valued functions just as we can with single-variable functions. For example, for the helix position vector

$$\mathbf{r}(t) = \langle 1 - \sin t, \cos t, \sin t \rangle$$

let us consider

$$\lim_{t \to \frac{\pi}{6}} \mathbf{r}(t).$$

The value of this limit should be a vector, namely the vector that the vector $\mathbf{r}(t)$ approaches as t approaches $\frac{\pi}{6}$. This is straightforward to compute, since to say that one vector is approaching another vector just means that the components of the first approach the components of the second, so that we just have to take the limit of each component of $\mathbf{r}(t)$:

$$\begin{aligned}
\lim_{t \to \frac{\pi}{6}} \mathbf{r}(t) &= \lim_{t \to \frac{\pi}{6}} \langle 1 - \sin t, \cos t, \sin t \rangle \\
&= \left\langle \lim_{t \to \frac{\pi}{6}} (1 - \sin t), \lim_{t \to \frac{\pi}{6}} \cos t, \lim_{t \to \frac{\pi}{6}} \sin t \right\rangle \\
&= \left\langle 1 - \sin \tfrac{\pi}{6}, \cos \tfrac{\pi}{6}, \sin \tfrac{\pi}{6} \right\rangle \\
&= \left\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \right\rangle.
\end{aligned}$$

Note that the reason why when computing the limit of each component we can simply evaluate at the point we are approaching is because each component is a continuous function.

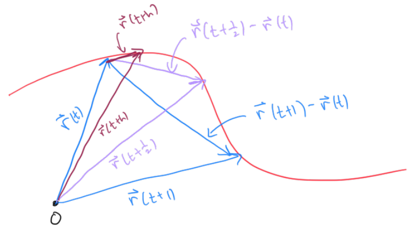
Derivatives and tangent vectors. Taking derivatives of vector-valued functions is also straightforward—we just take the derivative of each component:

$$\mathbf{r}(t) = \langle 1 - \sin t, \cos t, \sin t \rangle \rightsquigarrow \mathbf{r}'(t) = \langle -\cos t, -\sin t, \cos t \rangle .$$

But this derivative has an important geometric meaning in that it describes *tangent vectors* along the curve. Indeed, the derivative of a vector-valued function $\mathbf{r}(t)$ is officially defined via the same type of limit as what defines the single-variable derivative:

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

The numerator $\mathbf{r}(t+h) - \mathbf{r}(t)$ of the expression of which we are taking the limit describes a vector starting at the point with position vector $\mathbf{r}(t)$ on the curve and ending at the point with position vector $\mathbf{r}(t+h)$:

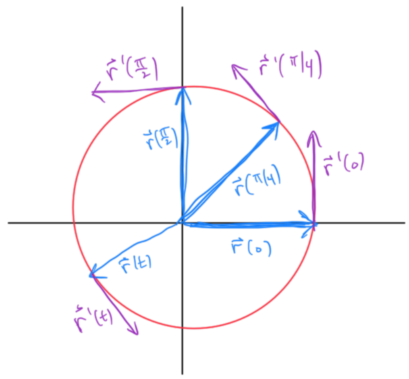


As $h \to 0$, the position vector $\mathbf{r}(t+h)$ moves closer to $\mathbf{r}(t)$, so that the vector $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ becomes closer and closer to being tangent to the curve, and hence in the limit it does become tangent.

Take for example the function $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ parametrizing a unit circle. We have

$$\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle ,$$

so that, say, $\mathbf{r}'(0) = \langle 0, 1 \rangle$ is tangent to the circle at the point $(1, 0)$ with position vector $\mathbf{r}(0) = \langle 1, 0 \rangle$, $\mathbf{r}'(\frac{\pi}{4}) = \left\langle -\sqrt{2}/2, \sqrt{2}/2 \right\rangle$ is tangent at position vector $\mathbf{r}(\frac{\pi}{4}) = \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle$, and $\mathbf{r}'(\frac{\pi}{2}) = \langle -1, 0 \rangle$ is tangent at $(0, 1)$:



Example. We find parametric equations for the tangent line to the curve with vector function

$$\mathbf{r}(t) = \langle 1 - \sin t, \cos t, \sin t \rangle$$

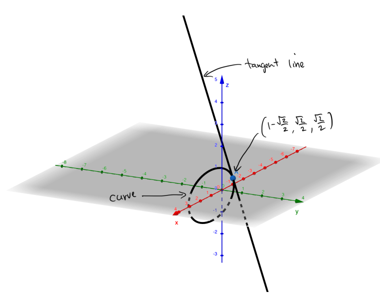
at the point $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Recall that to describe a line we need a point on the line and a direction vector for the line. For the point we take $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ as this is the point at which we want the tangent line, and for the direction vector we simply take the tangent vector at this point. The point $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is the point occurring at $t = \frac{\pi}{4}$. We have

$$\mathbf{r}'(t) = \langle -\cos t, -\sin t, \cos t \rangle , \text{ so } \mathbf{r}'(\tfrac{\pi}{4}) = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

is tangent to the curve at $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Thus with point $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and direction vector $\left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$, the desired tangent line has parametric equations

$$x = 1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t, \;\; y = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t, \;\; z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t.$$

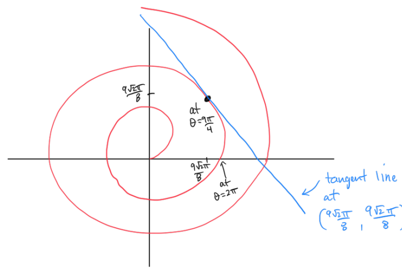
Here is the curve and this specific tangent line:



Another example. We find parametric equations for the tangent line to the curve with polar equation $r = \theta$ at the point corresponding to $\theta = \frac{9\pi}{4}$. First we need parametric equations for (x, y) points on this polar curve. We know that $x = r\cos\theta, y = r\sin\theta$, and here we are considering points where the value of r is just $r = \theta$, meaning that

$$x = r\cos\theta = \theta\cos\theta, \;\; y = r\sin\theta = \theta\sin\theta$$

are the (x, y) coordinates of points on this curve. These describe a spiral since, as θ increases, our points have distance $r = \theta$ to the origin that increases as well.



The value $\theta = \frac{9\pi}{4}$ thus gives the point $(x, y) = (\frac{9\pi}{4}\cos(\frac{9\pi}{4}), \frac{9\pi}{4}\sin(\frac{9\pi}{4})) = (\frac{9\sqrt{2}\pi}{8}, \frac{9\sqrt{2}\pi}{8})$ as the point at which we want the tangent line. With $\mathbf{r}(\theta) = \langle \theta\cos\theta, \theta\sin\theta\rangle$, tangent vectors are given by

$$\mathbf{r}'(\theta) = \langle \cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta \rangle ,$$

so the tangent vector at $(\frac{9\sqrt{2}\pi}{8}, \frac{9\sqrt{2}\pi}{8})$ is

$$\mathbf{r}'(\tfrac{9\pi}{4}) = \left\langle \cos(\tfrac{9\pi}{4}) - \tfrac{9\pi}{4}\sin(\tfrac{9\pi}{4}), \sin(\tfrac{9\pi}{4}) + \tfrac{9\pi}{4}\cos(\tfrac{9\pi}{4}) \right\rangle = \left\langle \frac{\sqrt{2}}{2} - \frac{9\sqrt{2}\pi}{8}, \frac{\sqrt{2}}{2} + \frac{9\sqrt{2}\pi}{8} \right\rangle .$$

Hence, the tangent line at the desired point has parametric equations

$$x = \tfrac{9\sqrt{2}\pi}{8} + (\tfrac{\sqrt{2}}{2} - \tfrac{9\sqrt{2}\pi}{8})t, \;\; y = \tfrac{9\sqrt{2}\pi}{8} + (\tfrac{\sqrt{2}}{2} + \tfrac{9\sqrt{2}\pi}{8})t.$$

Lecture 12: Integrals and Motion

Warm-Up 1. We find the tangent line to the curve given by

$$\mathbf{r}(t) = \left\langle e^{2t} - 1, t^3, e^{2t} \right\rangle$$

at the point where the curve intersects the surface $y = (x+1)^2 - z^2 + 27$. Points on the curve in question have coordinates

$$x = e^{2t} - 1, \;\; y = t^3, \;\; z = e^{2t},$$

so the curve intersects the given surface when these coordinates satisfy the equation of that surface:

$$y = (x+1)^2 - z^2 + 27 \rightsquigarrow t^3 = (e^{2t} - 1 + 1)^2 - (e^{2t})^2 + 27 \rightsquigarrow t^3 = 27 \rightsquigarrow t = 3.$$

(Note that the surface in question is a hyperbolic paraboloid, i.e. a saddle.) At $t = 3$ we get position $\mathbf{r}(3) = \langle e^6 - 1, 27, e^6 \rangle$, so we want the tangent line to the given curve at $(e^6 - 1, 27, e^6)$.

We have

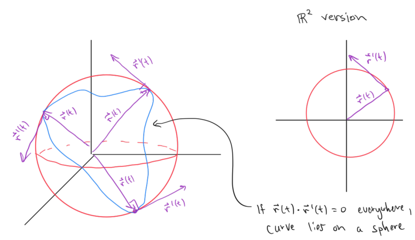
$$\mathbf{r}'(t) = \left\langle 2e^{2t}, 3t^2, 2e^{2t} \right\rangle, \text{ so } \mathbf{r}'(3) = \left\langle 2e^6, 27, 2e^6 \right\rangle$$

is tangent to the curve at the desired point. Hence we get

$$x = e^6 - 1 + 2e^6 t,\ y = 27 + 27t,\ z = e^6 - 2e^6 t$$

as parametric equations for the desired tangent line.

Warm-Up 2. Suppose a curve in $\mathbb{R}^3$ is parametrized by some $\mathbf{r}(t)$ with the property that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ orthogonal at all points. We justify the fact that this curves lies completely on a sphere:



Note that spheres (or circles in the 2-dimensional case!) do have the property that tangent vectors are always orthogonal to position vectors, so the point here is that *only* spheres (or circles) has this property. Our purpose for considering this problem is to illustrate that the "product rule" has analogues for vector-valued functions as well. In particular, if $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two vector-valued functions, then $\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)$ is a scalar-valued function depending on t (for input t we get as output the number that is the result of the given dot product), and the derivative of this scalar-valued function is

$$\frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) = \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t).$$

This indeed looks like the usual product rule, only with dot products of vectors instead of products of numbers. (A similar "product rule" holds for the cross product $\mathbf{r}_1(t) \times \mathbf{r}_2(t)$, but there'll be no need to consider this in our course.)

Back to the problem at hand. We take the derivative of $\mathbf{r}(t) \cdot \mathbf{r}(t)$ using the product rule:

$$\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{r}(t)) = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

Our assumption says that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal, so the final dot product above is zero:

$$\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{r}(t)) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \text{ for all } t.$$

But this means that the scalar-valued function $\mathbf{r}(t) \cdot \mathbf{r}(t)$ must be constant! This function gives the length of $\mathbf{r}(t)$ squared, and if this length squared is constant the length $|\mathbf{r}(t)|$ must be constant as well. Hence for all t, $|\mathbf{r}(t)|$ is the same number, which means that the curve parametrized by $\mathbf{r}(t)$ does indeed lie on a sphere, namely the sphere whose radius is this exact constant.

Integrals of vector functions. As with limits and derivatives, we can compute integrals of vector functions by taking the integral of each component. For example, if

$$\mathbf{r}(t) = \left\langle \cos t, t^2, e^t \right\rangle,$$

then

$$\int_0^2 \mathbf{r}(t)\, dt = \int_0^2 \left\langle \cos t, t^2, e^t \right\rangle\, dt$$

$$
\begin{aligned}
&= \left\langle \sin t, \tfrac{1}{3}t^3, e^t \right\rangle \Big|_0^2 \\
&= \left\langle \sin 2, \tfrac{8}{3}, e^2 \right\rangle - \left\langle \sin 0, 0, e^0 \right\rangle \\
&= \left\langle \sin 2, \tfrac{8}{3}, e^2 - 1 \right\rangle .
\end{aligned}
$$

We will not try to interpret such a thing as any type of "area", in this course at least.

Motion and acceleration. Integrals of vector-valued functions for us will only be used when discussing *motion*. The setup is that we have, say, a particle (or something else) moving along some curve parametrized by $\mathbf{r}(t)$, so that $\mathbf{r}(t)$ gives the position vector of the particle at time t. The derivative $\mathbf{r}'(t)$ (i.e., tangent vector) is then interpreted as the *velocity* $\mathbf{v}(t)$ of the particle at time t, and the length of velocity $|\mathbf{v}(t)| = |\mathbf{r}'(t)|$ gives the speed at time t. The derivative of velocity, or equivalently the second derivative of position, gives the *acceleration* $\mathbf{a}(t)$ at time t:

$$\mathbf{r}(t) = \text{position},\ \ \mathbf{v}(t) = \mathbf{r}'(t) = \text{velocity},\ \ \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \text{acceleration}.$$

Assume for example that a rocket moves through space with accelaration

$$\mathbf{a}(t) = \left\langle 1+t, e^{2t}, \tfrac{1}{t^2} \right\rangle$$

and that at time 1 it is at $(1,2,3)$ with position vector $\mathbf{r}(1) = \langle 1,2,3 \rangle$ and velocity $\mathbf{v}(1) = \langle -1,0,4 \rangle$. We want to determine at which position it will be at any time $t > 1$. To find this general position we must integrate velocity, and to find the general velocity we must integrate acceleration. So, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \left\langle 1+t, e^{2t}, \tfrac{1}{t^2} \right\rangle dt = \left\langle t + \tfrac{1}{2}t^2 + c_1, \tfrac{1}{2}e^{2t} + c_2, -\tfrac{1}{t} + c_3 \right\rangle ,$$

where c_1, c_2, c_3 are usual constants of integration, one for each component since they could be different! To find the values of these constants we use the given velocity $\mathbf{v}(1) = \langle -1,0,4 \rangle$ at time 1. We need

$$\mathbf{v}(1) = \left\langle 1 + \tfrac{1}{2} + c_1, \tfrac{1}{2}e^2 + c_2, -1 + c_3 \right\rangle \ \text{ to agree with } \mathbf{v}(1) = \langle -1, 0, 4 \rangle ,$$

and this requires $c_1 = -\frac{5}{2}, c_2 = -\frac{1}{2}e^2, c_3 = 5$. Thus the general velocity of the rocket is

$$\mathbf{v}(t) = \left\langle t + \tfrac{1}{2}t^2 - \tfrac{5}{2}, \tfrac{1}{2}e^{2t} - \tfrac{1}{2}e^2, -\tfrac{1}{t} + 5 \right\rangle .$$

Next we have

$$
\begin{aligned}
\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt \\
&= \int \left\langle t + \tfrac{1}{2}t^2 - \tfrac{5}{2}, \tfrac{1}{2}e^{2t} - \tfrac{1}{2}e^2, -\tfrac{1}{t} + 5 \right\rangle\, dt \\
&= \left\langle \tfrac{1}{2}t^2 + \tfrac{1}{6}t^3 - \tfrac{5}{2}t + d_1, \tfrac{1}{4}e^{2t} - \tfrac{1}{2}e^2 t + d_2, -\ln t + 5t + d_3 \right\rangle ,
\end{aligned}
$$

with again some to-be-determined constants of integration d_1, d_2, d_3. We are given that the position at time 1 is $\mathbf{r}(1) = \langle 1,2,3 \rangle$, so we need

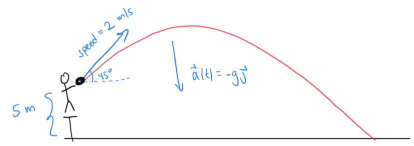
$$\mathbf{r}(1) = \left\langle \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{5}{2} + d_1, \tfrac{1}{4}e^2 - \tfrac{1}{2}e^2 + d_2, -\ln 1 + 5 + d_3 \right\rangle \ \text{ to equal } \langle 1,2,3 \rangle ,$$

so $d_1 = \frac{17}{6}, d_2 = 2 + \frac{1}{4}e^2, d_3 = -2$. Thus the position vector is

$$\mathbf{r}(t) = \left\langle \tfrac{1}{2}t^2 + \tfrac{1}{6}t^3 - \tfrac{5}{2}t + \tfrac{17}{6}, \tfrac{1}{4}e^{2t} - \tfrac{1}{2}e^2 t + 2 + \tfrac{1}{4}e^2, -\ln t + 5t - 2 \right\rangle ,$$

so at time t the rocket is at the point $(\frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{5}{2}t + \frac{17}{6}, \frac{1}{4}e^{2t} - \frac{1}{2}e^2 t + 2 + \frac{1}{4}e^2, -\ln t + 5t - 2)$.

Projectile motion. Suppose we throw a ball from a height of 5 meters at an angle of 45° above the horizontal with an initial speed of 2 meters per second. If the only force which affects the motion of the ball is gravity, we determine the path the ball will follow:



The starting point is that the acceleration of the ball due only to the effect of gravity is

$$\mathbf{a}(t) = -g\,\mathbf{j}$$

where g is a constant which is approximately 9.8 meters per second squared. (This comes from *Newton's second law of motion* in physics which says that force equals mass times acceleration, but we will simply take that this is the correct acceleration for granted since, after all, this is not a physics class. All we care about here is integrating so that we can find position from acceleration.) The initial position of the ball is $\mathbf{r}(0) = 5\,\mathbf{j}$ since it is starting at a height of 5 meters above the ground, and the initial velocity $\mathbf{v}(0)$ should have length $|\mathbf{v}(0)| = 2$ and angle $\frac{\pi}{4}$ with the horizontal direction, so using ideas from polar coordinates we have

$$\mathbf{v}(0) = \underbrace{|\mathbf{v}(0)|}_{r}\cos(\underbrace{\pi/4}_{\theta}) + \underbrace{|\mathbf{v}(0)|}_{r}\sin(\underbrace{\pi/4}_{\theta}) = 2(\tfrac{\sqrt{2}}{2})\,\mathbf{i} + 2(\tfrac{\sqrt{2}}{2})\,\mathbf{j} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}.$$

The velocity is

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (0\,\mathbf{i} - g\,\mathbf{j})\,dt = c_1\,\mathbf{i} + (-gt + c_2)\,\mathbf{j}.$$

With $\mathbf{v}(0) = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$, we must have

$$c_1\,\mathbf{i} + (0 + c_2)\,\mathbf{j} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j},$$

so $c_1 = \sqrt{2}, c_2 = \sqrt{2}$. Our velocity is thus

$$\mathbf{v}(t) = \sqrt{2}\,\mathbf{i} + (-gt + \sqrt{2})\,\mathbf{j}.$$

Next, the position is

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int [\sqrt{2}\,\mathbf{i} + (-gt + \sqrt{2})\,\mathbf{j}]\,dt = (\sqrt{2}t + d_1)\,\mathbf{i} + (-\tfrac{1}{2}gt^2 + \sqrt{2}t + d_2)\,\mathbf{j}.$$

Since $\mathbf{r}(0) = 0\,\mathbf{i} + 5\,\mathbf{j}$, we need

$$d_1\,\mathbf{i} + d_2\,\mathbf{j} = 0\,\mathbf{i} + 5\,\mathbf{j},$$

so $d_1 = 0, d_2 = 5$. Thus the position of the ball is

$$\mathbf{r}(t) = \sqrt{2}t\,\mathbf{i} + (-\tfrac{1}{2}gt^2 + \sqrt{2}t + 5)\,\mathbf{j},$$

so the ball follows the path with parametric equations

$$x = \sqrt{2}t, \;\; y = -\frac{1}{2}gt^2 + \sqrt{2}t + 5.$$

Note that these coordinates (since $t = \frac{x}{\sqrt{2}}$ from the first equation) satisfy $y = -\frac{1}{4}gx^2 + x + 5$, which is an upside-down parabola, just as our initial picture above and intuition of what should happen when you throw a ball suggests.

Lecture 13: Arclength

Warm-Up. A ball is thrown at a speed of 50 meters per second at an angle of $\frac{\pi}{3}$ above the horizontal from a height of 10 meters, with only gravity acting on the ball. We determine how far the ball travels downfield, and the maximum height it attains along the way:

The initial position is $\mathbf{r}(0) = \langle 0, 10\rangle$, and the initial velocity is $\mathbf{v}(0) = \left\langle 50\cos(\frac{\pi}{3}), 50\sin(\frac{\pi}{3})\right\rangle = \left\langle 25, 25\sqrt{3}\right\rangle$. With acceleration $\mathbf{a}(t) = \langle 0, -g\rangle$ due to gravity, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, -g\rangle\;dt = \langle c_1, -gt + c_2\rangle\,.$$

Since $\mathbf{v}(0) = \left\langle 25, 25\sqrt{3}\right\rangle$, we have $c_1 = 25, c_2 = 25\sqrt{3}$, so

$$\mathbf{v}(t) = \left\langle 25, -gt + 25\sqrt{3}\right\rangle.$$

Next we get

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \left\langle 25, -gt + 25\sqrt{3}\right\rangle\;dt = \left\langle 25t + d_1, -\tfrac{1}{2}gt^2 + 25\sqrt{3}t + d_2\right\rangle.$$

Since $\mathbf{r}(0) = \langle 0, 10\rangle$, we have $d_1 = 0, d_2 = 10$, so

$$\mathbf{r}(t) = \left\langle 25t, -\tfrac{1}{2}gt^2 + 25\sqrt{3}t + 10\right\rangle$$

is the general position vector of the ball.

Now, to determine how far the ball travels downfield we must determine when it this the ground. This happens when the vertical $\mathbf{j}$-component of position is 0, so when

$$-\frac{1}{2}gt^2 + 25\sqrt{3}t + 10 = 0.$$

With $g \approx 9.8$, we use a calculator (just this once!) to solve and get $t \approx 9.06$ seconds as the time when the ball hits the ground. The distance covered up to this point is given by the horizontal $\mathbf{i}$-component at this time, so the balls travels

$$25(9.06) = 226.55 \text{ meters}$$

downfield. The maximum height the ball reaches along the way occurs when the vertical component of position is at a maximum, so we find the time at which this occurs by setting the derivative of this vertical component equal to 0:

$$0 = (-\tfrac{1}{2}gt^2 + 25\sqrt{3}t + 10)' = -gt + 25\sqrt{3} \rightsquigarrow t = \tfrac{25\sqrt{3}}{g} \approx 4.42 \text{ seconds.}$$

The maximum height is thus the vertical component at this time, so

$$-\tfrac{1}{2}g(4.42)^2 + 25\sqrt{3}(4.42) + 10 \approx 105.66 \text{ meters}$$

is the maximum height attained.

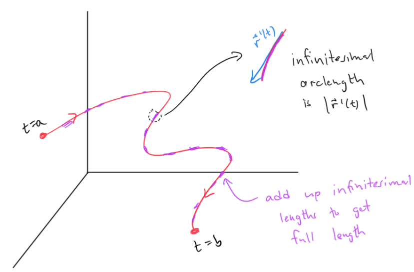
Arclength. The final quantity we'll want to compute given a curve is its *arclength*, which is just the length of the segment you would get if you were to straighten the curve out. In other words, if you were to travel from the starting point of the curve to the end point, the arclength is the distance you would travel. If a curve is parameterized by $\mathbf{r}(t)$ for $a \leq t \leq b$, its arclength is obtained by integrating the length of the tangent vector all along the curve:

$$\text{arclength} \;=\; \int_a^b |\mathbf{r}'(t)|\, dt.$$

This comes from thinking of the tangent vector as giving an "infinitesimal approximation" to the curve at each point: if $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ so

$$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

gives the "infinitesimal length" of an 'infinitestimal" piece of the curve, and thus the total length is obtained by "adding up" all of these infinitesimal quantities, which is what integrating these quantities over all values of t does:



An analogous integral gives the arclength for 2-dimensional curves as well. For example, take the circle $x^2 + y^2 = 16$, which is parametrized by

$$\mathbf{r}(t) = \langle 4\cos t, 4\sin t\rangle\,,\ \ 0 \leq t \leq 2\pi.$$

Of course, this is a circle of radius 4 and hence its circumference (which *is* its arclength) is $2\pi r = 8\pi$, so we don't need any fancy integral to compute this, but let us see that the integral gives the correct result anyway. We have

$$\mathbf{r}'(t) = \langle -4\sin t, 4\cos t\rangle\,, \text{ so } |\mathbf{r}'(t)| = \sqrt{16\sin^2 t + 16\cos^2 t} = \sqrt{16(\sin^2 t + \cos^2 t)} = \sqrt{16} = 4.$$

Thus the arclength is

$$\int_0^{2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2\pi} 4\, dt = 4t\Big|_0^{2\pi} = 8\pi$$

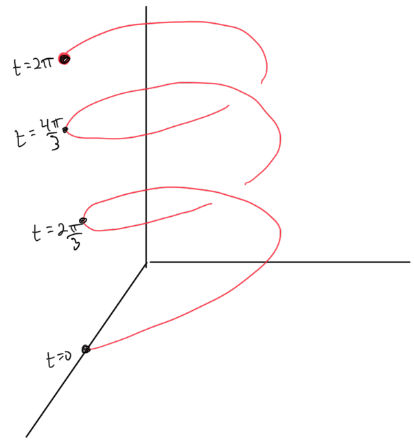
as expected.

Example. We find the arclength of the curve paramterized by

$$\mathbf{r}(t) = \left\langle \cos 3t, \sin 3t, 2t^{3/2}\right\rangle \quad 0 \leq t \leq 2\pi.$$

First, just to make sure we have a sense of what it is we are computing the arclength of, let us sketch this curve. The $x = \cos 3t$ and $y = \sin 3t$ coordinates of a point on this curve satisfy $x^2 + y^2 = 1$, so our curve lies on the cylinder $x^2 + y^2 = 1$, and as the curve wraps around the cylinder the

$z = 2t^{3/2}$ coordinate increases, so we have a helix. Now, as t runs between 0 and 2π, the input $3t$ into the cosine and sine functions in $\mathbf{r}(t)$ runs between 0 and 6π, which means that this curve wraps around the cylinder three times, or in other words that x and y complete three full revolutions in the "circular" direction:



To compute the arclength, we start with

$$\mathbf{r}'(t) = \left\langle -3\sin 3t, 3\cos 3t, 3t^{1/2} \right\rangle$$

and then

$$|\mathbf{r}'(t)| = \sqrt{9\sin^2(3t) + 9\cos^2(3t) + 9t} = \sqrt{9(\cos^3(3t) + \sin^2(3t) + 9t} = \sqrt{9+9t} = 3\sqrt{1+t}.$$

Thus the arclength of this curve is

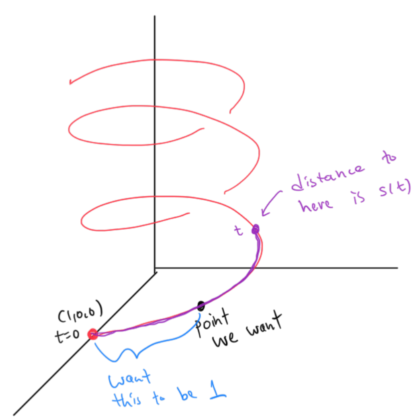
$$\int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} 3(1+t)^{1/2}\,dt = 2(1+t)^{3/2}\Big|_0^{2\pi} = 2(1+2\pi)^{3/2} - 2.$$

(We used a substitution $u = 1+t$ to compute $\int (1+t)^{1/2}\,dt = \int u^{1/2}\,du$.)

Arclength parameter function. Now saw we wish to find the point along the curve

$$\mathbf{r}(t) = \left\langle \cos 3t, \sin 3t, 2t^{3/2} \right\rangle, \ 0 \le t \le 2\pi$$

that is at a distance of 1 from $(1,0,0)$, where we measure distance along the curve itself:



In other words, we want the arclength of the portion of the curve between $(1,0,0)$ and the point we want to be 1. The point $(1,0,0)$ is the one occurring at the parameter value $t=0$, so we need to find the value of t such that

$$\int_0^t |\mathbf{r}'(u)|\,du = 1,$$

where we use now use u as the variable of integration so as to not confuse it with the upper limit t; with this value of t at hand, we plug into our parametric equations to find the point.

The function

$$s(t) = \int_0^t |\mathbf{r}'(u)|\,du$$

with t varying is called the *arclength parameter function* and measures distance along the curve from the point at $t=0$ to the point at some arbitrary $t>0$. Using the work we did above, we have

$$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 3(1+u)^{1/2}\,du = 2(1+u)^{3/2}\Big|_0^t = 2(1+t)^{3/2} - 2.$$

The point for which we are looking is thus the one occurring at t satisfying

$$2(1+t)^{3/2} - 2 = 1.$$

Solving gives

$$2(1+t)^{3/2} = 3 \rightsquigarrow (1+t)^{3/2} = \frac{3}{2} \rightsquigarrow 1+t = (\tfrac{3}{2})^{2/3}, \text{ so } t = (\tfrac{3}{2})^{2/3} - 1.$$

Thus, the point along the curve that is at a distance of 1 away from $(1,0,0)$ as measured along the curve is

$$(\cos(3(\tfrac{3}{2})^{2/3} - 1), \sin(3(\tfrac{3}{2})^{2/3} - 1), [(\tfrac{3}{2})^{2/3} - 1]^{3/2}).$$

Note that taking the derivative of

$$s(t) = \int_0^t |\mathbf{r}'(u)|\,du$$

with respect to t gives, by the Fundamental Theorem of Calculus, $s'(t) = |\mathbf{r}'(t)|$, which is why $|\mathbf{r}'(t)|$ is interpreted as speed: it is the rate of change of distance $s(t)$ with respect to time t.

Lecture 14: Multivariable Functions

Warm-Up 1. We find the arclength of the curve with polar equation $r = \sin\theta$. In fact this is a circle since

$$r = \sin\theta \rightsquigarrow r^2 = r\sin\theta \rightsquigarrow x^2 + y^2 = y \rightsquigarrow x^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{4}.$$

The radius is $\frac{1}{2}$, so the arclength/circumference is π, but we will derive this value from the arclength integral approach regardless. We can find parametric equations for this curve by taking $r = \sin\theta$ in the usual polar coordinate expressions for x and y:

$$x = r\cos\theta = \sin\theta\cos\theta,\ \ y = r\sin\theta = \sin\theta\sin\theta.$$

To get one revolution of this circle we take $0 \le \theta \le \pi$, since $\theta = 0$ gives the origin with $r = 0$, and then $\theta = \pi$ again gives $r = 0$, and hence the origin.

With $\mathbf{r}(\theta) = \langle \sin\theta\cos\theta, \sin^2\theta \rangle$, we have

$$\mathbf{r}'(\theta) = \langle \cos^2\theta - \sin^2\theta, 2\sin\theta\cos\theta \rangle,$$

so

$$\begin{aligned}
|\mathbf{r}'(\theta)| &= \sqrt{(\cos^2\theta - \sin^2\theta)^2 + 4\sin^2\theta\cos^2\theta} \\
&= \sqrt{\cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta + 4\sin^2\theta\cos^2\theta} \\
&= \sqrt{\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta} \\
&= \sqrt{(\cos^2\theta + \sin^2\theta)^2} \\
&= \sqrt{1} = 1.
\end{aligned}$$

Thus the arclength is

$$\int_0^\pi |\mathbf{r}'(\theta)|\, d\theta = \int_0^\pi 1\, d\theta = \pi$$

as expected.

Warm-Up 2. We find the arclength parameter function of $\mathbf{r}(t) = \langle \ln t, \frac{1}{2}t^2, \sqrt{2}t \rangle$ which measures the distance from $(0, \frac{1}{2}, \sqrt{2})$ in the direction of increasing t along the curve. We have

$$\mathbf{r}'(t) = \left\langle \tfrac{1}{t}, t, \sqrt{2} \right\rangle, \text{ so } |\mathbf{r}'(t)| = \sqrt{\tfrac{1}{t^2} + t^2 + 2} = \sqrt{(\tfrac{1}{t} + t)^2} = \tfrac{1}{t} + t.$$

The point $(0, \frac{1}{2}, \sqrt{2})$ occurs at parameter value $t = 1$, so the arclength parameter function measuring distance from this point is

$$s(t) = \int_1^t |\mathbf{r}'(u)|\, du = \int_1^t (\tfrac{1}{u} + u)\, du = (\ln u + \tfrac{1}{2}u^2)\Big|_1^t = \ln t + \tfrac{1}{2}t^2 - \tfrac{1}{2}.$$

Multivariable functions. Now that we have built up a good understanding of 3-dimensional space and describing objects (lines, planes, surfaces, curves) within it, we shift our focus to studying functions of several variables, which is what puts the "multi" in "multivariable calculus". Our eventual goal is to do calculus with such functions, meaning understand and use their derivatives, but before looking at derivatives we need a better sense of multivariable functions themselves.

What exactly do we mean by a function of several variables? Here's an example:

$$f(x, y) = x^2y + e^{xy}.$$

This is a function which takes two variables x and y as input (i.e., f is a function of two variables) and outputs the number $x^2y + e^{xy}$. The function

$$g(x, y, z) = xyz$$

is a function of three variables, which takes x, y, z as input and outputs the product xyz. Most phenomena in the "real world", or within the "mathematical world", depend on more than one variable or input, which must be described using multivariable functions.

Domain and range. A first basic question we can ask about multivariable functions is to determine their *domain* and *range*, which are concepts analogous to ones you would have seen in a single-variable calculus course. Take for example

$$f(x, y) = \ln(y - 3x).$$

The domain of f is the set of all possible (x, y) at which we can actually evaluate the function. For example, if we tried to evaluate this function at $(0, 2)$ we'd get

$$f(0, 2) = \ln(2 - 3(0)) = \ln 2,$$

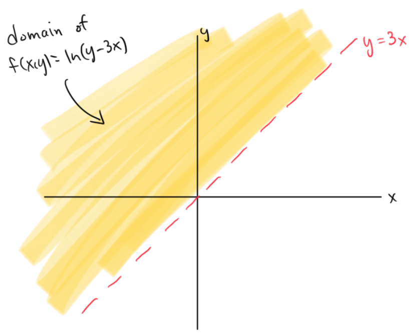
which makes perfect sense so $(0, 2)$ is in the domain of f. But for $(2, 0)$ we get

$$f(2, 0) = \ln(0 - 3(2)) = \ln(-6),$$

which is nonsense since we cannot take the logarithm of a negative number, so $(2, 0)$ is not in the domain of f. In this case, in order for the expression

$$\ln(y - 3x)$$

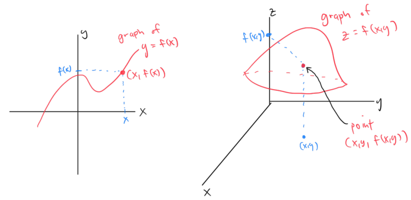
to make sense, $y - 3x$ must be positive, so the domain of f consists of all (x, y) satisfying $y - 3x > 0$. Visually, this is the part of the xy-plane that lies strictly above the line $y = 3x$:



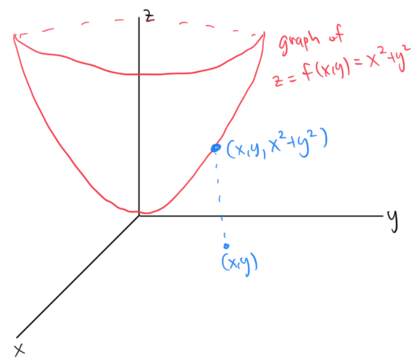
The range of $f(x, y) = \ln(y - 3x)$ consists of all numbers we get as actual outputs as the input (x, y) varies. For example, $\ln 2$ is in the range of f since it is the value obtained by evaluating $f(x, y)$ at $(x, y) = (0, 2)$. The natural log function gives as outputs every single possible number as the input varies (i.e, the graph of the single-variable function $h(x) = \ln x$ covers all possible values on the y-axis), so $f(x, y) = \ln(y - 3x)$ will give every possible number as an output as (x, y) varies throughout the domain. Hence the range of f is the interval $(-\infty, \infty)$, which contains every possible real number.

The function $g(x, y, z) = x^2 + y^2 + z^2$ of three variables has domain equal to all of $\mathbb{R}^3$ since all possible values of (x, y, z) will be ones for which the expression $x^2 + y^2 + z^2$ makes sense. The outputs in this case, however, can never be negative since $g(x, y, z) = x^2 + y^2 + z^2$ is always larger than or equal to 0, so the range is the interval $[0, \infty)$ consisting of nonnegative real numbers.

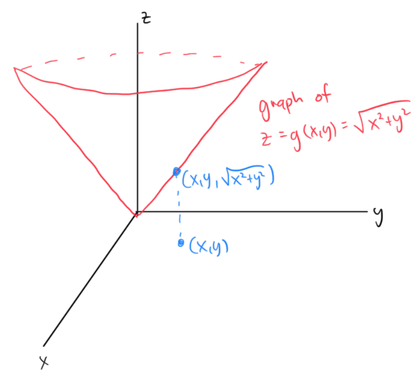
Graphs. As in the single-variable case, a key tool we'll use to understand the behavior of a multivariable function is its *graph*. In the single-variable case, the graph of $f(x)$ consists of the points in $\mathbb{R}^2$ whose y-coordinate is given by the value of the function at the x-coordinate, or in other words it is the curve defined by $y = f(x)$. For a function $f(x, y)$ of two variables, the analogous notion is the collection of points (x, y, z) in $\mathbb{R}^3$ whose z-coordinate is equal to the value of the function at the input (x, y), so it is the surface defined by $z = f(x, y)$:



For example, the graph of the function $f(x, y) = x^2 + y^2$ consists of all points satisfying $z = f(x, y) = x^2 + y^2$, which have seen before is the equation of a paraboloid opening upward:



The graph of $g(x, y) = \sqrt{x^2 + y^2}$ is defined by $z = \sqrt{x^2 + y^2}$, which gives the top half of the double cone $z^2 = x^2 + y^2$:



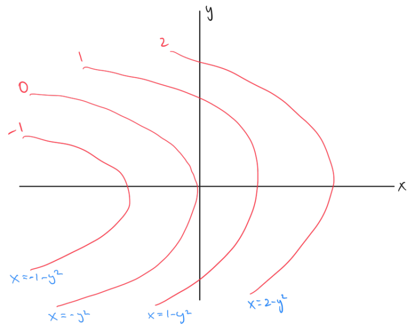
Again, the point in each of these is that any point on the graph has z-coordinate equal to the value of the function at the corresponding x, y-coordinates.

Level curves. Let us now consider the graph of $h(x, y) = y^2 + x$, which is defined by $z = y^2 + x$. This not the equation of a surface we've seen before, so in order to get a sense of what the graph looks like we default to the technique of looking at cross-sections, specifically cross-sections at different values of $z = k$. In the context of a function of two variables, these cross-sections are called *level curves* and give the piece of the graph that occurs at a certain "elevation".

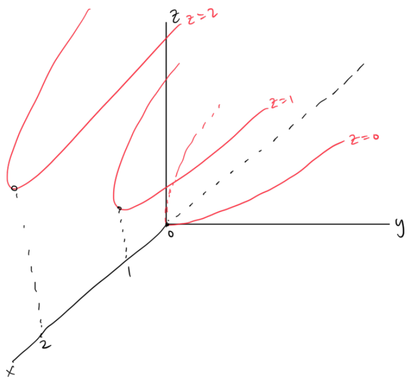
For $h(x, y) = y^2 + x$, the level curves at $z = k$ has equation

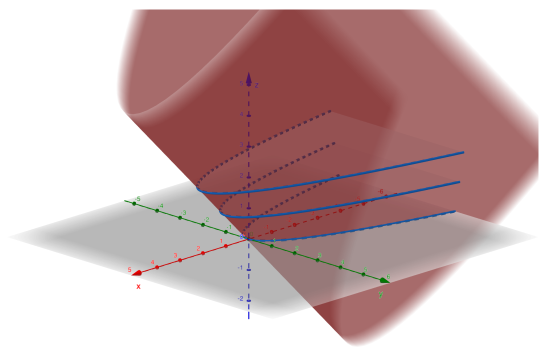
$$k = y^2 + x.$$

This gives a curve in the xy-plane, specifically a parabola in this case: the level curve at 0 is $0 = y^2 + x$, or $x = -y^2$, which passes through the origin; the level curve at $z = 1$ is $1 = y^2 + x$, or $x = 1 - y^2$, and has x-intercept at $x = 1$; the level curve at $z = 2$ is $2 = y^2 + x$, or $x = 2 - y^2$, and has x-intercept at $x = 2$; and so on



This picture with multiple level curves drawn is called a *contour map* for $h(x, y)$. View it as analogous to the type of 2-dimensional picture you typically see on a map for a mountain, where the heights at which specific level curves occur are labeled. By piecing these level curves together in 3-dimensions, we can get a sense for what the graph of our function looks like:



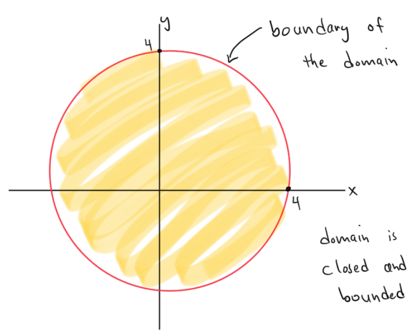


Lecture 15: More on Functions

Warm-Up 1. We determine the domain and range of the functions

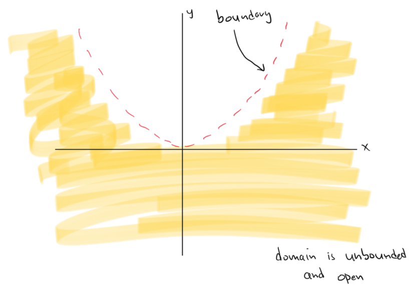
$$f(x, y) = \sqrt{16 - x^2 - y^2} \quad \text{and} \quad g(x, y) = \frac{1}{\sqrt{x^2 - y}}.$$

The expression $f(x, y) = \sqrt{16 - x^2 - y^2}$ is defined as long as we are not taking the square root of a negative number, so the domain consists of all points satisfying $16 - x^2 - y^2 \geq 0$. This can be written as $16 \geq x^2 + y^2$, which describes the region enclosed by a circle of radius 4 centered at the origin, including the circle $16 = x^2 + y^2$ on the boundary:



(By *boundary* here we just the mean the curve where the region "stops".) This is an example of a *bounded* region, which just means that there is a constraint on how far away from the origin points within it can be (i.e., it does not extend indefinitely in any direction), and is also a *closed* region, which just means that it contains its entire boundary. Since the value of $\sqrt{16-x^2-y^2}$ will never be negative but can take on any nonnegative value, the range of $f(x,y)=\sqrt{16-x^2-y^2}$ is the interval $[0,\infty)$ of nonnegative real numbers.

The expression $g(x,y)=\frac{1}{\sqrt{x^2-y}}$ makes sense as long as the denominator is nonzero, and because we are taking a square root, this requires that x^2-y be positive . The domain of $g(x,y)$ thus consists of all points satisfying $x^2-y>0$, or $x^2>y$, which is the region below the parabola $x^2=y$ but not including the parabola itself:

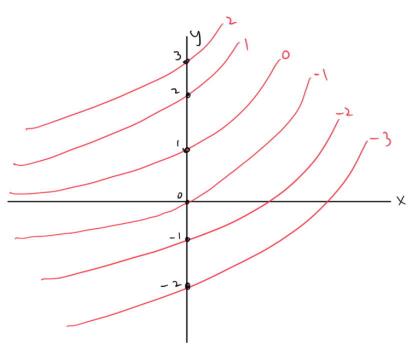


(We draw the boundary of this region, which is the parabola $x^2=y$, as dotted to indicate that these points are not meant to be included.) This region is *unbounded* since there are points within it that move further and further away from the origin, and it is an example of an *open* region, which means that it includes none of its boundary. The value of $\frac{1}{x^2-y}$ is always positive, and gets arbitrarily close to 0 as x^2-y gets larger, and arbitrarily large as x^2-y gets close to 0. Thus the range of $g(x,y)=\frac{1}{\sqrt{x^2-y}}$ is the open interval $(0,\infty)$ of positive real numbers.

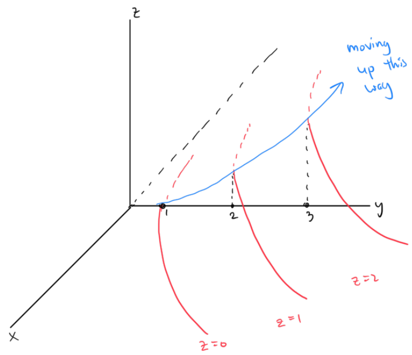
Warm-Up 2. We describe/sketch the graph of the function $h(x,y)=y-e^x$. We first consider some level curves. The level curve at $z=k$ is defined by

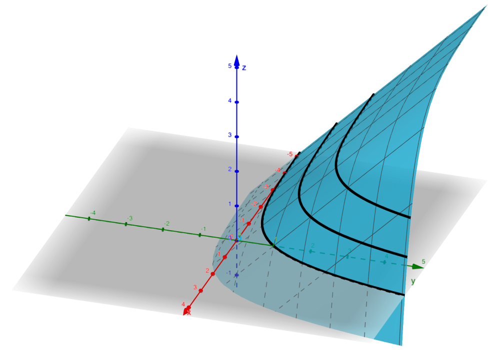
$$k=y-e^x, \text{ or } y=e^x+k.$$

These all look like the typical exponential curve $y=e^x$, only shifted up or down depending on the value of k:

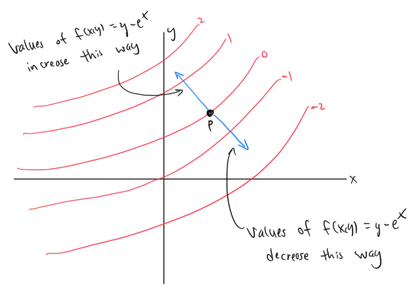


We can thus get a rough visualization of the graph by imagining these level curves occurring at the appropriate elevations:





Sketching graphs in general is not easy, but it is not actually something we'll be concerned with too much. Indeed, the basic point is that it is possible to determine much of the behavior of the function from the level curves and contour map alone without needing an accurate picture of the graph in 3-dimensions. For example, imagine we are at the following point P in the contour map of $h(x, y) = y - e^x$:



If we move away from P down and towards the right, we can see that the values of $h(x, y)$ should decrease since they get more negative. (Recall that the values of the function are the ones used to label the level curves; one of these entire curves is describing all the points at which we get one specific function value.) At the point P the value of $h(x, y)$ is 0 (since P lies on the level curve at 0), and when moving away from P down and towards the right we get function values that drop towards -1, then -2, and so on. This means that in this direction the graph of f should slope "downwards". If we move away from P up and towards the left, the function values increase, so the graph of $h(x, y)$ should slope upward in this direction away from P.

Example. Let us consider the function $f(x, y) = \frac{y}{x}$. Note first that no point with x-coordinate 0 is in the domain of f since $\frac{y}{x}$ is undefined when $x = 0$. The level curve at $z = 0$ is given by

$$0 = \frac{y}{x}, \text{ or } y = 0 \text{ (except } x = 0\text{);}$$

the level curve at $z = 1$ is given by

$$1 = \frac{y}{x}, \text{ or } y = x \text{ (except } x = 0\text{);}$$

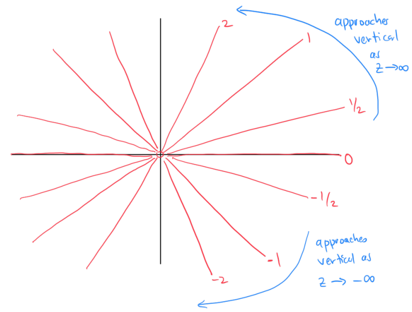
the level curve at $z = 2$ is given by

$$2 = \frac{y}{x}, \text{ or } y = 2x \text{ (except } x = 0\text{);}$$

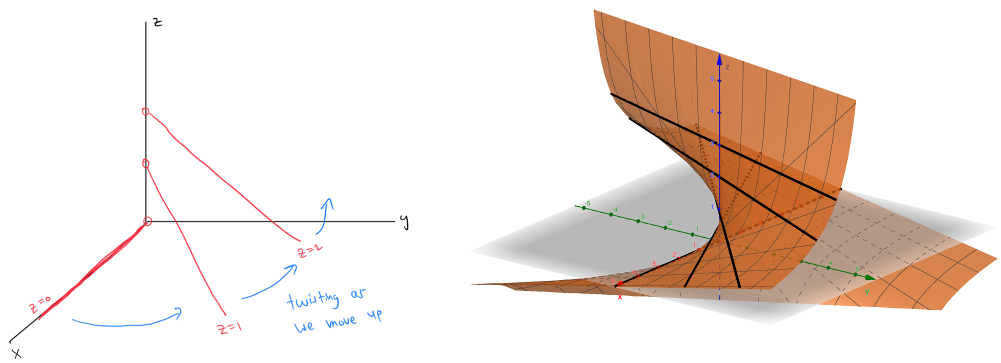
and the level curve at $z = -1$ is given by

$$-1 = \frac{y}{x}, \text{ or } y = -x \text{ (except } x = 0\text{).}$$

In fact, all level curves $k = \frac{y}{x}$ are lines $y = kx$ passing through the origin, only excluding the origin when $x = 0$. For $z > 0$ we get lines of positive slope which get closer to being vertical as $z \to \infty$, while for $z < 0$ we get lines of negative slope which get closer to being vertical as $z \to -\infty$:

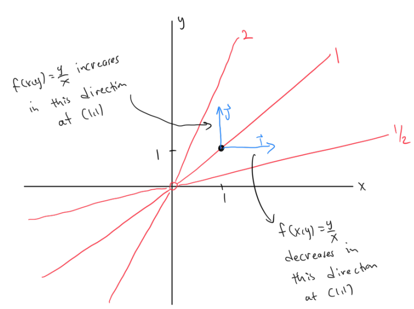


To obtain the graph, we take the line $y = 0$ (excluding the point at $x = 0$) at $z = 0$ and imagine twisting it towards the yz-plane as we move up and down:



(This is something like a 2-dimensional curve of a helix.)

From the contour map, if we are at the point $(1, 1)$ for example, we can see that the function should decrease if we move in the direction of the vector $\mathbf{i}$ at this point since the "elevations" decrease, while the function should increase if we move in the direction of the vector $\mathbf{j}$ at this point since elevations increase:



Another example. For the function $f(x, y) = \cos(x + y)$, note first that the range is the interval $[-1, 1]$ since these are the only values cosine can take on, so no part of the graph of $f(x, y)$ occurs above $z = 1$ and no part occurs below $z = -1$. The level curve at $z = 0$ consists of points satisfying

$$0 = \cos(x + y).$$

In order to have such a value for cosine, one possibility is to have the input $x + y$ equal $\frac{\pi}{2}$ since $\cos \frac{\pi}{2} = 0$. Thus $x + y = \frac{\pi}{2}$ makes up a portion of the level curve at $z = 0$. But cosine is also zero at, say, $\frac{3\pi}{2}$, so $x + y = \frac{3\pi}{2}$ is also part of the level curve at $z = 0$, and so is $x + y = \frac{5\pi}{2}$, or $x + y = -\frac{\pi}{2}$, etc. These are all lines of slope -1 with y-intercepts at odd integer multiples of $\frac{\pi}{2}$, which all *together* make up the level "curve" of $f(x, y) = \cos(x + y)$ at $z = 0$.

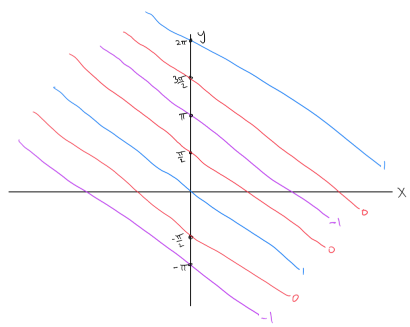
The level curve at $z = 1$ consists of points satisfying

$$1 = \cos(x + y).$$

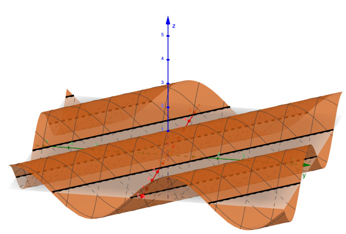
This value can be attained when $x + y = 0$, or $x + y = 2\pi$, or $x + y = -2\pi$, or more generally when $x + y$ is an integer multiple of 2π. These are also all lines of slope -1, which make up the entire level curve of $f(x, y)$ at $z = 1$. The level curve at $z = 2$, for example, is empty since no points satisfy $2 = \cos(x + y)$, and the level curve at $z = -1$ consists of lines

$$x + y = \pi, \; x + y = 3\pi, \; x + y = -\pi,$$

or more generally $x + y$ is an integer multiple of π. The contour map thus looks like:



If we are at the point $(0, \frac{\pi}{2})$, for example, on the level curve at $z = 0$, moving up and to the right (say in the direction of the vector $\mathbf{i} + \mathbf{j}$) causes the function $f(x, y) = \cos(x + y)$ to decrease in value towards -1, while moving down and to the left (in the direction of $-\mathbf{i} - \mathbf{j}$) causes the function to increase in value towards 1. The graph of $f(x, y) = \cos(x + y)$ looks like a 2-dimensional version of a cosine curve:



Level surface example. Finally we consider the 3-variable function $g(x, y, z) = x^2 + 2y^2 + 3z^2$. As in the 2-variable case, we consider inputs that given the same value of the function, so points satisfying

$$k = g(x, y, z) = x^2 + 2y^2 + 3z^2$$

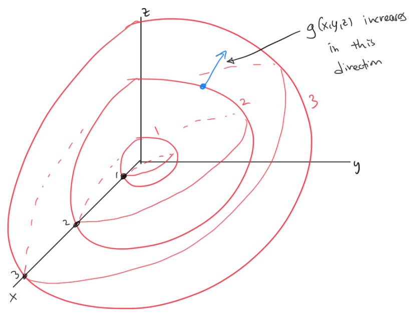
for a fixed k. Now, however, such equations describe surfaces and not curves, so we speak of the *level surfaces* of $g(x, y, z)$. The level surface at 1 has equation

$$1 = x^2 + 2y^2 + 3z^2,$$

which is the equation of an ellipsoid. The level surface at 2 is also an ellipsoid

$$2 = x^2 + 2y^2 + 3z^2,$$

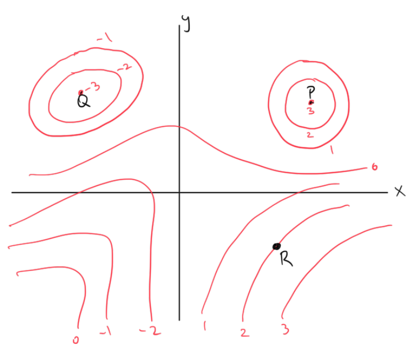
with the difference being that this one is larger than the one before. (Note that the ellipsoid at $z = 1$ has x-intercepts at $x = \pm 1$, while the level surface at $z = 2$ has x-intercepts which are further away from the origin at $x = \pm 2$.) The level surface at 0 consists of only the origin since $0 = x^2 + 2y^2 + 3z^2$ is only satisfied by $x = y = z = 0$, and the level surfaces at negative values are all empty since $g(x, y, z) = x^2 + 2y^2 + 3z^2$ is never negative. The level surfaces thus look like



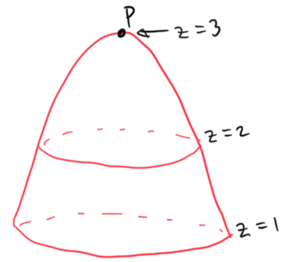
In this case we have no hope of visualizing the graph of $g(x, y, z) = x^2 + 2y^2 + 3z^2$ since this graph lives in 4-dimensions: we need three dimensions (i.e., axes) to keep track of the inputs x, y, z and another to keep track of the output. But, we can still get a sense of the behavior of $g(x, y, z)$ from the level sets alone: if we are on the level surface at $z = 2$, for example, moving away from the origin causes the value of $g(x, y, z)$ to increase, while moving towards the origin causes its value to decrease.

Lecture 16: Multivariable Limits

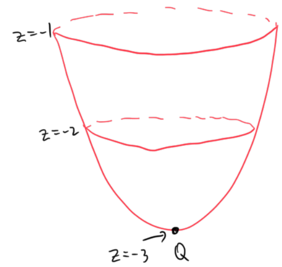
Warm-Up 1. Given the following contour map of $f(x, y)$, we describe what the graph of f roughly looks like near the given points P, Q, and R:



The value of f at P is 3 since, as labeled, the entire level curve at $z = 3$ consists solely of the point P. As we move away from P in any direction the values of f decreases towards $z = 2$ and then towards $z = 1$, so near P the graph looks something like



This is the type of picture which says that f has a *local maximum* at P since the value at P is larger than it is at any points nearby, but this is a concept we'll study in detail later. At Q the value if -3, and then the values *increase* (i.e., get less negative) as we move away from Q. Thus near Q the graph looks like



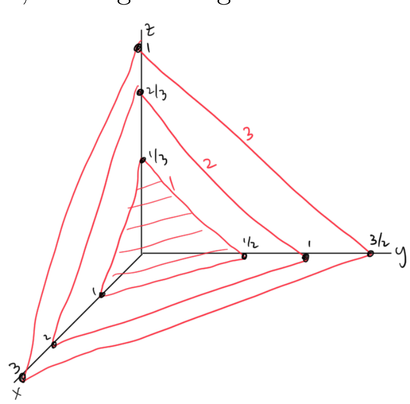
and we would say that f has a *local minimum* at Q.

The value of f at R is 2 since R sits on the level curve at $z = 2$. As we move through R horizontally (i.e., in the direction of increasing x), the values of f increase, so the graphs slopes upward in this direction, while if we move through R vertically (direction of increasing y), the values decrease so the graph slopes downward. If you imagine some type of "tangent plane" to the graph at R (also a concept we will study in more detail later), this plane would tilt upward in the x-direction but tilt downward in the y-direction.

Warm-Up 2. We describe the level surfaces of $g(x, y, z) = x + 2y + 3z$. These are given by equations of the form

$$k = x + 2y + 3z$$

where we hold the value of the function constant at k, and we have seen that such equations describe planes. Specifically, the level surface at 1 is the plane $1 = x + 2y + 3z$ with x, y, z-axes intercepts at $(1, 0, 0), (0, 1/2, 0), (0, 0, 1/3)$; the level surface at 2 is the plane $2 = x + 2y + 3z$ with x, y, z-axes intercepts at $(2, 0, 0), (0, 1, 0), (0, 0, 2/3)$; and the level surface at 3 is the plane $3 = x + 2y + 3z$ with intercepts $(3, 0, 0), (0, 3/2, 0), (0, 0, 1)$. As the value of $g(x, y, z) = x + 2y + 3z$ the level surfaces move further away from the origin, so we get things like



If we stand on one such plane, say in the first octant, and move away from the origin, the value of $g(x, y, z)$ increases, while moving towards the origin (still in the first octant) causes $g(x, y, z)$ to decrease in value.

Limits and continuity. We are almost at the point where we can introduce (finally!) the most important objects of study this quarter: multivariable derivatives. However, as is usual in calculus, before derivatives come limits, so we first focus on understanding multivariable limits. The basic idea is the same as it was in the single-variable case: the *limit* of $f(x, y)$ as (x, y) approaches (a, b) is meant to be the number (if it exists!) that $f(x, y)$ approaches as the inputs (x, y) get closer and closer to (a, b). We denote the value of this limit by

$$\lim_{(x,y)\to(a,b)} f(x, y),$$

and we can consider analogous notions for functions of more than two variables as well.

As a first example, let us compute

$$\lim_{(x,y)\to(1,2)} (xy + y^2).$$

As (x, y) approaches $(1, 2)$, we have x approaching 1 and y approaching 2, so xy approaches $1 \cdot 2$ and y^2 approaches 2^2, so altogether

$$\lim_{(x,y)\to(1,2)} (xy + y^2) = 1 \cdot 2 + 2^2 = 6.$$

In other words, in this case we can simply evaluate $f(x, y) = xy+y^2$ at the point we are approaching to get

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

This works here because $f(x, y) = xy + y^2$ is an example of a *continuous* function, which literally means that the limit as we approach a point will just be the value at that point. The fact that $f(x, y) = xy + y^2$ is continuous just comes from the fact that it is built of (using products and sums) of continuous functions: the functions x and y are continuous, so xy and y^2 are continuous, and hence so is $xy + y^2$.

Examples. Now we consider

$$\lim_{(x,y)\to(1,3)} \frac{xy + x - y - 1}{x - 1}.$$

Note here that $f(x, y) = \frac{xy+x-y-1}{x-1}$ is not continuous at $(1, 3)$ since it is not even defined there. This particular example is analogous to something like

$$\lim_{x\to1} \frac{x^2 - 1}{x - 1}$$

in the single-variable case, where the key point there is that we can find an alternative expression for our function $\frac{x^2-1}{x-1} = x+1$ valid for $x \neq 1$, which allows us to compute the limit using continuity. In the case at hand, we notice that $xy + x - y - 1$ factors as $xy + x - y - 1 = (x - 1)(y + 1)$, so

$$\lim_{(x,y)\to(1,3)} \frac{xy + x - y - 1}{x - 1} = \lim_{(x,y)\to(1,3)} \frac{(x - 1)(y + 1)}{x - 1} = \lim_{(x,y)\to(1,3)} (y + 1).$$

The resulting function $y+1$ is now perfectly continuous at $(1,3)$, so this final limit can be evaluated by plugging in:

$$\lim_{(x,y)\to(1,3)} \frac{xy+x-y-1}{x-1} = \lim_{(x,y)\to(1,3)} (y+1) = 3+1 = 4.$$

Suppose instead we consider

$$\lim_{(x,y)\to(1,3)} \frac{xy+x-y-2}{x-1}.$$

Again the function of which we are taking the limit is not continuous at $(1,3)$, and now no simple factorization will be possible. However, note in this case that even though the denominator approaches zero as $(x,y) \to (1,3)$, the numerator approaches $3+1-3-2=-1$. This means that the fraction

$$\frac{xy+x-y-2}{x-1}$$

gets larger and larger (in either the positive or negative direction) as $(x,y) \to (1,3)$, so it does not approach a finite value and hence the limit does not exist. (This is analogous to something like $\lim_{x\to 1} \frac{x}{x-1}$.) This was not the case in the previous example

$$\lim_{(x,y)\to(1,3)} \frac{xy+x-y-1}{x-1}$$

since both the denominator and numerator here approach 0 as $(x,y) \to (1,3)$, so the overall behavior of the fraction cannot be determined by focusing on the behavior of the numerator and denominator separately.

Sandwich theorem. For the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\cos(\tfrac{1}{x^2+y^2})$$

we need something new. An attempt to use something like

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\cos(\tfrac{1}{x^2+y^2}) = \left(\lim_{(x,y)\to(0,0)} (x^2+y^2)\right)\left(\lim_{(x,y)\to(0,0)} \cos(\tfrac{1}{x^2+y^2})\right)$$

does not work since

$$\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = \left(\lim_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim_{(x,y)\to(a,b)} g(x,y)\right)$$

only applies if both limits on the right exist, and in our case

$$\lim_{(x,y)\to(0,0)} \cos(\tfrac{1}{x^2+y^2})$$

does *not* exist: as $(x,y) \to (0,0)$, $\frac{1}{x^2+y^2}$ goes to ∞, which makes $\cos(\frac{1}{x^2+y^2})$ oscillate wildly with no well-defined limit. So, we cannot "break up" the limit in this way. (For an example of what can go wrong, note that if you tried to apply the same reasoning to $\lim_{x\to 0} x(\frac{1}{x})$ to get $(\lim_{x\to 0} x)(\lim_{x\to 0} \frac{1}{x}) = 0(\lim_{x\to 0} \frac{1}{x}) = 0$, the answer is nonsense since $x(\frac{1}{x}) = 1$ and hence $\lim_{x\to 0} x(\frac{1}{x})$ should be 1.)

The key idea here is that the cosine term is bounded since its values are always between -1 and 1, and so the fact that the x^2+y^2 term in front approaches 0 should force the entire product

$(x^2+y^2)\cos(\frac{1}{x^2+y^2})$ to approach 0 as well. (This doesn't work for $\lim_{x\to 0} x(\frac{1}{x})$ since the $\frac{1}{x}$ term is not bounded.) To make this precise, we note that

$$-(x^2+y^2) \leq (x^2+y^2)\cos(\tfrac{1}{x^2+y^2}) \leq x^2+y^2,$$

where on the right we replace the cosine term by the larger value 1, and on the left we replaced the cosine term by the smaller value -1:

$$-1 \leq \cos(\tfrac{1}{x^2+y^2}) \leq 1 \implies -(x^2+y^2) \leq (x^2+y^2)\cos(\tfrac{1}{x^2+y^2}) \leq x^2+y^2.$$

Since both the larger x^2+y^2 and smaller $-(x^2+y^2)$ bounds approach 0 as $(x,y)\to(0,0)$, the value $(x^2+y^2)\cos(\frac{1}{x^2+y^2})$ we care about must also approach 0 since it is "sandwiched" between two values that approach 0. Thus,

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\cos(\tfrac{1}{x^2+y^2}) = 0$$

by what we'll call the *sandwich theorem*, which is also sometimes called the *squeeze theorem*. (Sandwich is the term our book uses, so it's what we'll use as well.)

The sandwich theorem is a key tool that we'll also make use of in the context of polar coordinates (as we'll see next time), but note that in the book it is only briefly mentioned in the exercises alone and not in the main text. Nonetheless, it will be an important concept for us. When using the sandwich theorem we should be clear about the larger and smaller bounds we're using and what the final conclusion is. This is one point where explanations will be necessary.

Another example. We use the sandwich theorem to compute

$$\lim_{(x,y)\to(0,0)} (x^2+2y^4)\sin(\tfrac{e^{xy}}{x^2+y}).$$

We have

$$-1 \leq \sin(\tfrac{e^{xy}}{x^2+y}) \leq 1$$

since values of sine are never smaller than -1 nor larger than 1, so

$$-(x^2+2y^4) \leq (x^2+2y^4)\sin(\tfrac{e^{xy}}{x^2+y}) \leq x^2+2y^4.$$

Since the limits of both $-(x^2+2y^4$ and x^2+2y^4 as (x,y) approaches $(0,0)$ are zero, the sandwich theorem tells us that

$$\lim_{(x,y)\to(0,0)} (x^2+2y^4)\sin(\tfrac{e^{xy}}{x^2+y}) = 0.$$

Lecture 17: More on Limits

Warm-Up 1. We determine the value of c which makes the function

$$f(x,y) = \begin{cases} \frac{x^2-3x-y^2+3y}{x+y-3} & x+y-3 \neq 0 \\ c & x+y-3 = 0 \end{cases}$$

continuous at $(1,2)$. To be continuous at $(1,2)$ means that the limit as (x,y) approaches $(1,2)$ should be the value of the function at $(1,2)$:

$$\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2).$$

Since $(1,2)$ satisfies $x+y-3=0$, $f(1,2)$ has value c, so we need

$$\lim_{(x,y)\to(1,2)} f(x,y) = c.$$

Thus by computing the limit on the left we will have determined the value that c must be.

For $x+y-3=0$ we have $f(x,y)=c$ already, so what this boils down to is computing the limit for $x+y-3\neq 0$, which is

$$\lim_{(x,y)\to(1,2)} \frac{x^2-3x-y^2+3y}{x+y-3}.$$

The numerator of the fraction on the right factors as $x^2-3x-y^2+3y=(x+y-3)(x-y)$, so

$$\lim_{(x,y)\to(1,2)} \frac{x^2-3x-y^2+3y}{x+y-3} = \lim_{(x,y)\to(1,2)} \frac{(x+y-3)(x-y)}{x+y-3} = \lim_{(x,y)\to(1,2)} (x-y) = 1-2=-1,$$

where in the final step we use continuity of $x-y$ to find the limit by plugging in. Thus we need $c=-1$ in order to make $f(x,y)$ continuous at $(1,2)$.

Warm-Up 2. If $f(x,y)$ is a function satisfying $|f(x,y)|\leq [\ln(x^2+y^2+1)]^2$ at all points, we find the value of

$$\lim_{(x,y)\to(0,0)} e^{f(x,y)}.$$

This might not seem possible at first since we do not know what $f(x,y)$ is explicitly—we only know of some inequality that this unknown function is meant to satisfy. But we actually do have enough information to find the desired limit.

First, since the exponential function is continuous, we have

$$\lim_{(x,y)\to(0,0)} e^{f(x,y)} = e^{\lim_{(x,y)\to(0,0)} f(x,y)},$$

so all we must do is compute the limit of $f(x,y)$ as $(x,y)\to(0,0)$. From the inequality that $f(x,y)$ satisfies we get that

$$-[\ln(x^2+y^2+1)]^2 \leq f(x,y) \leq [\ln(x^2+y^2+1)]^2.$$

(In general, saying that $|a|\leq b$ for some expressions a and b means precisely that a itself is sitting between b and $-b$:

$$|a|\leq b \iff -b\leq a\leq b.$$

This is what we are using above to turn $|f(x,y)|\leq [\ln(x^2+y^2+1)]^2$ into the given string of inequalities.) Now, since the natural logarithm function is continuous, we have

$$\lim_{(x,y)\to(0,0)} \ln(x^2+y^2+1) = \ln(0^2+0^2+1) = \ln 1 = 0.$$

Thus both $-[\ln(x^2+y^2+1)]^2$ and $[\ln(x^2+y^2+1)]^2$ have limit 0 as (x,y) approaches $(0,0)$, so the sandwich theorem gives that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

as well. Hence the desired limit is

$$\lim_{(x,y)\to(0,0)} e^{f(x,y)} = e^{\lim_{(x,y)\to(0,0)} f(x,y)} = e^0 = 1.$$

Limits along different paths. Consider now

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x+2y}.$$

We claim that this limit does not exist, but no technique we've seen yet will give a way to justify this: both numerator and denominator approach 0, so we cannot deduce the behavior of the fraction from looking at the numerator and denominator separately, and there is no way we can algebraic simplify the given expression. The new idea we need is that if a limit where to exist, the manner in which we approach the given point should not make a difference. After all, we want to understand what $f(x,y)$ approaches as (x,y) approaches (a,b), and since (x,y) can approach (a,b) in many different ways, the limiting value we get for $f(x,y)$ should be the same regardless.

In this example consider first the limit as we approach $(0,0)$ using points only on the x-axis, which is defined by $y=0$. Such points look like $(x,0)$, so the value of the function at such points is given by

$$\frac{x+0}{x+2(0)}.$$

We want the limit of *this* single-variable expression (approaching along a specific curve will always turn our multivariable limit into a single-variable limit along that curve) as $x\to 0$, since $x\to 0$ is what it means for points of the form $(x,0)$ to approach $(0,0)$. We get

$$\lim_{(x,0)\to(0,0)} \frac{x+0}{x+2(0)} = \lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1.$$

So, if we try to determine the value of

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x+2y}$$

using points along the curve $y=0$ alone, we would expect the limit to be 1.

However, now we imagine approaching $(0,0)$ along the y-axis where $x=0$. Now we are using points of the form $(0,y)$, so we get

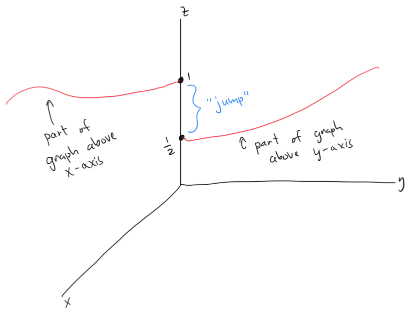
$$\lim_{(0,y)\to(0,0)} \frac{0+y}{0+2y} = \lim_{y\to 0}\frac{y}{2y} = \lim_{y\to 0}\frac{1}{2} = \frac{1}{2}.$$

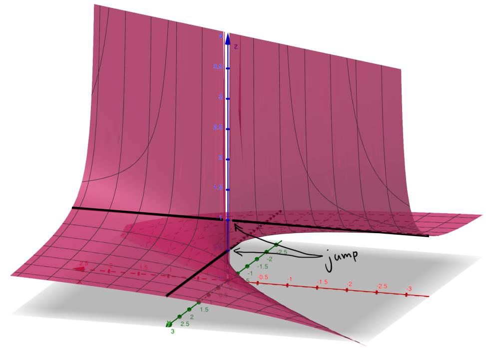
Thus using points on the curve $x=0$ to test the value of the limit, we would expect the limit to be $\frac{1}{2}$. Since we got different candidates for the limit when approaching $(0,0)$ along these two curves, it must be the case that the multivariable limit

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x+2y}$$

does not actually exist.

Geometrically, what is happening is that the graph as a type of "jump" at $(0,0)$ when we consider the piece where $x=0$ vs the piece where $y=0$, and this jump is what causes the limit to not exist:





Again, if the limit were to exist, it would not matter what piece we use to approach $(0,0)$—we'd get the same limiting value always. This is somewhat analogous to the idea of looking at left vs right-sided limits in the single-variable case, but not quite the same since when considering approach along a curve we are taking into account both sides of that curve, not just one side or the other.

Example. We justify the fact that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+3y^2}$$

does not exist. First we check the behavior along $y=0$:

$$\lim_{(x,0)\to(0,0)} \frac{0}{x^2+0} = \lim_{x\to 0} 0 = 0.$$

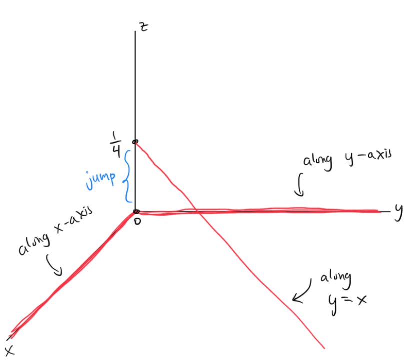
Along $x=0$ we have

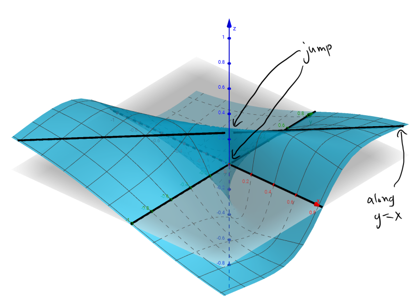
$$\lim_{(0,y)\to(0,0)} \frac{0}{0+3y^2} = \lim_{y\to 0} 0 = 0.$$

So far we cannot make any conclusions, but of course there are many other ways in which we can approach $(0,0)$ apart from solely along the x- or y-axis. If we approach instead along the line $y=x$, so that we consider points of the form (x,x), we have

$$\lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+3x^2} = \lim_{x\to 0} \frac{x^2}{4x^2} = \lim_{x\to 0} \frac{1}{4} = \frac{1}{4}.$$

Since we have found different ways of approaching $(0,0)$ that give different candidate values for the limit, the original multivariable limit does not exist. Geometrically, we have a "jump" like





Another example. For

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2},$$

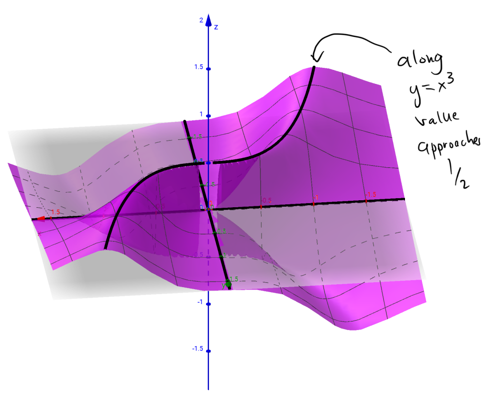
checking the limits along $x=0$ and $y=0$ is not enough since these will both give 0, as you can check. Checking the limit along $y=x$ is still not enough:

$$\lim_{(x,x)\to(0,0)} \frac{x^3x}{x^6+x^2} = \lim_{x\to 0} \frac{x^2x^2}{x^2(x^4+1)} = \lim_{x\to 0} \frac{x^2}{x^4+1} = \frac{0}{1} = 0.$$

Let us move beyond checking limits along lines alone, and see what happens if we approach $(0,0)$ along the curve $y=x^3$ instead. We have

$$\lim_{(x,x^3)\to(0,0)} \frac{x^3x^3}{x^6+(x^3)^2} = \lim_{x\to 0} \frac{x^6}{2x^6} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since we get a different limiting value along $y=x^3$ than, say, $x=0$, the multivariable limit does not exist:



Limits in polar coordinates. Finally we consider

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}.$$

Nothing we've done so far will work here (try it out!), so again we need a new idea. In this case, we can determine the behavior of the limit by converting to polar coordinates. In polar coordinates, the function of of which we are taking the limit looks like

$$\frac{x^3+y^3}{x^2+y^2} = \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = r(\cos^3\theta + \sin^3\theta).$$

The fact that we are meant to approach the origin can be phrased in terms of polar coordinates as requiring that r (i.e., distance to the origin) approach 0, without any constraints on what is happening to θ. Thus, we can write our multivariable limit in polar coordinates as

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta).$$

To determine the value of this polar limit, we use the sandwich theorem! The idea is that the r factor in front is approaching 0, which seems like it should force the entire expression to approach 0 as well, but to make this clear the sandwich theorem is needed since we are not able to "break up" the limit as

$$(\text{limit of } r) \text{ times } (\text{limit of } \cos^3\theta + \sin^3\theta)$$

since the second limit does not exist as it will depend on the behavior of θ. Instead, we note that since cosine and sine only give values between -1 and 1, we have

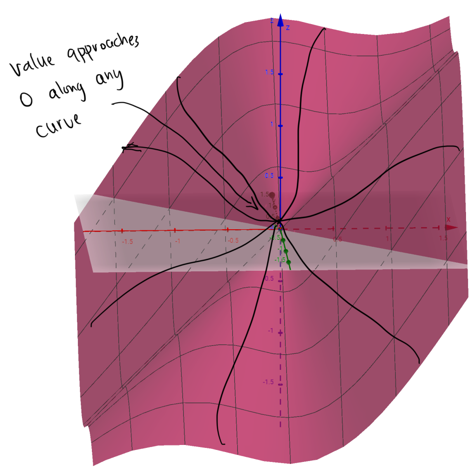
$$r(-1-1) \leq r(\cos^3\theta + \sin^3\theta) \leq r(1+1),$$

so that $r(\cos^3\theta+\sin^3\theta)$ is sandwiched between $-2r$ and $2r$. (Note that in this case $\cos^3\theta+\sin^3\theta$ will never actually have the value 1, nor -1, but this is OK: all we need are some "upper" and "lower" bounds on the value, not the most efficient bounds coming from the maximum and minimum values, which would take more work to find!)

Since $-2r$ and $2r$ both approach 0 as $r \to 0$, the sandwich theorem guarantees that

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0.$$

A picture of the graph does indeed suggest that the value (i.e., the z-coordinate) of the function is approaching 0 as (x,y) approaches the origin:



Lecture 18: Partial Derivatives

Warm-Up 1. We verify that

$$\lim_{(x,y)\to(0,1)} \frac{x\ln y}{y-x-1}$$

does not exist. (To be clear, the given function is not defined at points satisfying $y-x-1=0$, so we are only taking the limit using points not on the line $y=x+1$.) First, when approaching $(0,1)$ along the y-axis we have

$$\lim_{(0,y)\to(0,1)} \frac{0}{y-1} = \lim_{y\to 1} 0 = 0.$$

When approaching $(0,1)$ along the horizontal line $y=1$ (careful: we cannot approach along the x-axis since the x-axis does not pass through the point $(0,1)$, whereas $y=1$ does) we get

$$\lim_{(x,1)\to(0,1)} \frac{x\ln 1}{-x} = \lim_{x\to 0} \frac{0}{-x} = \lim_{x\to 0} 0 = 0,$$

so we move consider other curves.

If we approach $(0,1)$ along the curve $y=e^x$ (which does pass through $(0,1)$), we have

$$\lim_{(x,e^x)\to(0,1)} \frac{x\ln(e^x)}{e^x-x-1} = \lim_{x\to 0} \frac{x^2}{e^x-x-1}.$$

Since the numerator and denominator here both approach 0, we can use L'Hopital's rule for single-variable limits. (There is no analog of L'Hopital's rule for multivariable limits, but the point is that after restricting the points we consider to only those along a curve like $y=e^x$, we end up with a single-variable limits as a result.) L'Hopital's rule gives

$$\lim_{x\to 0} \frac{x^2}{e^x-x-1} = \lim_{x\to 0} \frac{2x}{e^x-1}.$$

We still have numerator and denominator approaching 0, so we need another application of L'Hopital's rule:

$$\lim_{x\to 0} \frac{x^2}{e^x-x-1} = \lim_{x\to 0} \frac{2x}{e^x-1} = \lim_{x\to 0} \frac{2}{e^x} = \frac{2}{e^0} = 2.$$

Since this gives a different candidate limit than we got along, say, $y=1$, the original multivariable limit does not exist.

Warm-Up 2. We determine whether or not

$$\lim_{(x,y)\to(0,0)} \frac{x-2xy+y^2}{\sqrt{x^2+y^2}}$$

exists. The form of the denominator suggests that converting to polar coordinates could be useful, so we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x-2xy+y^2}{\sqrt{x^2+y^2}} &= \lim_{r\to 0} \frac{r\cos\theta - 2r^2\cos\theta\sin\theta + r^2\sin^2\theta}{r} \\ &= \lim_{r\to 0} (\cos\theta - 2r\cos\theta\sin\theta + r\sin^2\theta).\end{aligned}$$

We claim that this limit does not exist. As a first step, note that we can compute the limit of the second and third parts using the sandwich theorem: since sine and cosine are always between -1 and 1, we have

$$-2r - r \le -2r\cos\theta\sin\theta + r\sin^2\theta \le 2r + r,$$

(note that $\sin^2\theta$ here cannot reach -1, but that is fine since all we need again are *some* bounds and not necessarily the most efficient; no need to work too hard to find the most precise bounds possible!) and since $-3r = -2r - r$ and $3r = 2r + r$ both approach 0 as $r\to 0$, we get that

$$\lim_{r\to 0} (-2r\cos\theta\sin\theta + r\sin^2\theta) = 0$$

by the sandwich theorem.

So we are left with determining the limit of the initial $\cos\theta$ term in

$$\lim_{r\to 0}(\cos\theta - 2r\cos\theta\sin\theta + r\sin^2\theta).$$

But recall that we placed no restrictions on what was happening θ since $r \to 0$ already forces (x, y) to approach $(0, 0)$. If we now go back and took $\theta = 0$, so that we approach the origin along the positive x-axis, we'd get

$$\lim_{r\to 0,\theta=0}(\cos\theta - 2r\cos\theta\sin\theta + r\sin^2\theta) = \cos 0 + 0 = 1,$$

but if we took $\theta = \frac{\pi}{4}$, thereby approaching the origin along $y = x$, we'd get

$$\lim_{r\to 0,\theta=\frac{\pi}{4}}(\cos\theta - 2r\cos\theta\sin\theta + r\sin^2\theta) = \cos(\tfrac{\pi}{4}) + 0 = \tfrac{\sqrt{2}}{2}.$$

Since we get different values for different θ-directions, the limit does not exist.

Partial derivatives. We have now arrived at the most fundamental notion of the quarter, that of the *partial derivatives* of a multivariable function. Let us use $f(x, y) = x^2y$ at the point $(1, -2)$ as an example. The partial derivative of $f(x, y) = x^2y$ *with respect to* x at $(1, -2)$ is denoted by $\frac{\partial f}{\partial x}(1, -2)$, and is defined by the limit

$$\frac{\partial f}{\partial x}(1, -2) = \lim_{h\to 0}\frac{f(1 + h, -2) - f(1, -2)}{h}.$$

Let us digest this. As $h \to 0$, we are looking at points $(1 + h, 2)$ getting closer to $(1, 2)$, but only via their x-coordinate since the y-coordinate remains 2 throughout. Thus, we are looking at how the function f changes "with respect to x" alone, and indeed the limit above is precisely what gives the ordinary single-variable derivative at $x = 1$ of the single-variable function $f(x, 2)$ obtained by holding y constant at 2 and only varying x. To compute this we thus literally treat y as constant and take a usual derivative with respect to x; x^2y is thought of as x^2 times a constant, so the derivative is $2x$ times that constant:

$$f(x, y) = x^2y \implies \frac{\partial f}{\partial x} = 2xy,$$

and thus evaluating at $(1, -2)$ gives $\frac{\partial f}{\partial x}(1, -2) = 2(1)(-2) = -4$.

This will be how we compute partial derivatives in general, but just this once let us work it out directly from the limit definition to get a feel for how it works. (After all, all derivative rules you've ever seen in your lives—such as the product and chain rules—are derived from the limit definition.) We have

$$\begin{aligned}
\frac{\partial f}{\partial x}(1, -2) &= \lim_{h\to 0}\frac{f(1 + h, -2) - f(1, -2)}{h} \\
&= \lim_{h\to 0}\frac{(1 + h)^2(-2) - 1^2(-2)}{h} \\
&= \lim_{h\to 0}\frac{-2 - 4h - 2h^2 + 2}{h} \\
&= \lim_{h\to 0}(-4 - 2h) \\
&= -4,
\end{aligned}$$

just as expected.

The partial derivative of $f(x,y) = x^2y$ with respect to y at $(1,-2)$ is also defined via a limit, only this time where we only vary the y-coordinate and keep the x-coordinate constant at 1:

$$\frac{\partial f}{\partial y}(1,-2) = \lim_{h \to 0} \frac{f(1,-2+h) - f(1,-2)}{h}.$$

This is the single-variable derivative with respect to y of the single-variable function $f(1,y)$ at $y = -2$, so we can compute this without having to use the limit definition just be treating x as if it were constant, differentiating, and then evaluating:

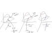
$$f(x,y) = x^2y \implies \frac{\partial f}{\partial y} = x^2 \implies \frac{\partial f}{\partial y}(1,-2) = 1.$$

(If x is constant, so is x^2, which is why the derivative of x^2 times y with respect to y is just the "constant" x^2.) We will also use the notation f_x and f_y for partial derivatives, where the subscript indicates the variable with differentiate with respect to, so

$$f_x(1,-2) = \frac{\partial f}{\partial x}(1,-2) \text{ and } f_y(1,-2) = \frac{\partial f}{\partial y}(1,-2).$$

Derivatives as slopes. Geometrically, partial derivatives compute slopes, just as in the single-variable case. The only difference is that now we consider slopes in different directions: $\frac{\partial f}{\partial x}$ gives the slope in the direction of increasing x, and $\frac{\partial f}{\partial y}$ gives the slope in the direction of increasing y.

In other words, the graph of $z = f(x,y)$ is some surface in $\mathbb{R}^3$. For the partial derivative at a point (a,b) with respect to x we hold $y = b$ fixed and vary the value of x to get some curve on the graph, namely the piece of the graph occuring in the plane $y = b$:



The slope of this curve at $x = a$ is precisely the partial derivative $\frac{\partial f}{\partial x}(a,b)$. Similarly, if we hold $x = a$ constant and vary y, we get the curve on the graph occuring in the $x = a$ plane, and the slope of this at $y = b$ is $\frac{\partial f}{\partial y}(a,b)$. In the example of $f(x,y) = x^2y$ at $(1,-2)$, the fact that $f_x(1,-2) = -4$ is negative means that the graph slopes downward at the point $(1,-2,-2)$ (the z-coordinate is the function value $f(1,-2) = 2$) when facing the direction of increasing x, and $f_y(1,-2) = 1$ being positive means the graph slopes upward at $(1,-2,-2)$ when facing the direction of increasing y. (We'll talk about how to find the slopes in *other* directions soon enough!)

Example. Consider $f(x,y) = xe^{xy}$. To compute f_x we think of our function as

$$xe^{x(\text{constant})}.$$

To differentiate this we need the product rule and then the (single-variable) chain rule:

$$\frac{d(xe^{x(\text{constant})})}{dx} = e^{x(\text{constant})} + xe^{x(\text{constant})}(\text{constant}).$$

Thus we get

$$f_x = \frac{\partial f}{\partial x} = e^{xy} + xe^{xy}y.$$

(To be clear, the final y at the end comes from differentiating the exponent xy of e^{xy} with respect to x.) For f_y we think of our function as

$$(\text{constant})e^{(\text{constant})y}.$$

No produce rule is needed now, just the chain rule:

$$\frac{d((\text{constant})e^{(\text{constant})y})}{dy} = (\text{constant})e^{(\text{constant})y}(\text{constant}).$$

Thus

$$f_y = \frac{\partial f}{\partial y} = xe^{xy}x = x^2e^{xy}.$$

At, say, the point $(2,4)$, we'd have

$$f_x(2,4) = e^8 + 8e^8 = 9e^8 \text{ and } f_y(2,4) = 4e^8,$$

so the graph of f slopes upward in both the x- and y-directions at the point $(2,4,2e^8)$ (but it is steeper in the x-direction than in the y-direction!), where $2e^8 = f(2,4)$ is the function value.

Second-order partial derivatives. There is no reason why we cannot compute partial derivatives of partial derivatives themselves, and this gives what are called the *second-order* partial derivatives of a function. Take again $f(x,y) = xe^{xy}$, whose *first-order* partial derivatives we computed before:

$$f_x = \frac{\partial f}{\partial x} = e^{xy} + xye^{xy} = (1+xy)e^{xy} \quad \text{and} \quad f_y = \frac{\partial f}{\partial y} = x^2e^{xy}.$$

We can now differentiate $f_x = \frac{\partial f}{\partial x} = (1+xy)e^{xy}$ either with respect to x or with respect to y:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^{xy} + (1+xy)e^{xy}y \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = xe^{xy} + (1+xy)e^{xy}x.$$

To be clear, in the first case we are applying the operation $\frac{\partial}{\partial x}$ of differentiation with respect to x to the function $\frac{\partial f}{\partial x}$, and in the second case we are applying the operation $\frac{\partial}{\partial y}$ of differentiation with respect to y to the function $\frac{\partial f}{\partial x}$. These are more commonly denoted by

$$\frac{\partial^2 f}{\partial x^2} = ye^{xy} + (1+xy)e^{xy}y \quad \text{and} \quad \frac{\partial^2 f}{\partial y\,\partial x} = xe^{xy} + (1+xy)e^{xy}x$$

respectively. In subscript notation, these are

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \quad \text{and} \quad f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x}.$$

Note that in subscript notation, we differentiate in the order written from left to right, while in the ∂ notation we differentiate in the order written from right to left: $\frac{\partial^2 f}{\partial y\,\partial x}$ means we differentiate with respect to x and then y (doing it in this order comes from thinking of this as applying $\frac{\partial}{\partial y}$ to $\frac{\partial f}{\partial x}$), while this would be denoted by f_{xy} in subscript form since x comes first and then y.

But we are not done, as we could also take $f_y = \frac{\partial f}{\partial y} = x^2e^{xy}$ and differentiate it with respect to either x or y. This gives

$$f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = 2xe^{xy} + x^2e^{xy}y \quad \text{and} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = x^2e^{xy}x.$$

Thus, $f(x,y) = xe^{xy}$ has four second-order partial derivatives:

$$f_{xx} = [y + (1+xy)y]e^{xy}, \; f_{xy} = [x + x(1+xy)]e^{xy}, \; f_{yx} = [2x + x^2y]e^{xy}, \; f_{yy} = x^3e^{xy}.$$

Note that, actually, after some algebraic rewriting, f_{xy} and f_{yx} above are exactly the same; this is no accident, as we'll clarify next time!

Lecture 19: Chain Rule

Warm-Up 1. We find the slope of the tangent line to the piece of the surface

$$z = x^3y^4 + y\sin(xy)$$

that lies in the plane $y = \frac{\pi}{2}$ at $(1, \frac{\pi}{2}, \frac{\pi^4}{16} + \frac{\pi}{2})$ (or rather the point on graph corresponding to this point), and the slope of the tangent line to the piece that lies in the plane $x = 1$ at this same point. The given surface is the graph of $f(x, y) = x^3y^4 + y\sin(xy)$, so the desired slopes are simply the partial derivatives of this function at $(1, \frac{\pi}{2})$. (Note that the z-coordinate $\frac{\pi^4}{16} + \frac{\pi}{2}$ of the given point is just the value of $f(x, y)$ at $(1, \frac{\pi}{2})$.) We have

$$\frac{\partial f}{\partial x} = 3x^2y^4 + y^2\cos(xy) \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x^3y^3 + \sin(xy) + xy\cos(xy).$$

Hence the slope of the tangent line to the piece of the given graph in the $y = \frac{\pi}{2}$ plane (note that on this plane we hold y constant and only vary x, which is why it is the partial derivative with respect to x and not y that gives the desired slope) at the given point is

$$\frac{\partial f}{\partial x}(1, \tfrac{\pi}{2}) = \frac{3\pi^4}{16}$$

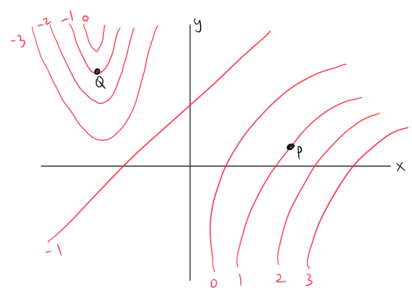
and the slope of the tangent line to the piece in the $x = 1$ plane (x constant and y varies) at the given point is

$$\frac{\partial f}{\partial y}(1, \tfrac{\pi}{2}) = \frac{4\pi^3}{8} + 1.$$

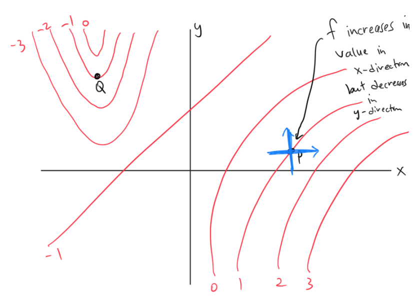
Note that since both of these are positive, the graph of $f(x, y)$ tilts upward at the given point in both the x- and y-directions.

Warm-Up 2. Given the following drawing of some level curves of a function $f(x, y)$ whose partial derivatives exist at all points, we determine the signs (i.e., are they positive, negative, or zero) of

$$f_x(P),\ f_y(P),\ f_x(Q),\ \text{and } f_y(Q).$$

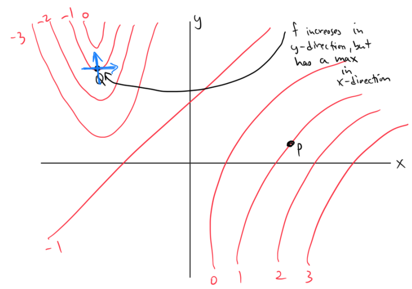


At P, if we hold the y-coordinate constant and only vary the x-coordinate, we see that the values of f move from being between 0 and 1 to the left to P, to 1 at P, to between 1 and 2 to the right of P:

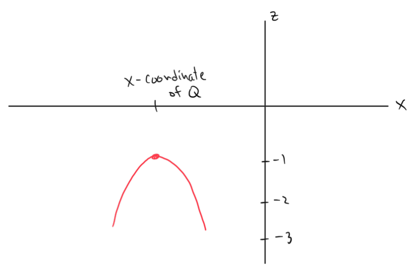


Thus f is increasing in value with respect to x at P, so $f_x(P)$ should be positive. If we instead fix the x-coordinate and vary y, the values of f decrease as we move vertically through P, so $f_y(P) < 0$.

Now, if we fix the x-coordinate of Q and vary y, the values of $f(x, y)$ increase through Q since get less negative: from being between -2 and -1 below Q, to -1 at Q, to between -1 and 0 above Q. Thus $f_y(Q) > 0$. With respect to x, however, Q is actually sitting at a local maximum since the values of $f(x, y)$ are smaller (more negative) than -1 to the left of Q *and* to the right of Q, while the value at Q is -1:



If we draw a picture of only z versus x (so ignore the y-direction for now), we'd get something like



which shows the local maximum behavior in the x-direction. At a local maximum the derivative should be zero, so $f_x(Q) = 0$.

Clairaut's theorem. We saw an example last time that f_{xy} and f_{yx} happened to give the same value. Here's another example of this. The first-order partial derivatives of $f(x, y) = x^2y^3 + xy^2$ are

$$\frac{\partial f}{\partial x} = 2xy^3 + y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + 2xy.$$

The second-order partial derivatives of f obtained by differentiating $\frac{\partial f}{\partial x}$ are

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2y^3 \quad \text{and} \quad f_{xy} = \frac{\partial^2 f}{\partial y\, \partial x} = 6xy^2 + 2y$$

and those obtained by differentiating $\frac{\partial f}{\partial y}$ are

$$f_{yx} = \frac{\partial^2 f}{\partial x\, \partial y} = 6xy^2 + 2y \quad \text{and} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6x^2y + 2x.$$

Indeed, we have $f_{xy} = f_{yx}$ in this example.

Second-order partial derivatives which involve differentiating with respect to different variables like this are called *mixed* second-order partial derivatives, and it is basic fact that these will agree under a continuity assumption. Specifically, what is known as *Clairaut's theorem* states that if the mixed second-order partial derivatives $\frac{\partial^2 f}{\partial y\,\partial x}$ and $\frac{\partial^2 f}{\partial x\,\partial y}$ are continuous (as they are in the examples we've seen), then

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}.$$

So, even though there are ostensibly four second-order partial derivatives for a function of two variables, two of them agree and there are only three distinct ones, assuming the continuity assumption in Clairaut's theorem is satisfied. This cuts down on the number of second-order partial derivatives one needs to actually compute.

We will use second-order partial derivatives soon enough to discuss linear and quadratic approximations to functions, but apart from this Clairaut's theorem will not really play a role this quarter, so we won't say much more about it. It will have some important consequences, however, for integration if you go on to take MATH 230-2. One could also ask how to interpret second-order partial derivatives geometrically. In the single-variable case second derivatives can be used to discuss concavity, and at least f_{xx} and f_{yy} have the same interpretation in the two variable case; the mixed second-order derivatives, however, require a bit more care to interpret geometrically. We'll save all such discussions to a written homework since they won't play a big role for us.

Three variable example. Everything we've done (including Clairaut's theorem) works just as well for a function of three variables. Take for example $g(x,y,z) = xe^{yz} + z$. This has three first-order partial derivatives, each obtained by holding two variables constant and differentiating with respect to the third:

$$\frac{\partial g}{\partial x} = e^{yz},\ \frac{\partial g}{\partial y} = xze^{yz},\ \frac{\partial g}{\partial z} = xye^{yz} + 1.$$

Now we can differentiate each of these in one of three ways, and get at first glance nine total second-order partial derivatives. But, many of these will be the same because of Clairaut's theorem, so we actually only get six distinct things. For example, we have

$$\frac{\partial^2 g}{\partial z\,\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x}\right) = ye^{yz} \quad \text{and} \quad \frac{\partial^2 g}{\partial x\,\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial z}\right) = ye^{yz},$$

which are the same. Also,

$$g_{yz} = xe^{yz} + xyze^{yz} \text{ is the same as } g_{zy} = xe^{yz} + xyze^{yz}.$$

The second-order partial derivative of g with respect to z twice has no "mixed" counterpart, and is

$$g_{zz} = \frac{\partial^2 g}{\partial z^2} = xy^2e^{yz}.$$

Multivariable chain rule. Suppose $f(x,y) = x\sin(xy)$, but that now x and y themselves are also functions that depend on some new variables u and v via

$$x = uv \quad \text{and} \quad y = 2u + u^2v.$$

Substituting these values into $f(x,y)$ will give an expression for f that depends on u and v:

$$f(x(u,v), y(u,v)) = uv\sin(uv[2u + u^2v]).$$

Our goal is to determine how f changes with respect to these new variables, or in other words to compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$. One approach is to take the expression we have for f above in terms of u

and v and differentiate it directly as we've been doing. This works fine in this example, although it will be a little messy since the expression for f in terms of u and v is not so nice. But, we will come across scenarios where such a direct substitution method is not going to work out, so we need something new.

What need is a multivariable version of the *chain rule*, since we can view the process above as *composing* $f(x, y)$ with the functions $x(u, v)$ and $y(u, v)$ for x, y in terms of u, v. Let us first recall the single-variable chain rule, and formulate in a way similar to that above where we treat a given variable as a function itself. The single-variable chain rule says that

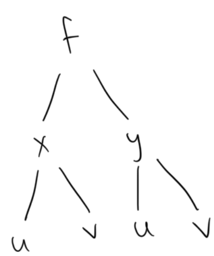
$$(g(f(x))' = g'(f(x))f'(x).$$

Think of $g(t)$ as a function of the variable t. Then in the composition $g(f(x))$ we set this variable $t = f(x)$ to itself be a function of x. The derivative of g with respect to the "new" variable x is

$$\frac{dg}{dx} = \frac{dg}{dt}\frac{dt}{dx}, \text{ where } \frac{dg}{dt} = g'(f(x)) \text{ and } \frac{dt}{dx} = f'(x).$$

The point is that in order to differentiate g with respect to the new variable x, we differentiate g with respect to the "intermediate" variable t, and multiply the result by the derivative of this intermediate variable t with respect to the new variable x. As x varies, the value of t will vary, which causes the value of g to vary, and so the rate at which g changes with respect to x should indeed depend on both the rate at which g varies with respect to t and the rate at which t varies with respect to x, which is what the chain rule $\frac{dg}{dx} = \frac{dg}{dt}\frac{dt}{dx}$ says.

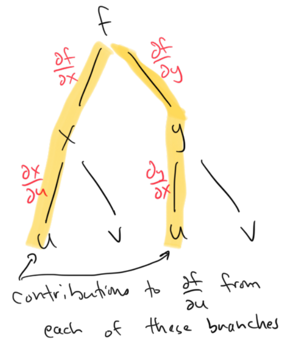
To get the correct expression for the multivariable chain rule, let us be clear about what depends on what: here f depends on x and y, and each of x and y depend on u and v, which we summarize in the diagram



To determine how the value of f changes with respect to u, note that changing the value of u changes the value of x *and* y, and each of these changes in turn changes the value of f. So, we should expect contributions to $\frac{\partial f}{\partial u}$ from both the resulting change in x and the resulting change in y; the correct statement is that

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}.$$

Each term on the right comes from one of the ways in which depends on u: the first comes from f depending on u through the "intermediate" variable x, and the second from f depending on u through the "intermediate" variable y. All we do is look at all the ways to get from f at the top of the diagram to u at the bottom, take a products of derivatives along each "branch" that occurs, and then add all such contributions together:



For $\frac{\partial f}{\partial v}$ we'd have one contribution through the "x" branch which looks like $\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}$, and another from through the "y" branch which looks like $\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$, so that $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$. The same idea works no matter how many variables and dependencies we have.

Example. For $f(x, y) = x\sin(xy)$ where

$$x = uv \quad \text{and} \quad y = 2u + u^2 v,$$

we thus have

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ &= [\sin(xy) + xy\cos(xy)]v + x^2\cos(xy)[2 + 2uv]\end{aligned}$$

and

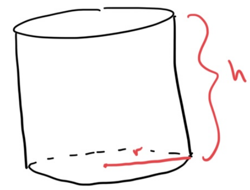
$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \\ &= [\sin(xy) + xy\cos(xy)]u + x^2\cos(xy)u^2.\end{aligned}$$

If we want to evaluate these partial derivatives specifically at, say, the values $(u, v) = (1, 2)$ of the "new" variables, we just note that at these values we have $x = uv = 2$ and $y = 2u + u^2 v = 6$, so that we can just evaluate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ at $u = 1, v = 2, x = 2, y = 6$:

$$\begin{aligned}\frac{\partial f}{\partial u}(1,2) &= \frac{\partial f}{\partial x}(2,6)\frac{\partial x}{\partial u}(1,2) + \frac{\partial f}{\partial y}(2,6)\frac{\partial y}{\partial u}(1,2) \\ &= [\sin(12) + 12\cos(12)]2 + 4\cos(12)[6] \\ &= 2\sin(12) + 48\cos(12) \\ \frac{\partial f}{\partial v}(1,2) &= \frac{\partial f}{\partial x}(2,6)\frac{\partial x}{\partial v}(1,2) + \frac{\partial f}{\partial y}(2,6)\frac{\partial y}{\partial v}(1,2) \\ &= [\sin(12) + 12\cos(12)] + 4\cos(12) \\ &= \sin(12) + 16\cos(12).\end{aligned}$$

Lecture 20: Directional Derivatives

Warm-Up 1. Suppose a cylindrical wax candle is melting. As it does, its height and radius change with respect to time. (It gets shorter as time goes on and perhaps the melting was falls down the sides and increases the radius.)



We want to give an expression for the rate at which the volume of the candle changes with respect time in terms of the change in the radius and the change in the height.

The base of the candle has area πr^2, so the volume of the candle, which is area of the base times the height, is

$$V = \pi r^2 h.$$

But now we have that r and h depend on time t, so we have the dependencies

The chain rule thus gives

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \\ &= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt},\end{aligned}$$

which is our desired expression. (Note that we here use the convention that single-variable derivatives are written using $\frac{d}{dt}$ notation so that ∂ is only used when working with partial derivatives. Here r and h are single-variable functions of t alone, which is why we use $\frac{dr}{dt}, \frac{dh}{dt}$ instead of $\frac{\partial r}{\partial t}, \frac{\partial h}{\partial t}$, but there is no harm in writing this as

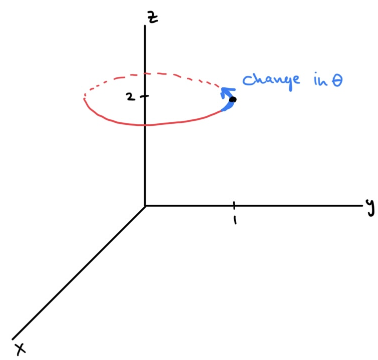
$$\frac{\partial V}{\partial t} = 2\pi r h \frac{\partial r}{\partial t} + \pi r^2 \frac{\partial h}{\partial t}$$

instead; it's just a matter of preference. In the end we are thinking of V as a single-variable function of t alone, which is why we also use $\frac{dV}{dt}$.) If we were given information about some of these actual values, say the numerical values of the radius and height at some given instance and the rate at which they are changing, we can find numerical values of $\frac{dV}{dt}$ as well.

Warm-Up 2. Suppose the temperature at each point in $\mathbb{R}^3$ is given by a function $T(x, y, z)$, and that specifically at $(0, 1, 2)$ we know that

$$T_x(0, 1, 2) = 1, \; T_y(0, 1, 2) = 2, \text{ and } T_z(0, 1, 2) = -3.$$

We determine the rate at which the temperature T is changing at $(0, 1, 2)$ with respect to the polar variable θ. As θ changes we are moving in an angular direction through $(0, 1, 2)$, and such a change in θ causes in change in x and y (not z!), so that T should change as well:

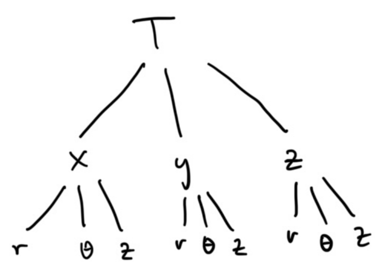


To be clear, we think of our new variables as (r, θ, z), where r, θ are the usual polar coordinates replacing x, y:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z \text{ remains } z.$$

(These are what are called *cylindrical coordinates* for $\mathbb{R}^3$, which you'll learn more about in MATH 230-2 when computing certain three-variable integrals. This is the only example we'll do with such coordinates in this class.) Note that we do not know what $T(x, y, z)$ actually is—we only know the rate at which T changes in the x-, y-, and z-directions, but in fact this is enough to find the change with respect to θ.

Here are our dependencies:



The multivariable chain rule thus gives

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial \theta},$$

one term for each branch leading to θ in our diagram. Using the polar expressions for x and y (and noting that z is independent of *theta*), we have

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x}(-r\sin\theta) + \frac{\partial T}{\partial y}(r\cos\theta) + \frac{\partial T}{\partial z}(0) = -r\sin\theta\frac{\partial T}{\partial x} + r\cos\theta\frac{\partial T}{\partial y}.$$

Now, at the point $(0, 1, 2)$, we have $r = 1$ and $\theta = \frac{\pi}{2}$, while z remains 2. Indeed, $(r, \theta) = (1, \frac{\pi}{2}0$ are the polar values that give $(x, y) = (0, 1)$, which are the proper x and y coordinates of $(0, 1, 2)$. Thus, to get the rate of change in T with respect to θ at this $(0, 1, 2)$, we evaluate at these values:

$$\begin{aligned}\frac{\partial T}{\partial \theta}\underbrace{(1, \tfrac{\pi}{2}, 2)}_{(r,\theta,z)} &= (-1\sin\tfrac{\pi}{2})\frac{\partial T}{\partial x}\underbrace{(0, 1, 2)}_{(x,y,z)} + (1\cos\tfrac{\pi}{2})\frac{T}{y}\underbrace{(0, 1, 2)}_{(x,y,z)} \\ &= (-1)(1) + (0)(2)\end{aligned}$$

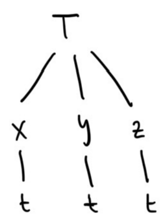
$$= -1,$$

where the values of $T_x(0,1,2)$ and $T_y(0,1,2)$ come from our initial setup. (The value of $T_z(0,1,2) = -3$ was not needed since this ended up being multiplied $\frac{\partial z}{\partial \theta} = 0$.) Thus, when moving in the angular direction through $(0,1,2)$ the temperature would be decreasing.

Differentiating along a path. Take the same setup temperature setup as above, but now assume we move through $(0,1,2)$ along the curve parametrized by

$$\mathbf{r}(t) = \left\langle \cos(\tfrac{\pi}{4}t), \sin(\tfrac{\pi}{4}t), t \right\rangle.$$

This describes a helix wrapping around the cylinder $x^2 + y^2 = 1$, and we want to determine the rate at which the temperature is changing at $(0,1,2)$ but now with respect to motion along this helix, or in other words with respect to the "time" parameter t on this curve. Our dependencies are now



where $x = \cos(\frac{\pi}{4}t), y = \sin(\frac{\pi}{4}t), z = t$ from the given parametrization, so the chain rule gives

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial T}{\partial x}(-\tfrac{\pi}{4}\sin(\tfrac{\pi}{4}t)) + \frac{\partial T}{\partial y}(\tfrac{\pi}{4}\cos(\tfrac{\pi}{4}t)) + \frac{\partial T}{\partial z}(1).\end{aligned}$$

The point $(0,1,2)$ we want occurs along the curve at time $t = 2$ (since then $x = \cos(\frac{\pi}{4}\cdot 2) = 0, y = \sin(\frac{\pi}{4}\cdot 2) = 1$, and $z = 2$), so the value of $\frac{dT}{dt}$ at our point is

$$\begin{aligned}\frac{dT}{dt}\bigg|_{t=2} &= T_x(0,1,2)(-\tfrac{\pi}{4}\sin\tfrac{\pi}{2}) + T_y(0,1,2)(\tfrac{\pi}{4}\cos\tfrac{\pi}{2}) + T_z(0,1,2) \\ &= (1)(-\tfrac{\pi}{4}) + (2)(0) + (-3) = -\tfrac{\pi}{4} - 3.\end{aligned}$$

Thus when moving along this path specifically, the temperature at $(0,1,2)$ would be decreasing at a rate of $\frac{\pi}{4} + 3$ (decrease because the derivative was negative).

Let us note that we can write the result of the chain rule above a little more compactly by interpreting the right side as a dot product:

$$\frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$$

The second vector $\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$ is just the tangent vector $\mathbf{r}'(t)$ along this path, and the first vector

$$\left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle$$

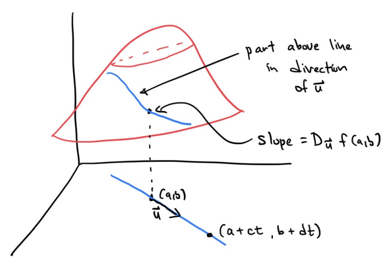
is important enough that we give it its own name: this is called the *gradient vector* of T and is denoted by ∇T. (The ∇ symbol is pronounced "nabla". We'll see why we use the term "gradient" for this vector soon enough.) The upshot is that when evaluating a function such as T only among points on a path with parametrization $\mathbf{r}(t)$, the derivative of T with respect to the path parameter t can be written as

$$\frac{d}{dt}T(\mathbf{r}(t)) = \nabla T(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Notice how, in a sense, this expression does look like "derivative of the outside" times "derivative of the inside", which are phrases you might have seen used for the single-variable chain rule.

Directional derivatives. Let us put the idea above to good use. Given a function, say for now of two variables $f(x, y)$, we have seen that $\frac{\partial f}{\partial x}$ gives the rate of change of f in the x-direction, or more precisely in the direction of the vector $\mathbf{i}$, and $\frac{\partial f}{\partial y}$ gives the rate of change of f in the y-direction, or more precisely direction of $\mathbf{j}$. That is, if we stand a point on the graph and face in one of these two directions, these partial derivatives give the slope of the part of the graph we see in that direction. But there are numerous other directions in which we might want to know the slope/rate of change, say the direction of $\mathbf{i}+\mathbf{j}$, or $\mathbf{i}-\mathbf{j}$, or whatever. How do we compute these?

To be precise, let us take the point (a, b) and a unit vector $\mathbf{u} = \langle c, d\rangle$ giving us the direction we care about. (We will see next time we require that $\mathbf{u}$ be a unit vector here.) By looking at the rate of change of f at (a, b) in the direction of $\mathbf{u}$, we mean to look at how f changes when evaluated among points strictly from the line passing through (a, b) in the direction of $\mathbf{u}$:



But we know how to describe this line explicitly, namely it has parametric equations

$$x = a + ct, \ \ y = b + dt,$$

so we are looking at the rate at which the values $f(a+ct, b+dt)$ are changing at (a, b)—or in other words at $t = 0$ since this is the value of t that gives the point (a, b)—since these are precisely the values of f along this line. We define the *directional derivative* of f at (a, b) in the direction of the unit vector $\mathbf{u}$ to be exactly the rate of change of these values $f(a + ct, b + dt)$ at $t = 0$, and we denote it by

$$D_{\mathbf{u}} f(a, b) = \frac{d}{dt}\Big|_{t=0} f(a + ct, b + dt).$$

This is precisely the same setup as the previous example where we differentiated along a curve, only that here the curve is a line parametrized by $\mathbf{r}(t) = \langle a + ct, b + dt\rangle$. According to the chain rule work we did previously, we can write this directional derivative as

$$D_u f(a, b) = \frac{d}{dt}\Big|_{t=0} f(a + ct, b + dt) = \nabla f(a, b) \cdot \mathbf{r}'(t) = \nabla f(a, b) \cdot \mathbf{u},$$

where the tangent vector of $\mathbf{r}(t) = \langle a + ct, b + dt\rangle$ is $\mathbf{r}'(t) = \langle c, d\rangle = \mathbf{u}$.

The upshot is that to compute a directional derivative all we have to do is take the gradient of our function (i.e., the vector whose components are the partial derivatives of our function) at the point we care about and dot it with the unit direction vector we care about. Note in particular that

$$D_{\mathbf{i}} f(a,b) = \nabla f(a,b) \cdot \mathbf{i} = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle 1, 0 \rangle = f_x(a,b)$$

and

$$D_{\mathbf{j}} f(a,b) = \nabla f(a,b) \cdot \mathbf{j} = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle 0, 1 \rangle = f_y(a,b),$$

so the directional derivatives in the directions of $\mathbf{i}$ or $\mathbf{j}$ (i.e., the slopes in x- and y-directions) are indeed just the usual partial derivatives.

Example. We compute the directional derivative of $f(x,y) = x^2y + y^2$ at $(1,2)$ in the direction of $\mathbf{i}+\mathbf{j}$. Note that $\mathbf{i}+\mathbf{j} = \langle 1,1 \rangle$ is not a unit vector, so before we can apply our directional derivative formula we must divide by its length to get a vector of length 1 in the direction we want:

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

The desired directional derivative is

$$D_{\mathbf{u}} f(1,2) = \nabla f(1,2) \cdot \mathbf{u}.$$

The gradient vector of f is

$$\nabla f = \langle f_x, f_y \rangle = \left\langle 2xy, x^2 + 2y \right\rangle,$$

so the gradient vector at $(1,2)$ specifically is

$$\nabla f(1,2) = \langle 4, 5 \rangle.$$

Thus the directional derivative we want is

$$D_{\mathbf{u}} f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \langle 4,5 \rangle \cdot \left\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \right\rangle = \tfrac{4}{\sqrt{2}} + \tfrac{5}{\sqrt{2}} = \tfrac{9}{\sqrt{2}}.$$

Hence at the point $(1,2)$ the function $f(x,y) = x^2y + y^2$ is changing at a rate of $\frac{9}{\sqrt{2}}$ when facing the direction of $\mathbf{i}+\mathbf{j}$. This is positive, so the graph at this point in this direction is sloping upward.

Instead if we want the directional derivative in the direction of, say, $2\mathbf{i} - \mathbf{j}$, we take

$$\mathbf{v} = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}$$

as our unit direction vector. The gradient vector of f at $(1,2)$ is still $\langle 4,5 \rangle$, so the directional derivative of f at $(1,2)$ in the direction of $\mathbf{v}$ is

$$D_{\mathbf{v}} f(1,2) = \nabla f(1,2) \cdot \mathbf{v} = \langle 4,5 \rangle \cdot \left\langle \tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}} \right\rangle = \tfrac{8}{\sqrt{5}} - \tfrac{5}{\sqrt{5}} = \tfrac{3}{\sqrt{5}}.$$

Thus the graph at this point in this direction is still sloping upward, but note that it is not sloping upward as steeply as it is in the direction of $\mathbf{i}+\mathbf{j}$ because this new rate of change is less positive than was the previous one.

Lecture 21: Gradient Vectors

Warm-Up. Set $f(x,y,z) = x^2y - y^3z + xz^2$ and $\mathbf{v} = \mathbf{i}+\mathbf{j}+\mathbf{k}$. We determine whether f is changing more rapidly in the direction of $\mathbf{v}$ at the point $(1,1,1)$ or at the point $(1,-3,0)$. We have

$$\nabla f = \left\langle 2xy + z^2, x^2 - 3y^2z, -y^3 + 2xz \right\rangle,$$

so $\nabla f(1,1,1) = \langle 3,-2,1\rangle$ and $\nabla f(1,-3,0) = \langle -6,1,27\rangle$. We divide $\mathbf{v}$ by its length to get a unit vector in the desired direction:

$$\mathbf{u} = \tfrac{1}{\sqrt{3}}\,\mathbf{i} + \tfrac{1}{\sqrt{3}}\,\mathbf{k} + \tfrac{1}{\sqrt{3}}\,\mathbf{k} = \left\langle \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \right\rangle.$$

Thus at the point $(1,1,1)$, f is changing at a rate of

$$D_{\mathbf{u}}f(1,1,1) = \nabla f(1,1,1)\cdot \mathbf{u} = \langle 3,-2,1\rangle \cdot \left\langle \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \right\rangle = \tfrac{2}{\sqrt{3}}$$

in the direction of $\mathbf{u}$, and at $(1,-3,0)$ the rate of change of f in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}}f(1,-3,0) = \nabla f(1,-3,0)\cdot \mathbf{u} = \langle -6,1,27\rangle \cdot \left\langle \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \right\rangle = \tfrac{22}{\sqrt{3}}.$$

Since $D_uf(1,-3,0) > D_{\mathbf{u}}f(1,1,1)$, f is changing more rapidly at $(1,-3,0)$ in the direction of $\mathbf{v}$ than it is at $(1,1,1)$. Note that both of these directional derivatives are positive, so f is increasing in this direction at both points, it's just that the rate of increase is even more positive at $(1,-3,0)$ than at $(1,1,1)$.

Geometric interpretations of gradients. Take $f(x,y) = x^2y + y^3$. We ask whether there is a direction in which the directional derivative of f at $(1,2)$ is, say, 20, or in other words whether there is a direction in which the slope of the graph of $f(x,y)$ at $(1,2,10)$ is 20. Algebraically, this comes down to determining whether there is a unit vector $\mathbf{u}$ such that

$$D_{\mathbf{u}}f(1,2) = \nabla f(1,2)\cdot \mathbf{u} = 20.$$

But to answer this what we really need to know is how large the value of $D_{\mathbf{u}}f(1,2)$ can be among *all* possible directions $\mathbf{u}$. That is, if we stand on the graph of $f(x,y)$ at $(1,2,10)$ and look all around, what is the steepest slope we'll see?

We have $\nabla f = \left\langle 2xy, x^2 + 3y^2\right\rangle$, so $\nabla f(1,2) = \langle 4,13\rangle$. If we take $\mathbf{u} = \langle c,d\rangle$ as our unit vector, thus assuming $c^2 + d^2 = 1$, then

$$D_{\mathbf{u}}f(1,2) = \nabla f(1,2)\cdot \mathbf{u} = \langle 4,13\rangle \cdot \langle c,d\rangle = 4c + 13d,$$

so we are looking to maximize $4c + 13d$ among points (c,d) satisfy $c^2 + d^2 = 1$. This is actually a type of optimization problem we will return to at the end of the quarter, but in this particular case we have a simple approach based on the geometric properties of dot products we saw at the beginning of the quarter. We have

$$D_{\mathbf{u}}f(1,2) = \nabla f(1,2)\cdot \mathbf{u} = |\nabla f(1,2)||\mathbf{u}|\cos\theta = |\nabla f(1,2)|\cos\theta$$

where θ is the angle between $\nabla f(1,2)$ and $\mathbf{u}$, and where we use the fact that $\mathbf{u}$ is assumed to be a unit vector to say that $|\mathbf{u}| = 1$. (*This* is why we restrict our attention to only unit vectors when computing directional derivatives, so that we can guarantee the rate of change of f in a given direction at a point depends only the function, the point we are at, and the direction as measured by an angle, but not on whether we use a vector of length 2 to describe that direction, or of length 5, or of length whatever.)

The value of $D_{\mathbf{u}}f(x, y) = |\nabla f(x, y)| \cos\theta$ is thus as large as possible when $\cos\theta$ is as large as possible, which happens when $\cos\theta = 1$ and hence when the angle θ between $\mathbf{u}$ and $\nabla f(x, y)$ is zero. But this means that $\nabla f(x, y)$ and $\mathbf{u}$ point in the same direction, so the first conclusion is that the direction of $\nabla f(x, y)$ itself *is* precisely the direction in which f increase most rapidly at (x, y). The second conclusion is that this most rapid rate of increase, or in other words the largest value that $D_{\mathbf{u}}f(x, y)$ can have among all possible direction, is

$$D_{\text{direction of } \nabla f(x,y)} f(x, y) = |\nabla f(x, y)| \cos(0) = |\nabla f(x, y)|,$$

so the length of the gradient at a point is itself that most rapid rate of increase. To summarize:

- $\nabla f(P)$ points in the direction in which f increases most rapidly at P, and
- $|\nabla f(P)|$ gives that most rapid rate of increase.

Thus, the gradient of $f(x, y)$ is not just some random vector made up of the partial derivatives, but it encodes much important geometric information as well via its direction and length. This interpretation is where the term "gradient" comes from, since gradients in general are things that describe something that changes, such as a color gradient or an elevation gradient on a mountain.

In the example above, we have that the largest possible rate of change of $f(x, y) = x^2y + y^3$ at $(1, 2)$ among all directions is $|\nabla f(1, 2)| = |\langle 4, 13\rangle| = \sqrt{173}$. Since 20 is larger than $\sqrt{173}$, we conclude that there is no direction in which the value of $D_{\mathbf{u}}f(1, 2)$ is 20. Moreover, the direction in which this maximum rate of change occurs is that of $\nabla f(1, 2) = \langle 4, 13\rangle$ itself.

Example. Suppose $T(x, y) = xe^{xy}$ gives the temperature at a point (x, y) in a lake, and that we are at the point $(2, 5)$. The direction in which the temperature increases most rapidly at $(2, 5)$ is $\nabla T(2, 5)$. We have

$$\nabla T = \left\langle e^{xy} + xye^{xy}, x^2e^{xy} \right\rangle, \text{ so } \nabla T(2, 5) = \left\langle 11e^{10}, 4e^{10} \right\rangle,$$

so the max rate of change of the temperature occurs in the direction of $\left\langle 11e^{10}, 4e^{10} \right\rangle$. The rate of change in this direction, or in the words the maximum rate of change at $(2, 5)$ itself, is

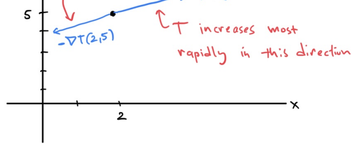
$$|\nabla T(2, 5)| = \left| \left\langle 11e^{10}, 4e^{10} \right\rangle \right| = \sqrt{121e^{20} + 16e^{20}} = e^{10}\sqrt{137}.$$

What about the direction in which the temperature *decreases* most rapidly at $(2, 5)$? This is the direction in which $D_{\mathbf{u}}\nabla T(2, 5)$ is as small as possible, and from

$$D_{\mathbf{u}}\nabla T(2, 5) = \nabla T(2, 5) \cdot \mathbf{u} = |\nabla T(2, 5)| \cos\theta,$$

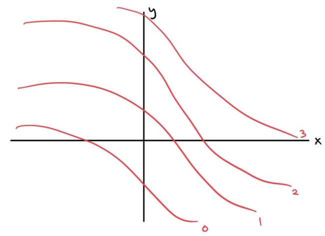
we see that this happens when $\cos\theta = -1$, so that $\theta = \pi$. Hence the direction in which the temperature decreases most rapidly is the direction opposite that of the gradient (since the angle between this direction and the gradient must be π), and so is

$$-\nabla T(2, 5) = \left\langle -11e^{10}, -4e^{10} \right\rangle.$$

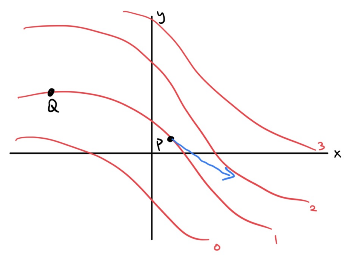


The rate of change in this direction, which is the smallest that $D_{\mathbf{u}}T(2,5)$ can be among all possible directions, is $|\nabla T(2,5)|\cos(\pi) = -|\nabla T(2,5)| = -e^{10}\sqrt{137}$.

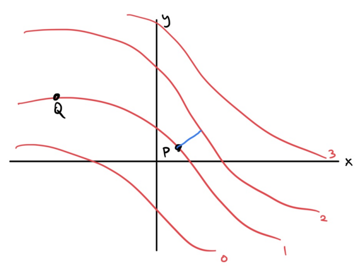
Another example. Suppose we are given the following level curves of a function $f(x, y)$:



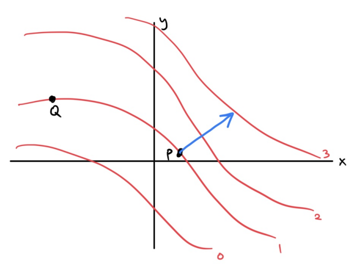
We want to (roughly) sketch the gradient vectors of f at P and at Q. The first thing to note is that $\nabla f(P)$ cannot point, say, down and to the left since in this direction f is decreasing in value (which we can see from the z-values on the level curves occurring in this direction), while we know $\nabla f(P)$ must be point in a direction in which f is increasing in value. Moreover, we cannot have the gradient of f at P look something like



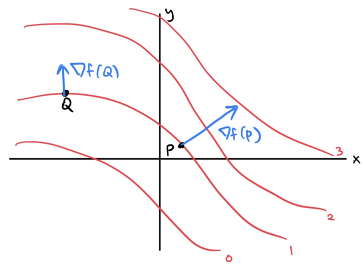
since, although f is increasing in this direction, it is not increasing by as much as it is in, say, the direction



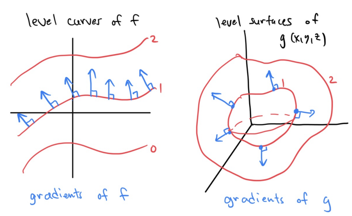
We can see that this is true by noting that in the first direction it takes f *longer* to increase by a value of 1 than it does in the second, since the level curves are closer to one another in latter direction than in the former. The closer level curves are to one another, the steeper the graph will be at those points. Thus $\nabla f(P)$ looks something like



We can draw $\nabla f(Q)$ in a similar way, where this gradient vector should point at Q towards the level curve at 3. But we can see a bit more, namely how long this gradient is versus the one at P. The lengths of these gradients give the slopes in the directions in which the graph is steepest at the given points, so we can determine which gradient is longer by determining where the graph is steeper. The fact that the level curves at P are closer to one another implies that the graph here is steeper than it is as Q, so the gradient at P should be longer than the gradient at Q:



Relation to level sets. There is one final important geometric observation we can make about the gradient. Note what happens if we ask for the directional derivative in a direction *tangent* to a level curve:



In this tangent direction $\mathbf{u}$, the value of $f(x, y)$ remains constant since moving along the level curve does not change the value of $f(x, y)$. (The level curve is, after all, the set of points at which f has one specific value.) Since f remains constant in this direction, the rate of change of f in this direction should be zero, so

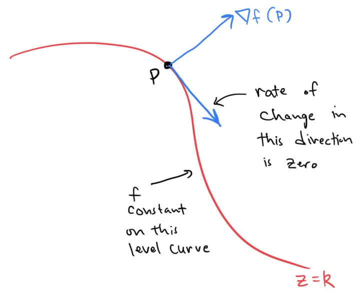
$$0 = D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u}.$$

But this means that $\nabla f(P)$ is orthogonal to the tangent direction $\mathbf{u}$. Another way saying this is that if we parametrize the level curve using some $\mathbf{r}(t)$, then $f(\mathbf{r}(t))$ is constant as t varies, so the derivative of this function which respect to t is 0. But we can write this derivative as

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

as we saw when discussing the idea of "differentiating along a curve" before, so $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$ and hence $\nabla f(P)$ is orthogonal to the tangent vector $\mathbf{r}'(t)$.

Everything works just as well for functions of more than two variables, so the upshot is at any point P, $\nabla f(P)$ will always be orthogonal to the level set of f containing P:



This helps to draw more accurate pictures of gradients, but also has another important use, as we will see next time.

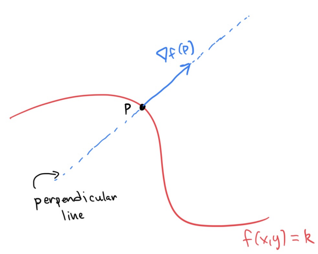
Final example. We find parametric equations for the line which is perpendicular to the curve

$$x^2y + y^3 = 10$$

at the point $(1, 2)$. To find this line we need a point on it, such as $(1, 2)$, and a vector giving the direction of the line, so in this case we need a vector that will be perpendicular to the curve at $(1, 2)$. But if we think of the given curve as a level curve of

$$f(x, y) = x^2y + y^3,$$

namely the level curve at the value $z = 10$, then we know that $\nabla f(1, 2)$ will indeed be orthogonal to this curve at $(1, 2)$. From a previous example we have $\nabla f(1, 2) = \langle 4, 13\rangle$, so we take this as a direction vector for the perpendicular line:



Hence the perpendicular line at $(1, 2)$ has parametric equations $x = 1 + 4t, y = 2 + 13t$.

Lecture 22: Tangent Planes

Warm-Up 1. For $f(x, y) = xe^{xy} + y^2$, we find the direction in which f increases most rapidly at $(-2, 1)$, and the direction in which it increases at a rate that is $\frac{\sqrt{2}}{2}$ times that most rapid rate of increase at $(-2, 1)$. The direction in which f increases most rapidly at a point is given the gradient of f at that point. So, we compute

$$\nabla f(x, y) = \left\langle e^{xy} + xye^{xy}, x^2e^{xy} + y^2\right\rangle, \text{ and } \nabla f(-2, 1) = \left\langle -e^{-2}, 4e^{-2} + 2\right\rangle.$$

Thus, at $(-2, 1)$, f increases most rapidly in the direction of $\left\langle -e^{-2}, 4e^{-2} + 2\right\rangle$.

Now, the rate of change in the direction computed above, or in other words the largest rate of change f has at $(-2,1)$ among all possible directions, is $|\nabla f(-2,1)|$. If we want directions in which the rate of change is $\frac{\sqrt{2}}{2}$ times this largest amount, we want unit vectors $\mathbf{u}$ such that

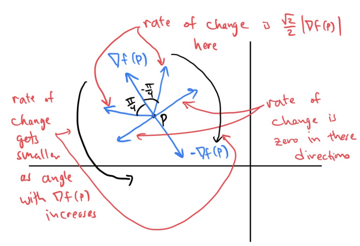
$$D_{\mathbf{u}} f(-2,1) = \tfrac{\sqrt{2}}{2}|\nabla f(-2,1)|.$$

But, using the geometric formula for the dot product, this is the same as

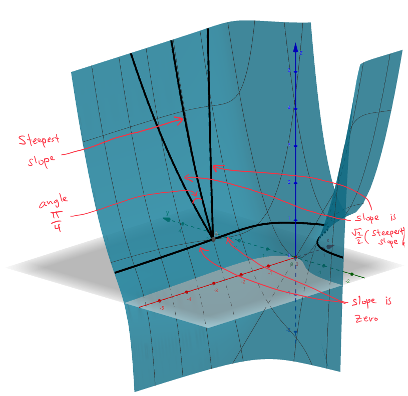
$$\underbrace{|\nabla f(-2,1)|\cos\theta}_{\nabla f(-2,1)\cdot\mathbf{u}} = \tfrac{\sqrt{2}}{2}|\nabla f(-2,1)|,$$

meaning we want vectors so that angle between it as $\nabla f(-2,1)$ satisfy $\cos\theta = \frac{\sqrt{2}}{2}$. Hence, the angle should be $\theta = \pm\frac{\pi}{4}$, so we have two directions in which the rate of change is $\frac{\sqrt{2}}{2}$ times the largest possible rate of change in any direction—namely the two directions obtained by rotating $\nabla f(-2,1)$ by $\frac{\pi}{4}$ or by $-\frac{\pi}{4}$.

This is indicative of what happens in general. The gradient at a point always points in the direction of most rapid increase, and the negative of the gradient (as we saw in an example last time) points in the direction of most rapid decrease. As we take a direction moving away from that of the gradient, the directional derivative gets smaller but remains positive for a while, hitting $\frac{\sqrt{2}}{2}$ times the maximum rate of change when we hit angles $\theta = \pm\frac{\pi}{4}$ away from the gradient direction:



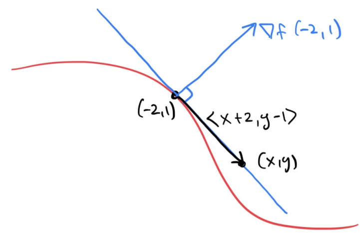
Moving further away still leads eventually to directions in which the rate of change is zero (these are precisely the directions tangent to the level curve containing the point of interest since they are the directions orthogonal to the gradient), and then we reach directions in which the rate of change becomes negative, until finally hitting the most negative the rate of change can be in the direction opposite the gradient. In the example of $f(x,y) = xe^{xy} + y^2$ at $(-2,1)$, these behaviors all look like



Warm-Up 2. We find a Cartesian equation for the tangent line to the curve

$$xe^{xy} + y^2 = -2e^{-2} + 1$$

at the point $(-2, 1)$. (Note that $(-2, 1)$ does satisfy this equation, which is why we are using this specific right-hand side.) We can view this curve as the level curve of $f(x, y) = xe^{xy} + y^2$ at the value $z = -2e^{-2} + 1$, so we know that $\nabla f(-2, 1) = \langle -e^{-2}, 4e^{-2} + 1 \rangle$ will be orthogonal to the given curve at $(-2, 1)$:



Now, the tangent line we want should be orthogonal to this gradient vector. A point (x, y) will thus be on this tangent line precisely when

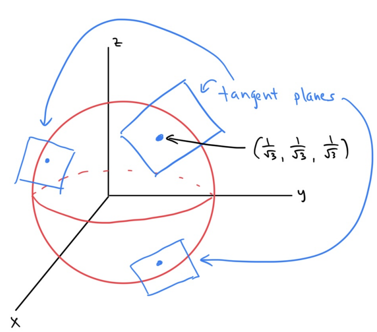
$$\nabla f(-2, 1) \cdot \langle x - (-2), y - 1 \rangle = 0.$$

Indeed, in order for (x, y) to be on this line, the vector $\langle x - (-2), y - 1 \rangle$ from $(-2, 1)$ to (x, y) must be parallel to this tangent line, which means it must perpendicular to the gradient. (This is exactly the same idea we used to derive the equation of a plane with a given normal vector way back when!) Thus, the Cartesian (meaning x, y) equation of this tangent line is

$$\langle -e^{-2}, 4e^{-2} + 2 \rangle \cdot \langle x + 2, y - 1 \rangle = 0,$$

or $-e^2(x + 2) + (4e^{-2} + 2)(y - 1) = 0$ after we compute the dot product.

Normal vectors and tangent planes. Consider the unit sphere $x^2 + y^2 + z^2 = 1$. We now seek to describe the *tangent plane* to the sphere at, say, the point $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$:



If we recall the method for finding equations of planes we discussed early in the quarter, what we need are a point on the plane (which we have!) and a vector normal to the plane.

But this normal vector can now be found using gradients! Indeed, the key is to interpret the given sphere as the level surface of a three-variable function, namely

$$f(x, y, z) = x^2 + y^2 + z^2$$

in this case. The unit sphere is the level surface of this function at the value 1, and we know from last time that the gradient of f at a point is indeed normal to the level set (surface in this case) of f containing that point. So, $\nabla f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ will be normal to the tangent plane we want. We have

$$\nabla f = \langle 2x, 2y, 2z\rangle\,, \text{ so } \nabla f(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}) = \left\langle \tfrac{2}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}, \tfrac{2}{\sqrt{3}} \right\rangle.$$

The equation of the tangent plane with this normal vector and containing $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is then

$$\left\langle \tfrac{2}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}, \tfrac{2}{\sqrt{3}} \right\rangle \cdot \left\langle x - \tfrac{1}{\sqrt{3}}, y - \tfrac{1}{\sqrt{3}}, z - \tfrac{1}{\sqrt{3}} \right\rangle = 0$$

in vector form, and

$$\tfrac{2}{\sqrt{3}}(x - \tfrac{1}{\sqrt{3}}) + \tfrac{2}{\sqrt{3}}(y - \tfrac{1}{\sqrt{3}}) + \tfrac{2}{\sqrt{3}}(z - \tfrac{1}{\sqrt{3}}) = 0$$

in scalar form.

For another example, take the surface with equation $xy + z^2 = 5$. We can view this as a level surface of the function

$$f(x, y, z) = xy + z^2,$$

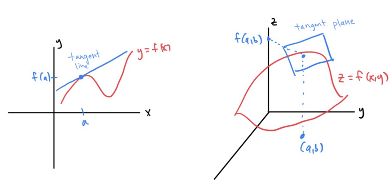
specifically the level surface at the function value 5. Thus at any point on this surface, the vector

$$\nabla f = \langle y, x, 2z\rangle$$

will be normal to the surface. The point $(1, 1, 2)$ is on this surface, for example, so $\nabla f(1, 1, 2) = \langle 1, 1, 4\rangle$ is normal to the tangent plane at this point, so this tangent plane has equation

$$(x - 1) + (y - 1) + 4(z - 2) = 0.$$

Tangent planes to graphs. Now consider the special case where the surface in question is the graph of a function $z = f(x, y)$ of two variables. (Note that here f denotes something different than what it did in the examples above: there f was a function of three variables of which the given surface was a level surface, but now f is denoting a function of two variables whose graph is the given surface. The difference can be seen in a simple example like the paraboloid $z = x^2 + y^2$, which is the graph of $f(x, y, z) = x^2 + y^2$, but which can also be seen as the level surface of the three-variable function $g(x, y, z) = x^2 + y^2 - z$ at the value 0.) The tangent plane to this graph at a point $(a, b, f(a, b))$ is an analog of the tangent line to the graph of $y = f(x)$ at a point $(a, f(a))$ you would have studied in a single-variable calculus course:



To find an equation of this tangent plane, we view the graph $z = f(x, y)$ as a level surface of $g(x, y, z) = f(x, y) - z$, namely the level surface at the value 0. Since $\nabla g = \langle f_x, f_y, -1\rangle$, a normal vector for the tangent plane at $(a, b, f(a, b))$ is then

$$\nabla g(a, b, f(a, b)) = \langle f_x(a, b), f_y(a, b), -1\rangle\,.$$

The tangent plane thus has vector equation

$$\langle f_x(a, b), f_y(a, b), -1\rangle \cdot \langle x - a, y - b, z - f(a, b)\rangle = 0,$$

and scalar equation

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0.$$

It is standard to rewrite this plane equation by isolating z to get

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Compare this resulting equation to the equation for the tangent line to $y = f(x)$ at $(a, f(a))$ from single-variable calculus, which is

$$y = f(a) + f'(a)(x-a).$$

In both equations the first constant term (i.e., the term with no variable attached) is the value of the function at the input point, and the terms with variables have values of derivatives as coefficients. Before $f'(a)$ gave the slope of the tangent line, and now we have two slopes to consider in the tangent plane, the slope in the x-direction $f_x(a,b)$ as the coefficient of the x term, and the slope in the y-direction $f_y(a,b)$ as the coefficient of the y term. In a sense, these two partial derivatives tell us how the tangent plane "tilts" in one direction or another. The upshot is that the tangent plane equation does look the same as the old tangent line equation, only now with two linear terms to account for the two variables on which our function depends.

Example. Let us find the equation for the tangent plane to the graph of $f(x,y) = x^2y^3 + x$ at $(2,1,6)$. (Note that $6 = f(2,1)$ is just the value of the function at $(2,1)$.) We have

$$f_x = 2xy^3 + 1 \quad \text{and} \quad f_y = 3x^2y^2,$$

so the tangent plane is

$$\begin{aligned} z &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 6 + 5(x-2) + 12(y-1). \end{aligned}$$

Linearization. Recall in the single-variable case that the tangent line to $y = f(x)$ at $x = a$ can be used to approximate the value of $f(x)$ for x near a. Indeed, the tangent line gives what's called the best *linear* approximation of f near a. The same is true in the two-variable case, where the tangent plane

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

gives the best linear approximation to $f(x,y)$ for (x,y) near (a,b). In this setting, giving the "best" linear approximation means that among all planes passing through the point $(a,b,f(a,b))$, the tangent plane is the one that comes closest to matching the graph of $z = f(x,y)$. Said another way, the function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

best approximates f near (a,b) among all possible linear functions, where "linear" in this context refers to the fact that x and y occur to a first power only. This function L is called the *linearization* of f at (a,b), and is simply the function whose graph is the tangent plane.

For example, for the function $f(x,y) = x^2y^3 + x$, the linearization of f at $(2,1)$ is

$$L(x,y) = 6 + 5(x-2) + 12(y-1).$$

With this we can approximate the value of, for example, $(1.9)^2(1.1)^3 + 1.9$. This is nothing but the value $f(1.9, 1.1)$ for the function f above, so the linearization gives

$$f(1.9, 1.1) \approx 6 + 5(1.9 - 2) + 12(1.1 - 1) = 6 + 5(-0.1) + 12(0.1) = 6.7$$

as the desired linear approximation. Since $(1.9, 1.1)$ is fairly close to the point $(2, 1)$ at which we computed the linearization/tangent plane at, this should give a fairly good approximation. The actual value of $f(1.9, 1.1) = (1.9)^2(1.1)^3 + 1.9$ is 6.70491, so 6.7 is indeed pretty good, and much quicker to compute by hand than $(1.9)^2(1.1)^3 + 1.9$ would be.

Error bound. We can judge how good of an approximation the linearization gives using the linear *error bound*. The fact is that the error in using the linear approximation is no larger than

$$|\text{linear error}| \leq \frac{1}{2}M(|\Delta x| + |\Delta y|)^2$$

where M denotes a bound on the magnitudes of the second derivatives of f:

$$|f_{xx}| \leq M, \ |f_{xy}| \leq M, \ |f_{yy}| \leq M.$$

The change $\Delta x = x - a$ in x and change $\Delta y = y - b$ in y denote the difference between the values of x and y giving the $f(x, y)$ we are wanting to approximate and the point (a, b) at which we took the linearization, and the bounds on the second derivatives above should be valid for all x, y that occur within these changes $\Delta x, \Delta y$. (We will give a sense as to where this error bound comes from next time.)

To see this in action in the example $f(x, y) = x^2y^3 + x$, for $(a, b) = (2, 1)$ and $(x, y) = (1.9, 1.1)$ we have

$$\Delta x = 1.9 - 2 = -0.1 \quad \text{and} \quad \Delta y = 1.1 - 1 = 0.1.$$

Also, since $f_x = 2xy^3 + 1$ and $f_y = 3x^2y^2$, we have

$$f_{xx} = 2y^3 \quad f_{xy} = 6xy^2 \quad f_{yy} = 6x^2y.$$

For x between 2 and 1.9 and y between 1 and 1.1, the absolute values of all of these second derivatives are no larger than, say, $M = 27$ (check what happens at the largest values of x and y in the ranges we are considering), so the error in approximating $(1.9)^2(1.1)^2 + 1.9$ by

$$6 + 5(1.9 - 2) + 12(1.1 - 1)$$

is no larger than

$$\frac{1}{2}M(|\Delta x| + |\Delta y|)^2 = \frac{27}{2}(0.1 + 0.1)^2 = \frac{27}{2}(0.2)^2 = 0.54.$$

As we saw above, the difference between the actual value 6.70491 of what we're trying to approximate and the approximate value 6.7 we found is 0.00491, which is indeed smaller than the error bound 0.54 we derived. The point is that finding the actual error is not always feasible since finding the exact value of $f(x, y)$ is not always possible, but having an approximate value together with a sense of how far off that approximate value is is often good enough.

Lecture 23: Quadratic Approximations

Warm-Up. We find an equation for the tangent plane to the graph of $z = ye^{3x}$ at $(0,1,1)$ and use it to approximate the value of $0.9e^{0.3}$. For $f(x,y) = ye^{3x}$, we have

$$f_x = 3ye^{3x} \quad \text{and} \quad f_y = e^{3x}.$$

Thus the tangent plane at $(0,1,1)$ is

$$\begin{aligned} z &= f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1) \\ &= 1 + 3x + (y-1). \end{aligned}$$

If we think of $0.9e^{0.3}$ as $0.9e^{3(0.1)}$, we see that we want to approximate $f(0.1, 0.9)$, so since $(0.1, 0.9)$ is fairly close to $(0,1)$, the tangent plane / linearization should give a fairly good approximation. We get

$$0.9e^{0.3} \approx 1 + 3(0.1) + (0.9 - 1) = 1.2.$$

The difference between this approximation and the actual value of $f(0.1, 0.9) = 0.9e^{3(0.1)}$ is no larger than the linear error bound

$$\frac{1}{2}M(|\Delta x| + |\Delta y|)^2$$

where M is a bound on the second derivatives of f. Since

$$f_{xx} = 9ye^{3x} \quad f_{xy} = 3e^{3x} \quad f_{yy} = 0,$$

for $0 \leq x \leq 0.1$ and $0.9 \leq y \leq 1$ we get

$$|f_{xx}| \leq 9e^{0.3} \text{ and } |f_{xy}| \leq 3e^{0.3},$$

so $M = 9e^{0.3}$ is a bound we can use. For us, $\Delta x = 0.1$ and $\Delta y = -0.1$, so the error in our approximation is no larger than

$$\frac{1}{2}(9e^{0.3})(0.1 + 0.1)^2 = 0.18e^{0.3}.$$

This error bound is about 0.243. The actual value of $0.9e^{0.3}$ is about 1.21487, so the difference between this and our approximate value of 1.2 does indeed fall within this error bound.

Taylor polynomials. We now seek to derive better approximations using expressions beyond simply linear ones, such as quadratic ones. To put this into the right context, we briefly introduce the notion of a Taylor polynomial. The *first-order Taylor polynomial* of $f(x,y)$ at (a,b) is nothing by the linearization from before:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

As we said last time, the point is that this is the linear polynomial which best approximates f near (a,b) among all linear polynomial.

The *second-order Taylor polynomial* of $f(x,y)$ at (a,b) is

$$Q(x,y) = L(x,y) + \frac{1}{2}[f_{xx}(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2].$$

Here, $L(x, y)$ denotes the first-order Taylor polynomial, so the second-order Taylor polynomial starts with this and adds on "second-order", or quadratic, terms. Just as the linear terms in the first-order Taylor polynomial have as their coefficients the corresponding partial derivatives, the coefficients of the quadratic terms in the second-order Taylor polynomial are the corresponding second-order partial derivatives. The term in the middle $2f_{xy}(x-a)(y-b)$ is really two terms in one: the corresponding quadratic terms are $(x-a)(y-b)$ and $(y-b)(x-a)$, with coefficients

$$f_{xy}(a, b) \quad \text{and} \quad f_{yx}(a, b)$$

respectively, so that really the quadratic piece altogether is

$$\frac{1}{2}[f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + f_{yx}(a,b)(y-b)(x-a) + f_{yy}(y-b)^2].$$

Equality of the mixed partial derivatives f_{xy} and f_{yx} then allows us to group these two mixed terms into one, so we just write it as $2f_{xy}(x-a)(y-b)$ instead.

The presence of the extra $\frac{1}{2}$ in front is not something we will explain in this course. If you've seen single-variable Taylor polynomials before (as would be covered in MATH 226 for example), the $\frac{1}{2}$ should look familiar to you. It's there because it has to be there so that differentiating $f_{xx}(a,b)(x-a)^2$, for example, twice gives $f_{xx}(a,b)$ instead of $2f_{xx}(a,b)$, but understanding this fully requires knowing more about Taylor polynomials in general than what we'll need. You can ask about it if you'd like to know some more of these details, but for the purposes of this course you can simply take it as given that the $\frac{1}{2}$ has to be there.

Example. Back to the example of $f(x, y) = ye^{3x}$. The first-order Taylor polynomial of f at $(0, 1)$ is the linearization

$$L(x, y) = 1 + 3x + (y - 1).$$

To compute the second-order Taylor polynomial of f at $(0, 1)$, we need the second-order partial derivatives:

$$f_{xx} = 9ye^{3x} \quad f_{xy} = 3e^{3x} = f_{yx} \quad f_{yy} = 0.$$

The second-order Taylor polynomial is then:

$$\begin{aligned} Q(x,y) &= L(x,y) + \frac{1}{2}[f_{xx}(0,1)(x-0)^2 + 2f_{xy}(0,1)(x-0)(y-1) + f_{yy}(0,1)(y-1)^2] \\ &= 1 + 3x + (y-1) + \frac{1}{2}[9x^2 + 6x(y-1)]. \end{aligned}$$

Quadratic approximations. Just as first-order Taylor polynomials provide the best linear approximations to a function, second-order Taylor polynomials provide the best quadratic approximations. That is, the second-order Taylor polynomial of f at (a, b) is the quadratic polynomial that best approximates f near (a, b) among all quadratic polynomials. So, in the example above, the best quadratic approximation to $f(x, y) = ye^{3x}$ near $(0, 1)$ is given by

$$Q(x,y) = 1 + 3x + (y-1) + \frac{9}{2}x^2 + 3x(y-1).$$

Before we used the first-order Taylor polynomial to approximate $f(0.1, 0.9) = 0.9e^{0.3}$, and got 1.2 as the approximate value. Now, the second-order Taylor polynomial gives

$$f(0.1, 0.9) \approx 1 + 3(0.1) + (0.9-1) + \frac{9}{2}(0.1)^2 + 3(0.1)(0.9-1) = 1.215.$$

The actual value of $0.9e^{0.3}$ was about 1.21487, so the quadratic approximation is indeed a better approximation than the linear approximation.

Plotting the graph of $z = ye^{3x}$, of $z = 1+3x+(y-1)$, and of $z = 1+3x+(y-1)+\frac{9}{2}x^2+3x(y-1)$ on a computer shows these approximations in action. (I won't include these drawings here since I can't find a nice way to make them clear enough, but you should try to graph these on your own on GeoGebra or a similar program!) The first-order Taylor polynomial gives an OK match to the graph of $z = ye^{3x}$ when close to $(0, 1)$, and the second-order Taylor polynomial gives a better approximation that is valid for a bit further away from $(0, 1)$ than was the case for the first-order approximation. The second-order approximation visually captures more of the "curvature" of the graph of $z = ye^{3x}$ as we start to move away from $(0, 1)$.

Another example. We compute the best linear and quadratic approximations (or in other words the first- and second-order Taylor polynomials) of $f(x, y) = e^{2x}\cos(3y)$ at $(0, \pi)$. We have

$$f_x = 2e^{2x}\cos(3y) \quad f_y = -3e^{2x}\sin(3y)$$

and then

$$f_{xx} = 4e^{2x}\cos(3y) \quad f_{xy} = -6e^{2x}\sin(3y) \quad f_{yy} = -9e^{2x}\cos(3y).$$

The best linear approximation near $(0, \pi)$ is given by

$$L(x, y) = f(0, \pi) + f_x(0, \pi)(x - 0) + f_y(0, \pi)(y - \pi) = -1 + 2x$$

and the best quadratic approximation is

$$\begin{aligned} Q(x, y) &= L(x, y) + \frac{1}{2}[f_{xx}(0, \pi)(x - 0)^2 + 2f_{xy}(0, \pi)(x - 0)(y - \pi) + f_{yy}(0, \pi)(y - \pi)^2] \\ &= -1 + 2x + \frac{1}{2}[-4x^2 + 9(y - \pi)^2]. \end{aligned}$$

Plotting all of these on a computer again shows the way in which the quadratic approximation is better than the linear approximation, and the quadratic one captures the way in which the surface "curves" as we move away from $(0, \pi)$.

Back to linear errors. We can now saying something about the expression we had last time for the linear error bound:

$$\frac{1}{2}M(|\Delta x| + |\Delta y|)^2.$$

The point is that this comes from the quadratic terms in the second-order Taylor polynomial. To be precise, it turns out that the linear error can be described *on the nose* by evaluating the second derivatives appearing in these quadratic terms at *some* point (c, d) on the line segment that run between (a, b) and (x, y). That is, for some such (c, d), the linear error is exactly

$$\frac{1}{2}[f_{xx}(c, d)(x - a)^2 + 2f_{xy}(c, d)(x - a)(y - b) + f_{yy}(c, d)(y - b)^2].$$

This result is known as *Taylor's theorem*, and understanding it in full is beyond the scope of this course. The expression above looks almost the same as the quadratic expressions in the second-order Taylor polynomial, only that in that case the second derivatives are evaluated at the point (a, b) at which we are taking the approximation, whereas here they are evaluated at some unknown (c, d). If you recall seeing the *mean value theorem* in single-variable calculus, a similar thing happens there, and indeed Taylor's theorem is just a higher-order version of that. (The single-variable version of Taylor's theorem is covered in MATH 226.)

The explicit form of the error given above is not so helpful, however, precisely because in general we have no information about c and d. However, often we don't need the explicit error, just a bound on the error, and this where the error bound we saw last time comes in: we bound the second derivatives by a single number M, so that

$$|f_{xx}(c,d)|, |f_{xy}(c,d)|, |f_{yy}(c,d)|$$

are all no larger than M, regardless of what the unknown (c,d) is. Then the linear error is bounded by

$$\frac{1}{2}[M|\Delta x|^2 + 2M|\Delta x||\Delta y| + M|\Delta y|^2]$$

where $\Delta x = x - a$ and $\Delta y = y - b$. Since

$$|\Delta x|^2 + 2|\Delta x||\Delta y| + |\Delta y|^2 = (|\Delta x| + |\Delta y|)^2,$$

we end up with precisely the formula for the error bound we had before. The takeaway is that this linear error bound appears as it does because of the behavior of the quadratic terms.

Higher-order approximations. To get better approximations, we use higher-order Taylor polynomials. The third-order Taylor polynomial, which provides the best cubic approximation, takes the second-order Taylor polynomial and adds on cubic terms like

$$(x-a)^3,\ (x-a)^2(y-b),\ (x-a)(y-b)^2,\ (y-b)^3.$$

The coefficients used for each of these are the corresponding third-order partial derivatives evaluated at (a,b), and in the end we put an extra factor of $\frac{1}{6}$ in front, analogous to the $\frac{1}{2}$ we used in the second-order Taylor polynomial. And so on, we add on higher-order terms with higher-order partial derivative as coefficients to keep going.

Just as the error in the linear approximation is controlled by the quadratic terms in the second-order Taylor polynomial, the error in the quadratic approximation is controlled by the *cubic* terms in the third-order Taylor polynomial, and so on. We'll give one example of this next time, but quadratic errors will not play a big role for us. Linear errors, however, are definitely things with which you should be familiar.

Lecture 24: Local Extrema

Warm-Up. We approximate the value of $\cos(\frac{\pi}{3} - 0.1)\sin(\frac{\pi}{3} + 0.2)$ using a linear and a quadratic approximation. Of course, you can just plug this into a calculator and see that the value is

$$0.5536512...$$

but the point is that Taylor polynomials are what calculators and computers use to come up with such values in the first place; your calculator has no idea what "sin" or "cos" mean, all it knows are the Taylor polynomials approximating them which are stored in its memory.

We find the second-order Taylor polynomial of $f(x,y) = \cos x \sin y$ at $(\pi/3, \pi/3)$. We have

$$f_x = -\sin x \sin y,\ f_y = \cos x \cos y$$

and then

$$f_{xx} = -\cos x \sin y,\ \ f_{xy} = -\sin x \cos y = f_{yx},\ \ f_{yy} = -\cos x \sin y.$$

Hence the second-order Taylor polynomial is

$$\begin{aligned} Q(x,y) &= f(\tfrac{\pi}{3},\tfrac{\pi}{3}) + f_x(\tfrac{\pi}{3},\tfrac{\pi}{3})(x-\tfrac{\pi}{3}) + f_y(\tfrac{\pi}{3},\tfrac{\pi}{3})(y-\tfrac{\pi}{3}) \\ &\quad + \tfrac{1}{2}[f_{xx}(\tfrac{\pi}{3},\tfrac{\pi}{3})(x-\tfrac{\pi}{3})^2 + 2f_{xy}(\tfrac{\pi}{3},\tfrac{\pi}{3})(x-\tfrac{\pi}{3})(y-\tfrac{\pi}{3}) + f_{yy}(\tfrac{\pi}{3},\tfrac{\pi}{3})(y-\tfrac{\pi}{3})^2] \\ &= \tfrac{\sqrt{3}}{4} - \tfrac{3}{4}(x-\tfrac{\pi}{3}) + \tfrac{1}{4}(y-\tfrac{\pi}{3}) + \tfrac{1}{2}[-\tfrac{\sqrt{3}}{4}(x-\tfrac{\pi}{3})^2 - 2(\tfrac{\sqrt{3}}{4})(x-\tfrac{\pi}{3})(y-\tfrac{\pi}{3}) - \tfrac{\sqrt{3}}{4}(y-\tfrac{\pi}{3})^2] \end{aligned}$$

From this we can extract the first-order Taylor polynomial by taking the linear terms:

$$L(x,y) = \tfrac{\sqrt{3}}{4} - \tfrac{3}{4}(x-\tfrac{\pi}{3}) + \tfrac{1}{4}(y-\tfrac{\pi}{3}).$$

The linear approximation to $\cos(\frac{\pi}{3}-0.1)\sin(\frac{\pi}{3}+0.2)$ is thus

$$\cos(\tfrac{\pi}{3}-0.1)\sin(\tfrac{\pi}{3}+0.2) \approx \tfrac{\sqrt{3}}{4} - \tfrac{3}{4}(-0.1) + \tfrac{1}{4}(0.2) = 0.5580127...$$

and the quadratic approximation is

$$\frac{\sqrt{3}}{4} - \frac{3}{4}(-0.1) + \frac{1}{4}(0.2) + \frac{1}{2}[-\frac{\sqrt{3}}{4}(-0.1)^2 - \frac{\sqrt{3}}{2}(-0.1)(0.2) - \frac{\sqrt{3}}{4}(0.2)^2] = 0.5558476....$$

Comparing the value given by a calculator we gave at the beginning, we see that both of these approximations are pretty good, with the quadratic approximation being better.

But even without the actual value to compare these approximations too, we can estimate the errors resulting from each. Note that all second-order partial derivatives of f computed above are bounded in absolute value by 1, so linear approximation is accurate to within

$$\frac{1}{2}(1)(\underbrace{0.1}_{|\Delta x|} + \underbrace{0.2}_{|\Delta y|})^2 = \frac{1}{2}(0.3)^2 = 0.045,$$

which we can see is indeed true using the actual value. The quadratic error is no larger than

$$\frac{1}{6}M(|\Delta x| + |\Delta y|)^3$$

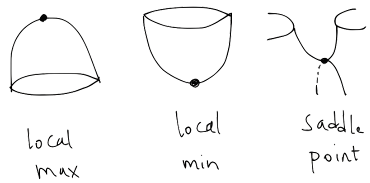
where M is now a bound on the third-order partial derivatives of f. (This is the only example of a quadratic error we will look at, so it is not important to memorize this expression. As briefly mentioned last time, it comes from the cubic terms in the third-order Taylor polynomial.) If we compute all third-order partial derivatives of f we will see that they too are all bounded by 1 (they all involve products of sines and cosines), so we take $M = 1$ and see that the quadratic approximations is accurate to within

$$\frac{1}{2}(1)(0.1+0.2)^3 = \frac{1}{2}(0.3)^3 = 0.0135,$$

which also makes sense given that we know the actual value.

Optimization. Our final goal this quarter is to study *optimization*, which deals with determining how large or small the values of a function can be, possibly subject to some constraints. This is, no doubt, one if not the main use of multivariable differential calculus in other fields, and is a nice way to finish off the course.

We start by understanding *local extrema* of functions of two variables, namely *local maximums* and *local minimums*:



(Ignore the notion of a saddle point for now.) Just as in the single-variable case, local maxima are points at which the function value is larger than (or equal to) what it is at points nearby, and local minima are points at which the function value is smaller than (or equal to) what it is at points nearby. Consider for example the function

$$f(x, y) = 4x + 6y - 12 - x^2 - y^2.$$

We can rewrite this by completing the square in both the x and y terms to get

$$f(x, y) = -(x - 2)^2 - (y - 3)^2 + 1.$$

Note that $f(2, 3) = 1$, and in fact for any other point we have $f(x, y) < 1$ since $-(x-2)^2-(y-3)^2$ is never positive, so the value of $f(x, y)$ in general is never larger than 1. Thus, $(2, 3)$ is a local maximum of f and the graph of f looks like the first picture in the image above, with the topmost point of the upside-down paraboloid occurring at $(2, 3, f(2, 3)) = (2, 3, 1)$.

Critical points. Finding the local extrema of the function in the example above was possible by doing some algebraic manipulation (i.e., completing the square), but this type of method only works in very specific cases, and so will not be suitable in general. We need a better way of finding and then classifying local extrema. The key observation is that at a local maximum or a local minimum of $f(x, y)$, it must in fact be true that both partial derivatives of f must be zero. Indeed, if we consider only the behavior of f in the x-direction at a local maximum, we see that we also have a local maximum of the single-variable function obtained from $f(x, y)$ by varying x only, so the derivative of this single-variable function f_x must be zero, and similarly for the function obtained when varying y only. The same is true in the local minimum case, so the upshot is that at a local maximum or a local minimum, we must have

$$f_x = 0 \quad \text{and} \quad f_y = 0.$$

A point satisfying this condition, where both partial derivatives are zero, is called a *critical point* of f. (You would have seen the same terminology used in the single-variable case.) So, local extrema are among the critical points. However, there is another type of critical point that is not a local extrema, which we call a *saddle point*. (See the third picture in the image above.) Instead of looking like a paraboloid or an upside-down paraboloid, the graph of f at a saddle point looks like a hyperbolic paraboloid, or in other words the surface of a saddle. The distinguishing feature of a saddle point is that it is sitting at a minimum in one direction, but at a maximum is another direction, as opposed to local maxima which sit at maximums in all directions or local minima which sit at minimums in all directions. Since a saddle point sits at a minimum one way but at a maximum another way, we still get that both partial derivatives are zero.

Examples. For the function $f(x, y) = 4x + 6y - 12 - x^2 - y^2$ from before, the critical points satisfy

$$f_x = 4 - 2x = 0 \quad \text{and} \quad f_y = 6 - 2y = 0.$$

Thus $x = 2$ and $y = 3$, so $(2, 3)$ is the only critical point of f. We argued before that in this case this critical point was a local maximum of f.

For $g(x, y) = x^2 - 2y^2 + 2x + 3$, the critical points are found by solving

$$g_x = 2x + 2 = 0 \quad \text{and} \quad g_y = -4y = 0,$$

so we get $(-2, 0)$ as the only critical point. This is an example where again some algebraic manipulation allows us to determine what type of critical point this is. After completing the square we get

$$g(x, y) = (x + 1)^2 - 2y^2 + 2.$$

This is the equation of a saddle (hyperbolic paraboloid) shifted by -1 in the x-direction, which can determine by comparing to something more standard like $z = x^2 - y^2$. Thus $(2, 3)$ is a saddle point of $g(x, y) = x^2 - 2y^2 + 2x + 3$.

Second derivative test. Determining that $(2, 3)$ was a saddle point above was possible to do using some algebra again because of the specific type of function we looked at. More generally, to determine what type of critical point we have, we can use the following *second derivative test.* At a critical point of $f(x, y)$ we compute the value of the expression

$$D = f_{xx} f_{yy} - (f_{xy})^2.$$

The second derivative test says that:

- if $D > 0$ and $f_{xx} < 0$, then our critical point is a local maximum;
- if $D > 0$ and $f_{xx} > 0$, then our critical point is a local minimum; and
- if $D < 0$, then our critical point is a saddle point.

If $D = 0$, the second derivative test gives us no information.

Now, justifying why this second derivative test works is beyond the scope of this course, but we will give a small sense of intuition next time. Conceptually though, we do expect that the behavior of critical points should be determined by something like "concavity", as is the case for single-variable functions, and "concavity" should be measured by second derivatives, so it makes sense that second-order partial derivatives are used in distinguishing between local maxima, local minima, and saddle points. Again, why the specific combination of second derivatives used in D above is the correct one to use is not something we will be able to justify in this course.

In the $f(x, y) = 4x + 6y - 12 - x^2 - y^2$ example with critical point $(2, 3)$, we have

$$f_{xx} = -2 \quad f_{xy} = 0 \quad f_{yy} = -2,$$

so $D = f_{xx} f_{yy} - (f_{xy})^2 = 4$. Since $D > 0$ and $f_{xx} < 0$, the second derivative test says that $(2, 3)$ is a local maximum of f. In the $g(x, y) = x^2 - 2y^2 + 2x + 3$ example with critical point $(-2, 0)$, we have

$$g_{xx} = 2 \quad g_{xy} = 0 \quad g_{yy} = -4.$$

Since $D = f_{xx} f_{yy} - (f_{xy})^2 = -8$ is negative, $(-2, 0)$ is indeed a saddle point as we justified earlier.

Example. We find and classify the critical points of $f(x, y) = x^2 + y^3 - x^2 y + y$. The critical points satisfy

$$f_x = 2x - 2xy = 0 \quad \text{and} \quad f_y = 3y^2 - x^2 + 1 = 0.$$

The first equation can be written as $2x(1 - y) = 0$, so $x = 0$ or $y = 1$. If $x = 0$, the second equation becomes

$$3y^2 + 1 = 0,$$

which has no solutions, so there are no critical points in the $x = 0$ case. If instead $y = 1$, the $f_y = 0$ equation becomes

$$3 - x^2 + 1 = 0, \text{ so } x = \pm 2.$$

Thus here we get two critical points: $(-2, 1)$ and $(2, 1)$.

To classify these critical points, we compute second derivatives:

$$f_{xx} = 2 - 2y \quad f_{xy} = -2x \quad f_{yy} = 6y.$$

At the critical point $(-2, 1)$, we get

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (0)(-12) - (4)^2 = -16 < 0,$$

so $(-2, 1)$ is a saddle point of f. At $(2, 1)$ we get

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (0)(12) - (-4)^2 = -16 < 0,$$

so $(2, 1)$ is also a saddle point of f. Thus $f(x, y) = x^2 + y^3 - x^2y + y$ has two critical points, both of which are saddle points.

Lecture 25: Absolute Extrema

Warm-Up. We find and classify the critical points of $f(x, y) = x^2 - y^3 - x^2y + y$. This is almost the same function we finished with last time, only with a sign change in the y^3 term. The critical points satisfy

$$f_x = 2x - 2xy = 0 \quad \text{and} \quad f_y = -3y^2 - x^2 + 1 = 0.$$

The first equation again gives $x = 0$ or $y = 1$. For $x = 0$, the second equation becomes

$$-3y^2 + 1 = 0, \text{ so } y = \pm\frac{1}{\sqrt{3}},$$

so so far we get two critical points: $(0, \frac{1}{\sqrt{3}})$ and $(0, -\frac{1}{\sqrt{3}})$. For $y = 1$, the $f_y = 0$ equation becomes

$$-3 - x^2 + 1 = 0,$$

which has no solutions. Thus the points found above are the only critical points.

We have

$$f_{xx} = 2 - y \quad f_{xy} = -2x \quad f_{yy} - 6y.$$

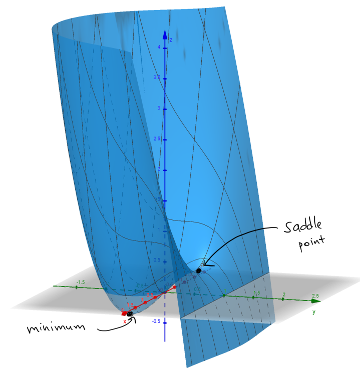
At $(0, -\frac{1}{\sqrt{3}})$, we get

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (2 + \tfrac{1}{\sqrt{3}})(\tfrac{6}{\sqrt{3}}) - 0 > 0,$$

so since $f_{xx} = 2 + \frac{1}{\sqrt{3}} > 0$ at this point, $(0, -\frac{1}{\sqrt{3}})$ is a local minimum of f. At $(0, \frac{1}{\sqrt{3}})$, we get

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (2 - \tfrac{1}{\sqrt{3}})(-\tfrac{6}{\sqrt{3}}) - 0 < 0$$

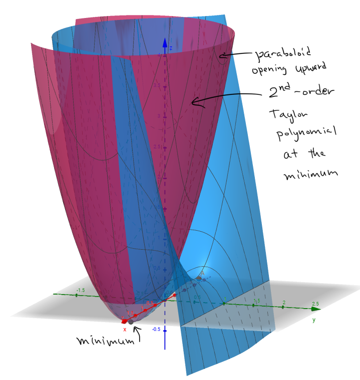
(note that $2 - \frac{1}{\sqrt{3}} > 0$), so $(0, \frac{1}{\sqrt{3}})$ is a saddle point of f. Having a computer plot the graph of $f(x, y) = x^2 - y^3 - x^2y + y$ should be it clear what is going on: near $(0, -\frac{1}{\sqrt{3}})$, the graph does look like a paraboloid opening upward, while near $(0, \frac{1}{\sqrt{3}})$ the graph does look like a saddle:



Why does the second derivative test work? We will not be able to fully justify the second derivative test, but let us now give at least some sense as to where it comes from. The key is in looking at the second-order Taylor polynomials at critical points. Let us use the function $f(x, y) = x^2 - y^3 - x^2y + y$ from above as an example. The second-order Taylor polynomial of f at $(0, -\frac{1}{\sqrt{3}})$ is

$$z = -\tfrac{2}{3\sqrt{3}} + \tfrac{1}{2}[(2 + \tfrac{1}{\sqrt{3}})x^2 + \tfrac{6}{\sqrt{3}}(y + \tfrac{1}{\sqrt{3}})^2].$$

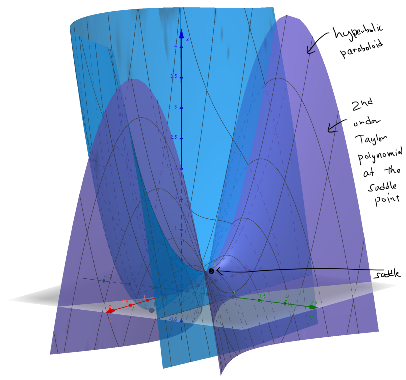
Because of the positive coefficients of the x^2 and y^2 terms, this quadric surface is indeed a paraboloid opening upward, so $(0, -\frac{1}{\sqrt{3}})$ is a local minimum of this second-order Taylor polynomial. And that is the point: the function $f(x, y)$ near $(0, -\frac{1}{\sqrt{3}})$ should be pretty similar to this second-order Taylor polynomial since this polynomial gives a good approximation to f near this point, and so $(0, -\frac{1}{\sqrt{3}})$ being a local minimum of this polynomial is why it is a local minimum of f as well:



The second-order Taylor polynomial of f at $(0, \frac{1}{\sqrt{3}})$ is

$$z = \tfrac{2}{3\sqrt{3}} + \tfrac{1}{2}[(2 - \tfrac{1}{\sqrt{3}})x^2 - \tfrac{6}{\sqrt{3}}(y - \tfrac{1}{\sqrt{3}})^2].$$

This is a quadric surface with coefficients of x^2 and y^2 being of opposite sign, so this surface is a hyperbolic paraboloid, or a saddle. Thus, again, the idea is that the behavior of $f(x, y)$ near the critical point $(0, \frac{1}{\sqrt{3}})$ is modeled by the behavior of its second-order Taylor polynomial at this point, so since this polynomial has a saddle point at $(0, \frac{1}{\sqrt{3}})$, so too does f:



In general, the second-order Taylor polynomial of a function f at a critical point (a, b) looks like

$$z = f(a, b) + \frac{1}{2}[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(x - a)(y - b) + f_{yy}(y - b)^2],$$

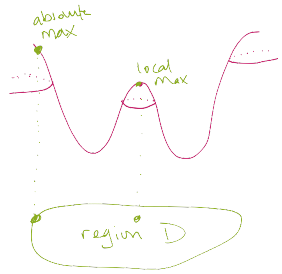
where there are no linear terms since these have coefficients $f_x(a, b) = 0 = f_y(a, b)$ given that (a, b) is a critical point of f. Near (a, b), the behavior of f is roughly the same as that of this Taylor polynomial, so the only question is what the graph of this Taylor polynomial looks like. The conditions in the second derivative test turn out to precisely determine this Taylor polynomial graph, and hence determine the nature of the critical point overall. There is still more to be said here, namely why specifically the quantity

$$D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$$

is of interest in determining the shape of the graph of the second-order Taylor polynomial, but that it all we will say in this course. To learn more you can look up material on what's called the *Hessian* if you're interested (the Hessian is the quantity we've denoted by D), and material on "diagonalizing quadratic forms", whatever that means.

Absolute extrema. Now we move away from the problem of finding the local extrema of a function f to that of finding its *absolute* or *global* extrema, which are the largest and/or smallest values a function can have overall. To make matters more interesting, we are interested in finding such values only over a restricted region D, meaning we ask for the absolute max/min values of f among points of D. The first thing to say is how we know that such absolute values exist. The answer usually comes from the *Extreme Value Theorem*, which states that any continuous function over a region that is both closed and bounded will have both an absolute maximum and an absolute minimum. (Recall that to be "closed" means that the region should include all its boundary.) So, if our function is continuous and our region is closed and bounded, our search for absolute extrema will not be done in vain.

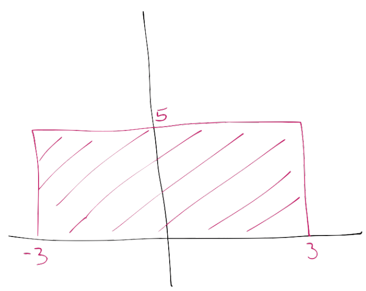
To find the absolute extrema, we start the same way as before by finding points where f possibly has a local max or min (we don't care about saddle points here), which means finding the critical points of f. After finding these critical points we can simply plug them into f to see which gives the largest value and which gives the smallest. However, this method does not account for the fact that the absolute max/min of f might actually occur along the *boundary* of D, since it is possible that a point on the boundary might give the largest or smallest value overall and yet not be a critical point. For instance, for a function and region looking like



we see that the maximum of f over D occurs on the boundary of D and not at the local maximum in the interior of D; in this case the partial derivatives of f at the boundary point are not zero, so the boundary point is not a critical point of f. This analogous to the issue of optimizing a single variable function over an interval, where the endpoints of the interval should also be considered in the end even though they might not be critical points.

So, after finding critical points of f we still have to check for any possible maximums/minimums on the boundary. Usually this means that we use the equation(s) of the boundary to come up with a simplified version of f along the boundary, and optimize that simplified function instead. The following examples show how this all works.

Example 1. We find the absolute extrema of $f(x, y) = x^2 + xy + y^2 - 6y$ over the rectangle described by $-3 \leq x \leq 3$ and $0 \leq y \leq 5$, which looks like:



The function f is continuous, and this region is closed and bounded, so the extreme value theorem guarantees that the absolute extrema we seek do exist.

First we find critical points. We have

$$\nabla f = (2x + y, x + 2y - 6),$$

so critical points satisfy

$$2x + y = 0 \text{ and } x + 2y - 6 = 0.$$

The first equation gives $y = -2x$ and substituting into the second gives

$$x + 2(-2x) - 6 = 0, \text{ so } -3x - 6 = 0.$$

Thus $x = -2$ and as a result $y = -2x = 4$, so $(-2, 4)$ is the only critical point. Note that at this point we can use the second derivative test to determine that $(-2, 4)$ is a local minimum, but this is not necessary since in the end we'll just test all points we find anyway to determine which give the absolute max and min.

Now we check the boundary of the rectangle, which consists of four different line segments. The bottom has equation $y = 0$, so the function f along the bottom edge becomes

$$f(x, 0) = x^2.$$

This is now just a function of one variable, which we optimize using techniques from single variable calculus. In this case the only (single-variable) critical point is at $x = 0$, giving $(0, 0)$ as a candidate point for the absolute max and min overall. The right edge has equation $x = 3$, so the function becomes

$$f(3, y) = 9 + 3y + y^2 - 6y = y^2 - 3y + 9.$$

Then $f_y = 2y - 3$ along the right edge, so $(3, \frac{3}{2})$ is a candidate max/min point along the right edge. The top edge is $y = 5$ so f becomes

$$f(x, 5) = x^2 + 5x - 5.$$

Then $f_x = 2x + 5$ along the top, so $(-\frac{5}{2}, 5)$ is another candidate max/min. Finally, the left edge is $x = -3$, so f becomes

$$f(-3, y) = y^2 - 9y + 9,$$

which gives $(-3, \frac{9}{2})$ as another candidate.

To recap, so far we have

$$(-2, 4),\ (0, 0),\ (3, \tfrac{3}{2}),\ (-\tfrac{5}{2}, 5),\ (-3, \tfrac{9}{2})$$

as possible points where the absolute maximum and minimum occur. But these aren't the only possible points since checking each boundary edge does not take into account what happens at the corners of the rectangle! For instance, along the right edge we had

$$f(3, y) = y^2 - 3y + 9,$$

which has its maximum value along the right edge at the corner $(3, 5)$, and yet this point is not a critical point of the function f restricted to the right edge. In other words, for the same reason why finding critical points of $f(x, y)$ does not necessarily give candidate max/min point along the boundary, finding critical points of f restricted to each boundary piece does not necessarily the candidate max/min points which occur at the corners of each boundary piece. So, we have to include the four corners

$$(3, 0),\ (3, 5),\ (-3, 5),\ (-3, 0)$$

among the candidate points for an absolute max/min.

In total then we have nine points to test: the one critical point, the four points we found along the boundary pieces, and the four corner points. Plugging all of these into the function gives:

$$\begin{array}{ccc} f(-2,4) = -12 & f(0,0) = 0 & f(3,3/2) = 6.75 \\ f(-5/2,5) = -11.25 & f(-3,9/2) = -11.25 & f(3,0) = 9 \\ f(3,5) = 19 & f(-3,5) = -11 & f(-3,0) = 9, \end{array}$$

so the absolute maximum value of f is 19, which is attained at $(3, 5)$, while the absolute minimum value of f is -12, which is attained at $(-2, 4)$.

Example 2. We find the absolute extrema of the function $f(x, y) = x^2y$ over the region described by $3x^2 + 4y^2 \leq 12$, which is just the region enclosed by the ellipse $3x^2 + 4y^2 = 12$. Again, f is continuous and this region is closed and bounded, so absolute extrema exist by the extreme value theorem. First,

$$f_x = 2xy \quad f_y = x^2,$$

which are both 0 only when $x = 0$. Thus points on the y-axis are the critical points of f. Note, however, that along the y-axis, the value of $f(0, y)$ is 0, and 0 will be neither the absolute maximum nor the absolute minimum of f since points in the first quadrant give values larger than 0 and points in the fourth quadrant give values smaller than 0. Hence the critical points on the y-axis won't matter in the end.

Now, the points on the boundary satisfy $3x^2 + 4y^2 = 12$, so $x = \pm\sqrt{4 - \frac{4}{3}y^2}$. Hence along the boundary the function f becomes

$$f(\pm\sqrt{4 - \tfrac{4}{3}y^2}, y) = \left(4 - \tfrac{4}{3}y^2\right) y = -\tfrac{4}{3}y^3 + 4y.$$

This has derivative $-4y^2 + 4$, so only $y = \pm 1$ give critical points. Then $x^2 = 4 - \frac{4}{3}y^2 = 4 - \frac{4}{3}$, so $x = \pm\sqrt{\frac{8}{3}}$. However, we have to be careful again about not missing any points we need to consider. In particular, after solving for x in terms of y above, the resulting expressions $x = -\sqrt{4 - \frac{4}{3}y^2}$ for the left half of the ellipse and $x = \sqrt{4 - \frac{4}{3}y^2}$ for the right half are only valid for $-\sqrt{3} \leq y\sqrt{3}$, so we are missing the potential behaviors at these endpoints $y = \pm\sqrt{3}$. (These are analogous to the "corners" in the rectangle case.) At each of these, however, we get that $x = 0$, so f is zero and we already said that 0 would be neither the absolute maximum nor the absolute minimum.

Hence the candidate max/min points along the boundary ellipse are

$$\left(\sqrt{\tfrac{8}{3}}, 1\right), \ \left(-\sqrt{\tfrac{8}{3}}, 1\right), \ \left(-\sqrt{\tfrac{8}{3}}, -1\right), \ \left(\sqrt{\tfrac{8}{3}}, -1\right).$$

Plugging in these points together with the critical points on the y-axis, we find that the absolute maximum value of f is $\frac{8}{3}$, which is attained at $(\sqrt{8/3}, 1)$ and $(-\sqrt{8/3}, 1)$, and the absolute minimum value is $-\frac{8}{3}$, which is attained at $(-\sqrt{8/3}, -1)$ and $(\sqrt{8/3}, 1)$.

Lecture 26: Lagrange Multipliers

Warm-Up 1. We find the absolute extrema of $f(x, y) = 1 + (x + 1)^2 - 2(x + 1)(y - 1) - (y - 1)^2$ over the region enclosed by the triangle with vertices $(0, 0), (0, 1)$, and $(1, 0)$. This region is closed and bounded, and f is continuous, so the absolute extrema exist.

The critical points of f satisfy

$$f_x = 2(x + 1) - 2(y - 1) = 0 \quad \text{and} \quad f_y = -2(x + 1) - 2(y - 1) = 0.$$

Adding these two equations together gives $-4(y - 1) = 0$, so $y = 1$. Then using $y = 1$ in the $f_x = 0$ equation gives $2(x + 1) = 0$, so $x = -1$. Thus $(-1, 1)$ is the only critical point of f, but since this does not fall within our given triangular region, we can simply ignore it.

Now we check for candidates for the absolute extrema on the boundary triangle. The bottom edge has equation $y = 0$, so along this edge our function becomes

$$f(x, 0) = 1 + (x + 1)^2 + 2(x + 1) - 1 = x^2 + 4x + 3.$$

This has single-variable critical points when $2x+4=0$, so at $x=-2$, which does not give a point in our region. The left edge of the triangle has equation $x=0$, so our function along this edge becomes

$$f(0,y) = 1+1-2(y-1)-(y-1)^2 = -y^2+3.$$

This has single-variable critical points when $-2y=0$, so when $y=0$. This gives the origin $(0,0)$ as a candidate point, which is a corner point we would have considered anyway. Finally, along the diagonal edge of the triangle where $y=1-x$, our function values are

$$f(x,1-x) = 1+(x+1)^2-2(x+1)(-x)-(-x)^2 = 4x^2+4x+2.$$

Differentiating gives $8x+4=0$, so $x=-\frac{1}{2}$, which falls outside our region.

Thus the only points we need to consider are the corner points of the triangle: $(0,0)$, $(1,0)$, and $(0,1)$. Evaluating our function at each of these gives

$$f(0,0)=3 \quad f(1,0)=8 \quad f(0,1)=2,$$

so the absolute maximum value of f over the given region is 8 and occurs at $(1,0)$, while the absolute minimum value is 2 and occurs at $(0,1)$.

Warm-Up 2. We find the point on the plane $x+2y+5z=1$ that is closest to the point $(1,0,2)$. To find the point *closest* to $(1,0,2)$ we need to minimize the function which gives distance from a point (x,y,z) to $(1,0,2)$, which is

$$f(x,y,z) = \sqrt{(x-1)^2+y^2+(z-2)^2}.$$

But we are only considering points (x,y,z) that lie on the plane $x+2y+5z=1$, meaning we are only considering points whose x-coordinate satifies

$$x = 1-2y-5z.$$

The point is that with this condition we can turn our three-variable distance function above into a two-variable function by substituting in for x, so the function we need to minimize is

$$f(1-2y-5z,y,z) = \sqrt{(1-2y-5z-1)^2+y^2+(z-2)^2}.$$

A final simplification is to note that a square root is optimized when the expression of which we are taking the square root is optimized, so it is enough to minimize the two-variable function

$$f(y,z) = (-2y-5z)^2+y^2+(z-2)^2.$$

(This will avoid having to differentiate a square root, which will avoid a bit of messy algebra.) The upshot is that finding the absolute minimum of this one function incorporates both the original distance function we wanted and the constraint that our point should lie on the given plane.

There are no boundary points to consider in this case since we are considering points (y,z) that vary throughout the entire yz-plane, so the absolute minimum must occur at a critical point. Note also that the extreme value theorem does not apply since the yz-plane is not bounded, so we have to rely on our geometric intuition to conclude that an absolute minimum should exist. The critical points satisfy

$$f_y = 2(-2y-5z)(-2)+2y=0 \quad \text{and} \quad f_z = 2(-2y-5z)(-5)+2(z-2)=0,$$

which simplify to

$$10y + 20z = 0 \quad \text{and} \quad 20y + 52z - 4 = 0.$$

The first equation vies $y = -2z$, and the second then gives

$$20(-2z) + 52z - 4 = 0, \text{ or } 12z = 4.$$

Hence $z = \frac{1}{3}$, so $y = -2(\frac{1}{3}) = -\frac{2}{3}$. The absolute minimum of $f(y,z) = (-2y-5z)^2 + y^2 + (z-2)^2$ thus occurs when $(y,z) = (-\frac{2}{3}, \frac{1}{3})$. The corresponding value of x is $x = 1 - 2(-\frac{2}{3}) - 5(\frac{1}{3}) = \frac{2}{3}$, so the point on $x + 2y + 5z = 1$ that is closest to $(1,0,2)$ is $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$.

Lagrange multipliers. The second Warm-Up above is an example of *constrained optimization*, where we optimize a function among points satisfying some constraining equation; in that case, we were optimizing (in the original phrasing)

$$f(x,y,z) = \sqrt{(x-1)^2 + y^2 + (z-2)^2}$$

subject to the constraint

$$x + 2y + 5z = 1.$$

We also had an example of this last time, when we were optimizing

$$f(x,y) = x^2y$$

subject to the constraint

$$3x^2 + 4y^2 = 12$$

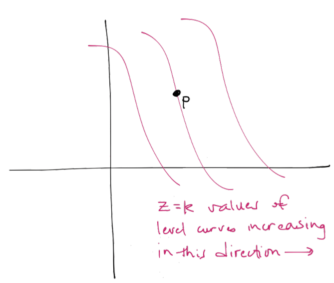
describing the ellipse which enclosed the region of interest. In both of these examples the approach was to use the constraint to eliminate a variable, and then optimize the resulting function dependeding on one less variable.

However, such an approach becomes challenging to carry out in other examples as soon as we work with more complicated functions and constraints. Instead, we consider a new approach, given by what's called the method of *Lagrange multipliers*. The goal of this method is, as described above, to optimize (meaning maximize or minimize) a function subject to a constraint. In the two-variable case we have a function $f(x,y)$ we want to optimize and the constraint is described by an equation of the form

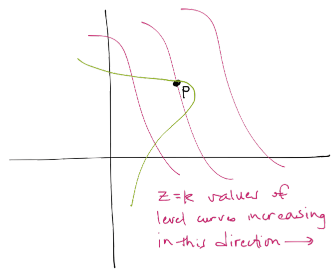
$$g(x,y) = k.$$

In the three-variable case, we have a three-variable function to optimize and the constraint will be described by a three-variable function as well, and so on.

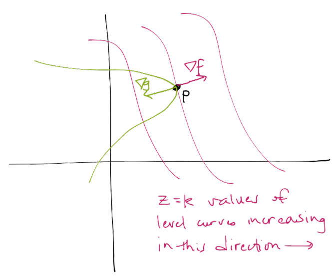
Here is the key geometric picture to have in mind, at least in the two-variable case. Say that the level curves of f look like



with the maximum of f among points satisfying the constraint occurring at the point P. The question is: what does the constraint curve have to look like in relation to these level curves? It should certainly pass through P if we are assuming P satisfies the constraint, but we can say more. The constraint curve cannot look like



since this would lead to points satisfying the constraint curve which give a *larger* value for f than P does, which is not possible if we are saying that P is where the maximum occurs. Thus, the constraint curve can only look like



with the point being that at a maximum the constraint curve and level curve must be *tangent* to each other. A similar reasoning shows that the same is true at a minimum.

Now, $\nabla f(P)$ is perpendicular to the level curve of f containing P and $\nabla g(P)$ is perpendicular to the constraint curve at P, so since these two curves are tangent to each other, these two gradients must be parallel to each other. Hence the conclusion is:

> At a point which gives the maximum or minimum value of f subject to the constraint determined by a function g, we must have $\nabla f = \lambda \nabla g$ for some scalar λ.

Thus, solving $\nabla f = \lambda \nabla g$ gives us the candidate points for the maximum/minimum of f subject to the constraint $g = k$. All this works for three-variable optimization problems as well. The scalar λ that shows up here is called the "Lagrange multiplier" of the problem, which is where the name of the method comes from. This scalar itself has a certain interpretation, but not one we will cover in this course.

Example. Let us revisit the problem of finding the absolute extrema of $f(x, y) = x^2y$ among points on the ellipse $3x^2 + 4y^2 = 12$. We thus take the constraint curve to be

$$g(x, y) = 3x^2 + 4y^2 = 12.$$

By the method of Lagrange multipliers, the absolute extrema we seek should be among the points satisfying the equation

$$\nabla f = \lambda \nabla g$$

for some scalar λ.

The equation above in this case becomes

$$\left\langle 2xy, x^2 \right\rangle = \lambda \left\langle 6x, 8y \right\rangle.$$

Comparing each component on both sides gives two equations, and the constraint gives a third:

$$\begin{aligned} 2xy &= \lambda 6x \\ x^2 &= \lambda 8y \\ 3x^2 + 4y^2 &= 12. \end{aligned}$$

The goal is to find (x, y) that satisfy all three of these, for some λ. (There is no set way of doing so that will work every single time, so it is important to get practice with these types of equations.) Here we note that dividing the first equation by the second gives

$$\frac{2xy}{x^2} = \frac{\lambda 6x}{\lambda 8y}, \text{ or } \frac{2y}{x} = \frac{6x}{8y}.$$

Now, anytime we divide we should be careful to not divide by zero. But here x, y, or λ might be zero, so such a manipulation would not be valid in these cases. However, if x or y is zero, our function value is just $f(0, y) = 0$ or $f(x, 0) = 0$, and 0 is not the absolute maximum nor minimum we want since x^2y can take on positive and negative values at other points on the ellipse $3x^2 + 4y^2 = 1$. Moreover, if $\lambda = 0$, then $2xy = \lambda 6x$ implies that at least one of x or y is zero, which we just pointed out does not give the absolute extrema we need. Thus, we may as well assume that none of x, y, λ are zero.

If none of x, y, λ are zero, then the equation $\frac{2y}{x} = \frac{6x}{8y}$ is valid, and hence $16y^2 = 6x^2$. Thus $8y^2 = 3x^2$, and plugging into the constraint gives

$$8y^2 + 4y^2 = 12.$$

Thus $y = \pm 1$, and then $3x = 8y^2 = 8$, so $x = \pm\sqrt{\frac{8}{3}}$. Hence we get four points satisfying the Lagrange multiplier equations overall, namely

$$(\sqrt{\tfrac{8}{3}}, 1),\ (\sqrt{\tfrac{8}{3}}, -1),\ (-\sqrt{\tfrac{8}{3}}, 1),\ (-\sqrt{\tfrac{8}{3}}, -1).$$

The absolute extrema of $f(x, y) = x^2y$ among points satisfying $3x^2 + 4y^2 = 12$ are those among these points, and we simply plug into to see which is which; we get the same answer as last time, where the absolute maximum is $\frac{8}{3}$ and occurs at the points with positive y-coordinate, and the absolute minimum is $-\frac{8}{3}$, occurring at the points with negative y-coordinate.

Lecture 27: More on Multipliers

Warm-Up 1. We find the point on $x + 2y + 5z = 1$ that is closest to $(1, 0, 2)$, now using the method of Lagrange multipliers. We thus want to minimize the function

$$f(x, y, z) = (x - 1)^2 + y^2 + (z - 2)^2$$

subject to the constraint

$$g(x, y, z) = x + 2y + 5z = 1.$$

(The distance from (x, y, z) to $(1, 0, 2)$ is actually the square root of the function defining f, but, as we point out before, minimizing this square root is the same as minimizing f.) By the method of Lagrange multipliers, the minimum occurs among the points satisfying

$$\nabla f = \lambda \nabla g, \text{ which is } \langle 2(x-1), 2y, 2(z-2)\rangle = \lambda \langle 1, 2, 5\rangle .$$

By comparing components and including the constraint, we get the equations

$$\begin{aligned} 2(x-1) &= \lambda \\ 2y &= 2\lambda \\ 2(z-2) &= 5\lambda \\ x + 2y + 5z &= 1. \end{aligned}$$

The first equation gives $x = 1 + \frac{1}{2}\lambda$; the second gives $y = \lambda$; and the third gives $z = 2 + \frac{5}{2}\lambda$. Plugging these into the constraint gives

$$(1 + \tfrac{1}{2}\lambda) + 2\lambda + 5(2 + \tfrac{5}{2}\lambda) = 1,$$

which gives $\lambda = -\frac{2}{3}$. Thus

$$x = 1 + \tfrac{1}{2}(-\tfrac{2}{3}) = \tfrac{2}{3}, \; y = -\tfrac{2}{3}, \text{ and } z = 2 + \tfrac{5}{2}(-\tfrac{2}{3}) = \tfrac{1}{3}.$$

Hence we get that the point on $x + 2y + 5z = 1$ closest to $(1, 0, 2)$ is $(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$, which agrees with the answer we found before.

Warm-Up 2. We find the largest possible product among three positive numbers x, y, z whose sum is 100. So, we want to maximize the function

$$f(x, y, z) = xyz$$

subject to the constraint

$$g(x, y, z) = x + y + z = 100.$$

The maximum occurs among points satisfying $\lambda f = \lambda \nabla g$, which in this case is

$$\langle yz, xz, xy\rangle = \lambda \langle 1, 1, 1\rangle .$$

Thus the equations to solve, including the constraint, are

$$\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \\ x + y + z &= 100. \end{aligned}$$

The first two equations immediately give $yz = xz$. Now, we are only considering positive numbers, so we may assume that each of x, y, z is nonzero. (Even if one were zero, the value of f would then by 0, which is not the maximum we want since in particular $x = 1, y = 1, z = 98$ satisfy the constraint and give $xyz = 98$.) Thus we can divide by z to get $x = y$. Similarly, we have $xz = xy$, and dividing by the nonzero number y gives $x = z$. Hence, the product xyz is maximized when x, y, z are all the same. Plugging into the constraint gives

$$x + x + x = 100, \text{ so } x = \frac{100}{3},$$

and thus $y = z = \frac{100}{3}$ as well. We know these values give a maximum instead of a minimum because the product in this case is $\frac{100^3}{3^3}$, which is larger than the value 98 we get when $x = 1 = y, z = 98$.

Another example. Consider a rectangular box. We want to determine the dimension of the box which result in the maximum possible volume among those boxes with surface area 100. Denoting the dimensions by x, y, z (z is height) we thus want to maximize the volume function $f(x, y, z) = xyz$ subject to the constraint

$$g(x, y, z) = 2xy + 2yz + 2xz = 100,$$

which comes from figuring out the surface area of the box. Then $\nabla f = \lambda \nabla g$ becomes

$$(yz, xz, xy) = \lambda(2y + 2z, 2x + 2z, 2x + 2y).$$

Equating components and including the constraint gives the equations

$$\begin{aligned} yz &= \lambda(2y + 2z) \\ xz &= \lambda(2x + 2z) \\ xy &= \lambda(2x + 2y) \\ 2xy + 2yz + 2xz &= 100. \end{aligned}$$

To solve these, note that the left sides of the first three equations are pretty similar and become equal after multiplying the first equation through by x, the second by y, and the third by z:

$$\begin{aligned} xyz &= \lambda(2xy + 2xz) \\ xyz &= \lambda(2xy + 2yz) \\ xyz &= \lambda(2xz + 2yz). \end{aligned}$$

Then subtracting the first two equations gives

$$0 = 2\lambda z(x - y).$$

None of the dimensions can be zero since this certainly wouldn't give a maximum volume (we wouldn't even really have a box at all), and λ can't be zero since this would imply that some of the dimensions were zero. Thus we must have

$$x - y = 0, \text{ so } x = y.$$

Subtracting the first and third equations from before gives

$$0 = 2\lambda y(x - z).$$

Again, λ and y are not zero so $x - z = 0$ and hence $x = z$. Thus so far we know that the dimensions of the box we're looking for will result in the length, width, and height all being the same.

Now we find the exact values of $x.y.z$. Substituting $x = y = z$ into the constraint gives

$$2y^2 + 2y^2 + 2y^2 = 100, \text{ so } y = \frac{10}{\sqrt{6}}.$$

(We ignore the negative square root since y should be a positive width.) Hence we have

$$x = \frac{10}{\sqrt{6}}, \ y = \frac{10}{\sqrt{6}}, \ z = \frac{10}{\sqrt{6}}.$$

To show that these dimensions indeed give a maximum volume and not a minimum volume, we argue as follows. Consider shrinking the height and width of the box but at the same time increasing the length so that the surface area stays fixed at 100. Then the volume, because the height and width are approaching 0, will approach zero as well. Since we can make the volume arbitrarily small while keeping the surface area at 100, there is no minimum volume so the dimensions we found must give a maximum volume.

Final example. Suppose we are constructing an open (i.e. no lid) can in the shape of a cylinder, where the material for the base costs $\$5/\text{cm}^2$ and the material for the upright side costs $\$2/\text{cm}^2$. We determine the dimensions which minimize the cost of constructing the can if we want the volume to be 40π cm^3.

Letting r, h denote the radius and height, the total cost of making the can is

$$f(r,h) = 5\pi r^2 + 4\pi rh,$$

which is obtained by multiplying the area of the base and side by the corresponding cost per unit area. Thus we want to minimize f subject to the constraint $g(r,g) = \pi r^2 h = 40\pi$. Lagrange multipliers gives the equation

$$(10\pi r + 4\pi h, 4\pi r) = \lambda(2\pi rh, \pi r^2),$$

so the dimensions we want must satisfy

$$\begin{aligned} 10\pi r + 4\pi h &= 2\lambda\pi rh \\ 4\pi r &= \lambda\pi r^2 \\ \pi r^2 h &= 40\pi. \end{aligned}$$

We can assume r and h are nonzero since otherwise the volume could not 40π, and hence we can also assume $\lambda \neq 0$ since otherwise the second equation above would give $r = 0$. The second equation then gives

$$r = \frac{4}{\lambda}.$$

Substituting into the first equation gives

$$10\pi\left(\frac{4}{\lambda}\right) + 4\pi h = 2\lambda\pi\left(\frac{4}{\lambda}\right)h,$$

which simplifies to $h = \frac{10}{\lambda}$. Comparing $r = \frac{4}{\lambda}$ and $h = \frac{10}{\lambda}$ gives

$$h = \frac{5}{2}r,$$

and plugging this into the constraint gives

$$\pi r^2\left(\frac{5}{2}r\right) = 40\pi, \text{ so } r^3 = \frac{80}{5}.$$

Hence $r = \sqrt[3]{80/5}$ and $h = \frac{5}{2}\sqrt[3]{80/5}$ are the dimensions which minimize cost.

To be sure that this gives a minimum and not a maximum, note that we can increase r and decrease $h = \frac{40}{r^2}$ accordingly to keep the volume at 40π, and this will lead to larger and larger costs since increasing the area of the base has a greater effect on cost than decreasing the height. Thus there can be no maximum cost, so the dimensions we found indeed give minimum cost.