Math 290-3: Linear Algebra & Multivariable Calculus
Northwestern University, Lecture Notes

Written by Santiago Cañez

These are notes which provide a basic summary of each lecture for Math 290-3, the third quarter of "MENU: Linear Algebra & Multivariable Calculus", taught by the author at Northwestern University. The book used as a reference is the 4th edition of *Vector Calculus* by Colley. Watch out for typos! Comments and suggestions are welcome.

Contents

Lecture 1: Double Integrals

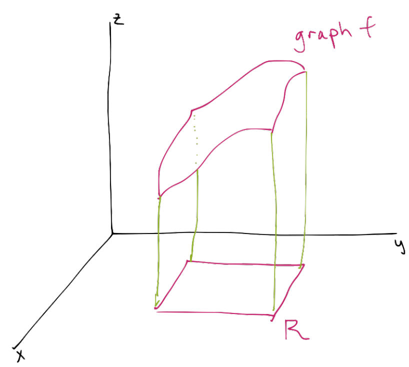
Final quarter of the year! Today I gave a brief overview of the course, which is all about integration. We then started with material on *double integrals*, the first types of integrals we will consider.

Double integrals. Recall that in single-variable calculus integrals compute areas, namely the area of the region under the graph of a (single-variable) function and above some interval. Analogously, integrals of functions of two variables compute volumes in the following sense.

Say that f is a non-negative (meaning its graph is always above the xy-plane) function of two variables and that R is a rectangle in the xy-plane. The *double integral* of f over R, denoted by

$$\iint_R f(x,y)\,dA$$

gives the volume of the region under the graph of f and above R:



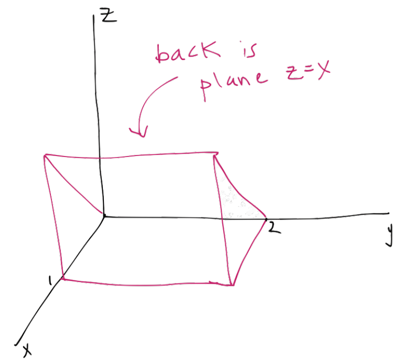
For now the "dA" is merely a notational device, but it will help to think of it as representing an "infinitesimal" piece of area. We'll say something about how such integrals are precisely defined next time, but their interpretation in terms of volumes is what will be most important.

This also makes sense even for a function whose graph dips below the xy-plane, only we consider the "volume" of such a piece to be negative, so that a double integral in general gives a type of "net" volume.

Example 1. We compute $\iint_{[0,1]\times[0,1]} 2\,dA$. The notation $[0,1]\times[0,1]$, pronounced "[0, 1] cross [0, 1]" denotes the rectangle consisting of points with x-coordinate between 0 and 1 and y-coordinate also between 0 and 1. This double integral should give the volume under the graph of $f(x,y) = 2$, which is just a horizontal plane at a height of 2, and above $[0,1] \times [0,1]$. But this region is just a rectangular box of height 2 and length and width both 1, so its volume is 2. Thus

$$\iint_{[0,1]\times[0,1]} 2\,dA = 2.$$

Example 2. Now we consider $\iint_{[0,1]\times[0,2]} x\,dA$. The graph of $f(x,y) = x$ is the slanted plane $z = x$, which is obtained by taking the line $z = x$ in the xz-plane and sliding it out in the y-directions, so the region we're wanting the volume of looks like:



Note that in this case our rectangle consists of points whose x-coordinates lies between 0 and 1 and y-coordinate lies between 0 and 2.

The point is that we can determine this volume geometrically even though we don't yet know how to compute double integrals in general. Note that the region in consideration is exactly half of a rectangular box of height and width 1 and length 2. This box has volume 2, so our region has volume 1:

$$\iint_{[0,1]\times[0,2]} x\,dA = 1.$$

We can also see this by noting that the volume of this region is precisely the area of one of its side triangles times the length of the region. The side triangle on the xz-plane has area $\frac{1}{2}$, which is half the base times height, so the entire region has volume $\frac{1}{2}\cdot 2 = 1$, as we saw before.

Iterated integrals. Using geometry to compute these volumes is nice, but will not always work since our regions might be more complicated. Instead, we can always compute double integrals using *iterated integrals*. For instance, the double integral in Example 2 can also be computed using the expression:

$$\iint_{[0,1]\times[0,2]} x\,dA = \int_0^2 \int_0^1 x\,dx\,dy.$$

The expression on the right side is an iterated integral where we have chosen to integrate with respect to x first and then with respect to y, based on the order of the dx and dy terms. The bounds on these iterated integrals match up with the order we are integrating with respect to: the bounds on the "inner" integral with respect to x are the bounds on x in the rectangle and the bounds on the "outer" integral with respect to y are the bounds on y coming from the rectangle.

We compute an iterated integral by first evaluating the "inner" integral and then "outer" one. In this case, the inner integral is

$$\int_0^1 x\,dx = \frac{x^2}{2}\bigg|_0^1 = \frac{1}{2},$$

where we integrate with respect to x as you normally would in single-variable calculus. (If our function involved y we would simply treat y as a constant when integrating with respect to x.) Then we're left with computing the outer integral with respect to y, so overall we have:

$$\begin{aligned}\iint_{[0,1]\times[0,2]} x\,dA &= \int_0^2 \left(\int_0^1 x\,dx\right)\,dy \\ &= \int_0^2 \frac{x^2}{2}\bigg|_0^1\,dy\end{aligned}$$

$$= \int_0^2 \frac{1}{2}\, dy$$
$$= \frac{y}{2}\Big|_0^2$$
$$= 1,$$

which agrees with the value we found previously using geometry.

Instead we could have integrated with respect to y first and then x:

$$\iint_{[0,1]\times[0,2]} x\, dA = \int_0^1 \int_0^2 x\, dy\, dx.$$

Note that the bounds change accordingly. For the inner integral with respect to y, x is treated as a constant so we get:

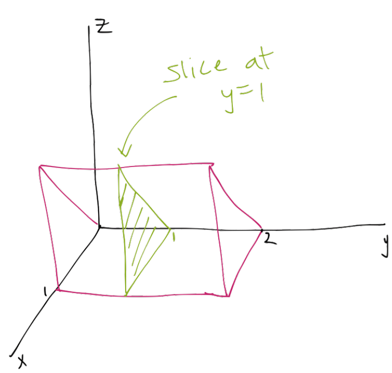
$$\begin{aligned}\iint_{[0,1]\times[0,2]} x\, dA &= \int_0^1 \left(\int_0^2 x\, dy\right) dx \\ &= \int_0^1 xy\Big|_0^2 dx \\ &= \int_0^1 2x\, dx \\ &= x^2\Big|_0^1 \\ &= 1\end{aligned}$$

which again agrees with the value found before. (In the third step we substituted 2 and 0 in for y only since that is what we integrated with respect to before that point.)

Why do iterated integrals work? This is all well and good, but we should make clear why it is that iterated integrals give a valid way to compute double integrals. The point is that when computing an iterated integral we are essentially "slicing" the region in question into pieces, finding the areas of each of these slices, and then adding these areas together to get the total volume. For instance, consider the iterated integral

$$\int_0^2 \int_0^1 x\, dx\, dy$$

from before, giving the volume of the region drawn in Example 2. The inner integral here is giving the areas of the triangles obtained by slicing this region up along different values of y:



so the iterated integral should be thought of as

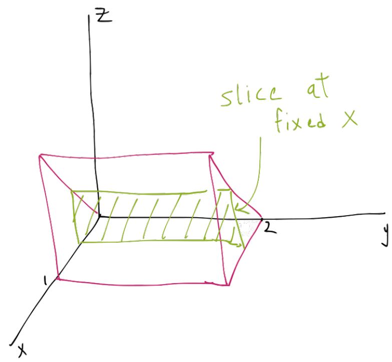
$$\int_0^2 \int_0^1 x\,dx\,dy = \int_0^2 (\text{area of slice at some fixed } y)\,dy.$$

Now, recalling that an integral should intuitively be thought of as some kind of infinite sum, this last expression says that we should add together the areas of all these slices as y ranges from 0 to 2, and it makes sense that doing so should give the volume of our entire region since these slices all together fill out that volume.

Similarly, the iterated integral

$$\int_0^1 \int_0^2 x\,dy\,dx$$

uses slices in the x-direction. Indeed, the inner integral gives the area of the rectangle obtained by slicing our region up at different values of x:



Thus

$$\int_0^1 \int_0^2 x\,dy\,dx = \int_0^1 (\text{area of slice at some fixed } x)\,dx$$

and again it makes sense that adding up the areas of these slices as x varies from 0 to 1 should give the overall volume since these slices also fill out the entire volume.

Fubini's Theorem. Under some assumptions which will always be satisfied by the types of functions we will consider in this course, for a function f of two variables and a rectangle $[a,b]\times[c,d]$ we have

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Geometrically this says that the volume of the region under the graph of f and above the rectangle can be obtained either by slicing that region up in the y-direction (as in the first iterated integral) and adding together the areas of the resulting slices, or by slicing that region up in the x-direction (as in the second iterated integral) and adding together the areas of the resulting slices.

Example 3. We compute $\iint_{[1,2]\times[-1,0]} 12x^2y^3\,dA$. Using an iterated integral with respect to x first we have:

$$\iint_{[1,2]\times[-1,0]} 12x^2y^3\,dA = \int_{-1}^0 \int_1^2 12x^2y^3\,dx\,dy$$

$$
\begin{aligned}
&= \int_{-1}^{0} 4x^3y^3\Big|_1^2 \, dy \\
&= \int_{-1}^{0} 28y^3 \, dy \\
&= 7y^4\Big|_{-1}^{0} \\
&= -7.
\end{aligned}
$$

Instead, integrating with respect to y first gives:

$$
\begin{aligned}
\iint_{[1,2]\times[-1,0]} 12x^2y^3 \, dA &= \int_1^2 \int_{-1}^{0} 12x^2y^3 \, dy \, dx \\
&= \int_1^2 3x^2y^4\Big|_{-1}^{0} \, dx \\
&= \int_1^2 -3x^2 \, dx \\
&= -x^3\Big|_1^2 \\
&= -7.
\end{aligned}
$$

Note that it makes sense we get a negative value for this double integral: for y in the interval $[-1, 0]$, y^3 is always ≤ 0 so since x^2 is always non-negative the function $12x^2y^3$ only takes on values ≤ 0 over the given rectangle, meaning that its graph lies below the xy-plane.

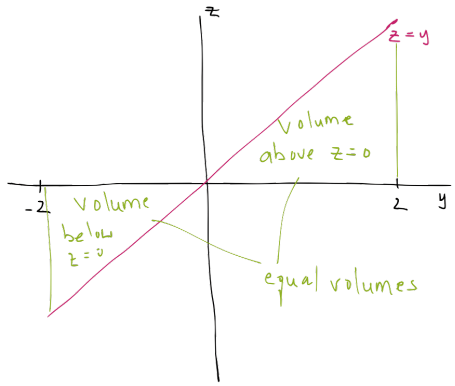
Lecture 2: Riemann Sums

Today we took a step back and spoke about how double integrals are actually defined via Riemann sums. Although this won't be too important of a concept in this course, the precise definition along with the notion of what it means for a function to be *integrable* is very important in later courses. (Take a real analysis course, such as the Math 320 sequence, to truly understand what integration means.)

Warm-Up 1. Without doing any computation, we want to give a geometric reason as to why

$$
\iint_{[0,100]\times[-2,2]} y \, dA = 0.
$$

The graph of $f(x, y) = y$ is the plane $z = y$, whose intersection with the yz-plane is the line $z = y$. Now, the right side of this is above the xy-plane while the left is below, so the region we're considering lies above the xy-plane for $y > 0$ and below the xy-plane for $y < 0$:



(The region in questions extends out from the yz-plane at $x = 0$ out to $x = 100$.) The point is that the volume of the right side of this region is the same as that of the left side (since $z = y$ is symmetric across the z-axis), but these two "volumes" are counted with opposite signs so they cancel each other out. Thus the "net" volume of our region is indeed 0.

The moral is: using geometry or symmetry can make certain (but not all) integral computations simpler.

Warm-Up 2. We determine the volume of the region under the surface $z = xe^{xy}$ and above the rectangle $[1,2] \times [1,3]$ in the xy-plane. The given surface is the graph of $f(x,y) = xe^{xy}$, so this volume is given by the double integral

$$\iint_{[1,2]\times[1,3]} xe^{xy}\,dA.$$

We'll try to compute this using iterated integrals in both possible orders.

Integrating with respect to y first gives:

$$\begin{aligned}\iint_{[1,2]\times[1,3]} xe^{xy}\,dA &= \int_1^2 \int_1^3 xe^{xy}\,dy\,dx \\ &= \int_1^2 e^{xy}\Big|_1^3\,dx \\ &= \int_1^2 (e^{3x} - e^{x})\,dx \\ &= \left(\frac{1}{3}e^{3x} - e^{x}\right)\Big|_1^2 \\ &= \frac{1}{3}e^6 - e^2 - \frac{1}{3}e^3 + e.\end{aligned}$$

Note that in the second step, since we treat x as a constant when integrating with respect to y, the anti-derivative of xe^{xy} with respect to y is simply e^{xy}.

Integrating with respect to x first gives:

$$\begin{aligned}\iint_{[1,2]\times[1,3]} xe^{xy}\,dA &= \int_1^3 \int_1^2 xe^{xy}\,dx\,dy \\ &= \int_1^3 \left(\frac{x}{y}e^{xy} - \frac{1}{y^2}e^{xy}\right)\Big|_1^2\,dy\end{aligned}$$

where we had to use integration by parts. Specifically, setting $u = x$ and $dv = e^{xy}\,dx$, we have

$$du = dx \text{ and } v = \frac{1}{y}e^{xy},$$

so the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

gives

$$\int xe^{xy}\,dx = \frac{x}{y}e^{xy} - \int \frac{1}{y}e^{xy} = \frac{x}{y}e^{xy} - \frac{1}{y^2}e^{xy}$$

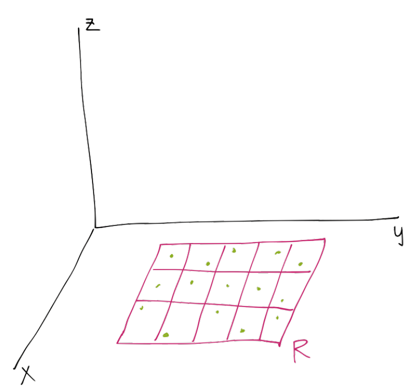
as we used above. Continuing on we have:

$$\int_1^3 \left(\frac{x}{y}e^{xy} - \frac{1}{y^2}e^{xy}\right)\Big|_1^2 dy = \int_1^3 \left(\frac{2}{y}e^{2y} - \frac{1}{y^2}e^{2y} - \frac{1}{y}e^{y} + \frac{1}{y^2}e^{y}\right) dy$$

and now we're stuck as not even integration parts will help us. Luckily, we don't have to continue further since we already computed the double integral we wanted above when integrating with respect to y first.

The moral is: choosing a good order of integration often times makes all the difference in the world.

Riemann sums. As in single-variable calculus, the precise definition of a double integral is given in terms of Riemann sums, which make clear why it is that double integrals actually compute volume. Suppose that f is a two-variable function and that R is some rectangle. We imagine splitting R up into smaller rectangles somehow, and over each smaller piece we choose a "sample" point:



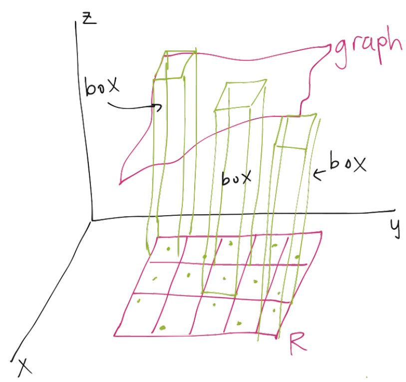
Over each smaller rectangle we construct a rectangular box whose base is that smaller rectangle and whose height is given by the value of the function at the corresponding sample point; the volume of this box is then

$$f(\text{sample point}) \cdot (\text{area of smaller rectangle}).$$

The associated *Riemann sum* is then the sum of these individual volumes, so:

$$\text{Riemann sum} = \sum_{\text{smaller rectangle}} f(\text{sample point}) \cdot (\text{area of smaller rectangle}).$$

Geometrically, these rectangular boxes look like:

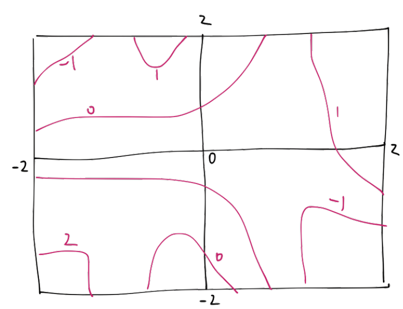


and the point is that their combined volume should provide an approximation to the actual volume under the graph of f. (The book has a better picture of such boxes.) Breaking our original rectangle into even smaller pieces results in an even better approximation, and the double integral of f over R is thus defined to be the "limit" of these Riemann sums as the smaller rectangles get arbitrarily small:

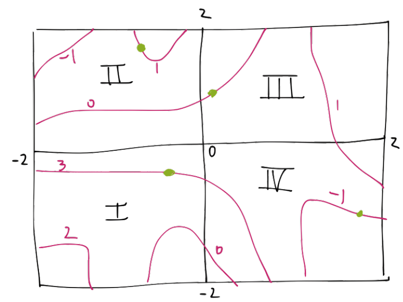
$$\iint_R f(x,y)\,dA = \lim_{\text{as small rectangles get smaller and smaller}} \text{of Riemann sums.}$$

We won't get more precise than this in this here, but again this is something you would study more carefully in a real analysis course.

Example 1. Consider the rectangle $[-2,2]\times[-2,2]$ and suppose we have a function f whose level curves look like:



Break this rectangle up into four smaller rectangles using the x- and y-axes, and in each of these pick the following sample points (in green):



We determine the Riemann sum corresponding to these choices.

Rectangle I has area 4 and for the chosen sample point we have

$$f(\text{sample point}) = 1.$$

Thus this rectangle contributes

$$f(\text{sample point}) \cdot \text{area} = 1 \cdot 4 = 4$$

to the Riemann sum. Similarly, rectangle II contributes

$$f(\text{sample point}) \cdot \text{area} = 0 \cdot 4 = 0,$$

rectangle III contributes

$$f(\text{sample point}) \cdot \text{area} = 3 \cdot 4 = 12,$$

and rectangle IV contributes

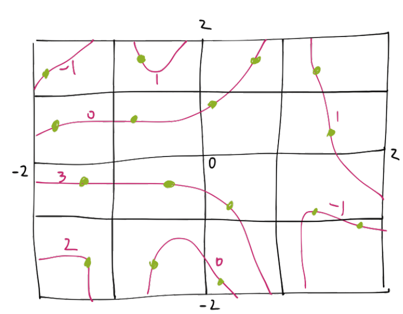
$$f(\text{sample point}) \cdot \text{area} = -1 \cdot 4 = -4.$$

The associated Riemann sum is thus

$$4 + 0 + 12 - 4 = 12,$$

which is one approximation to the value of $\iint_{[-2,2]\times[-2,2]} f(x,y)\,dA$.

To get a better approximation let's divide our original rectangle into even smaller pieces by dividing each of the four rectangles above into four smaller ones. This gives 16 total small rectangles and we choose the following sample points from each:



Each of these smaller rectangles has area 1. The associated Riemann sum will now have 16 terms, which are (starting in the upper left rectangle and moving to the right):

$$-1\cdot 1+1\cdot 1+0\cdot 1+1\cdot 1+0\cdot 1+0\cdot 1+0\cdot 1+1\cdot 1+3\cdot 1+3\cdot 1+3\cdot 1-1\cdot 1+2\cdot 1+0\cdot 1+0\cdot 1-1\cdot 1,$$

which gives a value of 11 for this Riemann sum.

This might seem like a lot of work, but the point is that since in this case we don't know explicitly what f is—all we have are some of its level curves—we can't compute its double integral directly and approximating it using Riemann sums is the best we can do. This is definitely something you may come across in later courses.

Example 2. Let's use Riemann sums to justify some basic properties of double integrals. First, say we have two functions f and g with $f(x,y)\leq g(x,y)$ for every point (x,y). Then

$$\iint_R f(x,y)\,dA \leq \iint_R g(x,y)\,dA,$$

meaning that if two functions obey some inequality then their double integrals do as well. Indeed, when we break up R into smaller pieces and choose some sample points, over each smaller rectangle we'll have

$$f(\text{sample point})\cdot\text{area} \leq g(\text{sample point})\cdot\text{area}$$

since $f(\text{sample point})\leq g(\text{sample point})$ and the areas are the same. Thus the sum of such expressions also obeys the same inequality, so

$$\text{Riemann sum for } f \leq \text{Riemann sum for } g$$

and hence the limits of such sums also obeys the same inequality, which gives the desired inequality among double integrals.

Now, it is also true that

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$$

Indeed, a Riemann sum for $f+g$ on the left looks like

$$\sum_{\text{smaller rectangle}} [f(\text{sample point} + g(\text{sample point})]\cdot(\text{area of smaller rectangle}),$$

which after distributing becomes

$$\sum_{\text{smaller rectangle}} f(\text{sample point})\cdot\text{area} + \sum_{\text{smaller rectangle}} g(\text{sample point})\cdot\text{area}.$$

Thus

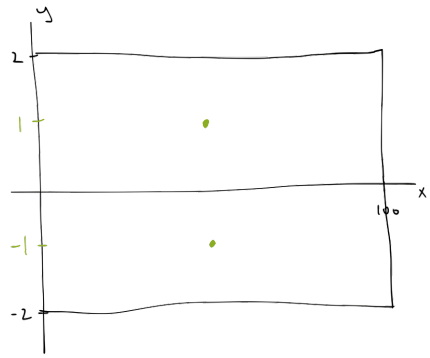
$$\text{Riemann sum for } f+g \;=\; \text{Riemann sum for } f \;+\; \text{Riemann sum for } g,$$

and taking limits of both sides as the smaller rectangles get smaller gives the desired equality among double integrals.

Example 3. Finally, let's use Riemann sums to give another way to approach the first Warm-Up. Recall that we wanted to determine

$$\iint_{[0,100]\times[-2,2]} y\,dA.$$

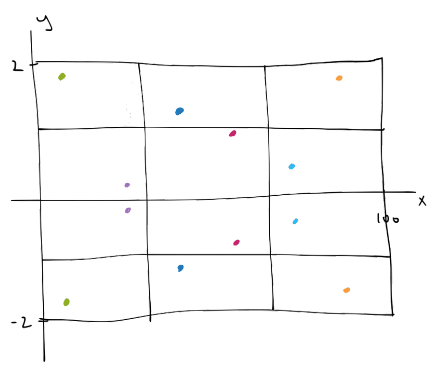
To start, let's break our rectangle up into two pieces: the top half and bottom half. In each of these let's choose as a sample point the center:



The value of the function $f(x, y) = y$ at the same sample point is 1 and its value at the bottom sample point is -1, so since the areas of both smaller rectangles is 200 we get zero for the value of the Riemann sum:

$$1 \cdot 200 + (-1) \cdot 200 = 0.$$

Now imagine we keep dividing our rectangle up into smaller and smaller pieces. For any sample point we pick in one of the "upper" rectangles let's pick as a sample point in a corresponding "lower" rectangle a point whose y-coordinate (i.e. value of $f(x, y) = y$) is negative that of the "upper" sample point, so something like:



where sample points of the same color are the ones which have opposite y-coordinates. Note that the contribution to the total Riemann sum from the rectangles with the green sample points is

$$(\text{some } y\text{-value}) \cdot \text{area} + (\text{negative that same } y\text{-value}) \cdot \text{area} = 0.$$

Similarly, for any term in the associated Riemann sum because of the sample points we chose we can find another term which will cancel out the first one, which will give a total Riemann sum equal to 0. Since we can do this no matter how small our rectangles get, the limit of Riemann sums should also be zero and we get

$$\iint_{[0,100]\times[-2,2]} y \, dA = 0$$

as desired.

Important. Apart from showing up in the actual definition of a double integral, Riemann sums are useful for approximating integrals and in justifying basic properties of integrals.

Integrability. There is one point which we haven't touched on yet, and which indeed is usually never mentioned in a single-variable calculus course: if an integral is defined as a limit of Riemann sums, how can we be certain that limit exists? The answer is that we can't, which leads to what it means for a function to be "integrable":

> Given a rectangle R, a function of two variables f is said to be *integrable* over R if the limit of Riemann sums used to define the double integral of f over R actually exists.

This is never an issue for us since we really only deal with continuous functions, and continuous functions are always integrable (as we'll see in a bit). But, this again is definitely something which is important in latter applications. (And here I'll give one final plug for real analysis, a truly amazing course.)

For instance, consider the function

$$f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

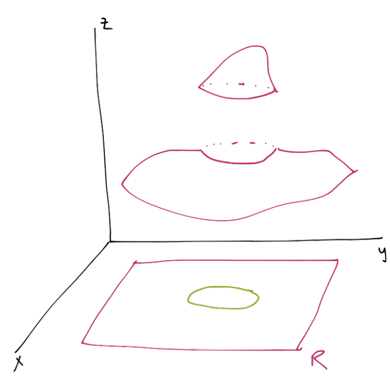
It turns out that for any rectangle R which contains the origin, f is not integrable over R, meaning it does not have a well-defined double integral over R. The reason is that f is *unbounded*, meaning that its values get larger and larger as $(x,y) \to (0,0)$. The fact that you can find points close to $(0,0)$ which give any large value of f you want will actually imply that the limit of Riemann sums does not exist. (This is not something we'll have to understand further in this course.)

Here then is our saving grace:

> **Theorem.** If f is bounded and the set of points in R where it is not continuous has zero area, then f is integrable over R.

In particular, if f is continuous then the set of points where it is not continuous is "empty" and the empty set definitely has zero area, so continuous functions are ALWAYS integrable. The upshot is that this is the reason why we'll never have to actually worry about whether or not integrals exist in this class, since all functions we'll consider will be continuous.

Final example. Here's an example of a non-continuous function that still has a well-defined integral. Say we have a function f whose graph looks like:



So, the graph of f "jumps" from the lower surface to the higher surface along the green circle. This function is bounded, and the set of points where it is not continuous are precisely those points on the green circle; this circle as zero area, so f is integrable over the given rectangle R. The value of its double integral would be the volume under the lower surface plus the volume under the higher surface.

One clarification: when we say that the circle has zero area we do NOT mean that the region which it *encloses* has zero area (this region has positive area), but rather that the curve consisting of only the circle itself has zero area. Indeed, *any* 1-dimensional curve has zero area even though it might enclose a region which does not.

Lecture 3: More on Double Integrals

Today we got back to the business of computing double integrals, in particular double integrals over non-rectangular regions. The bulk of the work here goes into setting up the correct bounds of integration.

Warm-Up. We justify the following basic property of integrals:

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA,$$

where c is a scalar, using Riemann sums. After dividing R up into smaller rectangles, a Riemann sum for the left side looks like

$$\sum cf(\text{sample point})\cdot\text{area}.$$

But each term in this sum is being multiplied by the same constant c, so we can pull c out of the sum to get

$$c\sum f(\text{sample point})\cdot\text{area}.$$

Now we see that this last expression is just c times a Riemann sum for f, so we get that

$$\text{Riemann sum for } cf \; = c(\text{Riemann sum for } f).$$

Hence after taking limits of both sides as our smaller rectangles get smaller and smaller, we get the desired equality among the double integrals above.

Back to Fubini's Theorem. Luckily for us instead of having to actually use Riemann sums to compute double integrals, we have Fubini's Theorem available, which says that we can compute double integrals using iterated integrals. Previously we gave an intuitive geometric reason for this based on the method of "slicing", but now note how surprising this theorem actually is when viewed from the point of view of Riemann sums.

If you fully unwind the precise definition of an iterated integral like

$$\int_a^b\int_c^d f(x,y)\,dy\,dx,$$

you'll see the inner integral requires one limit of Riemann sums while the outer integral requires another, so that in the end the iterated integral becomes a "limit of Riemann sums of limits of Riemann sums". On the other hand,

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dA$$

requires one single limit of (two-variable) Riemann sums, and at a quick glance there is no reason to expect that

limit of two-variable Riemann sums = limit of Riemann sums of limits of Riemann sums

should be true. But of course, this is precisely what Fubini's Theorem says, which is why this fact is so-deserving of it's own name.

The full version of Fubini's Theorem actually requires some assumptions about the function $f(x, y)$—check the book—but these will always be satisfied by the continuous functions we'll consider, which is why we'll never have to justify the fact that Fubini's Theorem is applicable.

Double integrals over general regions. Double integrals over non-rectangular regions have the same interpretation as double integrals over rectangles: given a function $f(x, y)$ and a region D in the xy-plane, the double integral

$$\iint_D f(x, y)\, dA$$

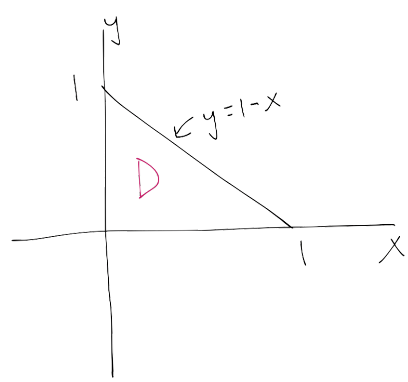
gives the "volume" of the solid bounded by the graph of f and the region D. Fubini's Theorem still applies to say that we can compute these using iterated integrals, and the twist now comes in dealing with the bounds of integration, which no longer need be constant as we'll see in some examples.

Important. For a region D in the xy-plane and a function f, we have

$$\iint_D f(x, y)\, dA = \int_{\text{min value of outer}}^{\text{max value of outer}} \int_{\text{lower bound of inner in terms of outer}}^{\text{upper bound of inner in terms of outer}} f(x, y)\, d(\text{inner})\, d(\text{outer})$$

where "inner" denotes whatever we pick to be the inside variable and "outer" whatever we pick to be the outside variable. In particular, note that the outer bounds will **ALWAYS** be constant and the inner bounds can **ONLY** depend on the outer variable.

Example 1. Let's find the volume of the solid region under the graph of $f(x, y) = 2xy$ and above the triangle D bounded by the x- and y-axes and the line $y = 1 - x$:



This volume is given by the double integral

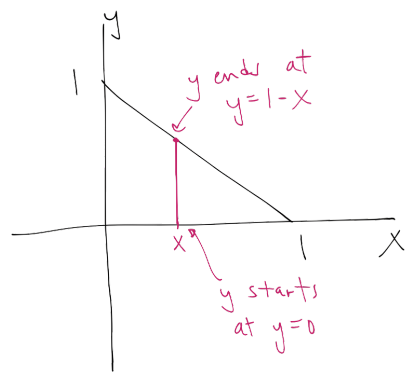
$$\iint_D 2xy\, dA.$$

Let's integrate with respect to y first, so we'll have an iterated integral of the form

$$\iint_D 2xy\, dA = \int_?^? \int_?^? 2xy\, dy\, dx.$$

The outer bounds simply come from the min and max values which the outside variable x takes over the entire region D, which in this case are 0 and 1 respectively.

Now, for the inner bounds we ask: at a fixed value of the outside variable x, where does y start at and where does y end at in our region? In this case, at any x the value of y always starts at $y = 0$ on the x-axis and increases until it hits the line $y = 1 - x$:



Thus the inner bounds on y should be 0 to $x - 1$, so we get

$$\iint_D 2xy\, dA = \int_0^1 \int_0^{1-x} 2xy\, dy\, dx$$

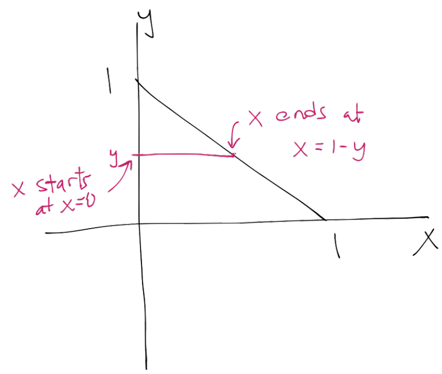
as the expression for our double integral. Note that we shouldn't take the upper bound on y to be 1 since this would say that we are including points above the line $y = x - 1$ and below $y = 1$ in our region, which would only happen if we were integrating over the rectangle $[0, 1] \times [0, 1]$. Finally, we compute this iterated integral using the same method as before, by first computing the inner integral and then the outer one:

$$\begin{aligned}
\iint_D 2xy\, dA &= \int_0^1 \int_0^{1-x} 2xy\, dy\, dx \\
&= \int_0^1 xy^2 \Big|_0^{1-x} dx \\
&= \int_0^1 x(1-x)^2\, dx \\
&= \int_0^1 (x^3 - 2x^2 + x)\, dx \\
&= \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 \right) \Big|_0^1 \\
&= \frac{1}{12}.
\end{aligned}$$

Let's also verify that we get the same result integrating with respect to the opposite order, so we want an iterated integral of the form:

$$\iint_D 2xy\, dA = \int_?^? \int_?^? 2xy\, dx\, dy.$$

The bounds on the outer variable y are again the min and max value of y in D (so 0 and 1 again), and for the inner bounds on x we determine where x starts at on the left and where x ends at on the right at a fixed y:



At any y, x starts on the y-axis at $x = 0$ and increases (to the right) until it hits the line $y = 1 - x$, which is the same as $x = 1 - y$. Hence the inner bounds on x should be 0 to $1 - x$, so we get

$$\iint_D 2xy\,dA = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx.$$

Note that in this case switching the order of integration just had the effect of exchanging x and y in the bounds, but this is really just a coincidence that happens in this particular example and is NOT something which will usually be true. (To be precise, this only works in this example because D is symmetric about the line $y = x$.) Computing this gives

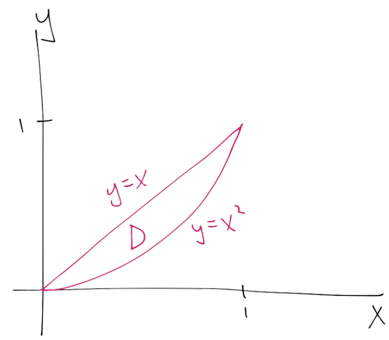
$$\begin{aligned}
\iint_D 2xy\,dA &= \int_0^1 \int_0^{1-y} 2xy\,dx\,dy \\
&= \int_0^1 x^2y\Big|_0^{1-y}\,dy \\
&= \int_0^1 (1-y)^2 y\,dy \\
&= \int_0^1 (y^3 - 2y^2 + y)\,dy \\
&= \left(\frac{1}{4}y^4 - \frac{2}{3}y^3 + \frac{1}{2}y^2\right)\Big|_0^1 \\
&= \frac{1}{12},
\end{aligned}$$

just as before. Hence the volume of the solid region under the surface $z = 2xy$ and above the triangle D is 1/12.

Example 2. We compute

$$\iint_D (2y - x)\,dA$$

where D is the region in the first quadrant bounded by $y = x$ and $y = x^2$, which looks like:



First we integrate in the order $dx\,dy$. The outer bounds on y are 0 to 1 since these are the smallest and largest values of y in D. At a fixed value of y, the value of x in our region starts on the left along the line $x = y$ and increases to the right until it hits $y = x^2$, or $x = \sqrt{y}$. Thus the inner bounds on x are y to $\sqrt{y}$, so

$$\begin{aligned}\iint_D (2y - x)\,dA &= \int_0^1 \int_y^{\sqrt{y}} (2y - x)\,dx\,dy \\ &= \int_0^1 \left(2xy - \frac{1}{2}x^2\right)\Big|_y^{\sqrt{y}}\,dy \\ &= \int_0^1 \left(2y^{3/2} - \frac{1}{2}y - 2y^2 + \frac{1}{2}y^2\right)\,dy \\ &= \left(\frac{4}{5}y^{5/2} - \frac{1}{4}y^2 - \frac{1}{2}y^3\right)\Big|_0^1 \\ &= \frac{4}{5} - \frac{1}{4} - \frac{1}{2} \\ &= \frac{1}{20}.\end{aligned}$$

As a check, let us all integrate in the order $dy\,dx$. The outer bounds on x are 0 to 1 since these are the smallest and largest values of x in D. At a fixed x, the values of y in D start along $y = x^2$ on the bottom and move up until $y = x$, so the inner bounds on y are x^2 to x. Thus we get:

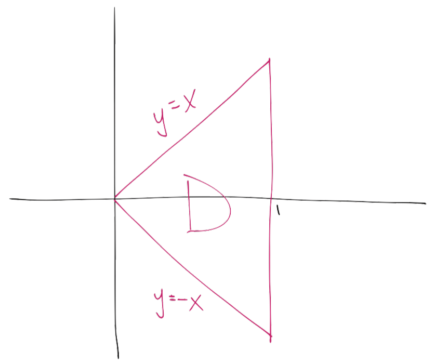
$$\begin{aligned}\iint_D (2y - x)\,dA &= \int_0^1 \int_{x^2}^{x} (2y - x)\,dy\,dx \\ &= \int_0^1 (y^2 - xy)\Big|_{x^2}^{x}\,dx \\ &= \int_0^1 (x^2 - x^2 - x^4 + x^3)\,dx \\ &= \left(-\frac{1}{5}x^5 + \frac{1}{4}x^4\right)\Big|_0^1 \\ &= -\frac{1}{5} + \frac{1}{4} \\ &= \frac{1}{20},\end{aligned}$$

agreeing with our answer before.

Example 3. Now we compute

$$\iint_D dA$$

where D is triangle with sides formed by the lines $y = x, y = -x$, and $x = 1$:



To be clear, the integrand is $dA = 1\,dA$, so we are integrating the constant function $f(x, y) = 1$. Geometrically, this should give the volume under the plane $z = 1$ and above D, and since this region is obtained by sliding D up this volume is

$$\text{area } D \times \text{height} = \text{area } D \times 1 = \text{area } D.$$

So we are actually simply computing the area of D. Using geometry, this area should be 1, so we expect that $\iint_D dA = 1$.

Important. For any region D in the xy-plane,

$$\iint_D dA = \iint_D 1\,dA = \text{area of } D.$$

Back to Example 3. First we integrate with respect to the order $dy\,dx$. The outer bounds on x are 0 to 1, and at a fixed x the value of y in D starts on the bottom line $y = -x$ and increases to the top line $y = x$. Thus our integral is

$$\begin{aligned}\iint_D dA &= \int_0^1 \int_{-x}^{x} dy\,dx \\ &= \int_0^1 2x\,dx \\ &= 1.\end{aligned}$$

Now notice what happens if we want to integrate with respect to x first. In this case the outer bounds on y are -1 to 1, and the inner bounds on x should come from the leftmost boundary of D to the rightmost boundary. The right boundary of D is $x = 1$, but the point is that the left boundary is actually given by two separate lines: $y = x$ on top and $y = -x$ on the bottom. Hence we do not have one single equation describing the inner lower bound on x, so to integrate in this order we have to "split" our region up into pieces:

$$\iint_D dA = \iint_{\text{top half of } D} dA + \iint_{\text{bottom half of } D} dA.$$

(Actually, the leftmost boundary can in fact be described by the single equation $x = |y|$ but to compute an integral involving this we'd eventually have to split up our region anyway.)

Over the top half of D, the outer bounds on y would be 0 to 1 and the inner bounds on x start on the left line $x = y$ and move to the right line $x = 1$, while over the bottom half the outer bounds on y are -1 to 0 and the inner bounds on x go from $x = -y$ on the left to $x = 1$ on the right. Thus

$$\begin{aligned}\iint_D dA &= \iint_{\text{top half of } D} dA + \iint_{\text{bottom half of } D} dA \\ &= \int_0^1 \int_y^1 dx\, dy + \int_{-1}^0 \int_{-y}^1 dx\, dy \\ &= \int_0^1 (1-y)\, dy + \int_0^{-1} (1+y)\, dy \\ &= \left(y - \frac{1}{2}y^2\right)\Big|_0^1 + \left(y + \frac{1}{2}y^2\right)\Big|_{-1}^0 \\ &= \left(1 - \frac{1}{2}\right) + \left(-(-1) - \frac{1}{2}\right) \\ &= 1.\end{aligned}$$

Regions of Type I, II, and III. We've seen that for certain regions integrating with respect to one order requires that we split up the region into pieces, while this is not necessary when integrating with respect to the other order. The book distinguishes these scenarios by defining regions of "type I" and regions of "type II":

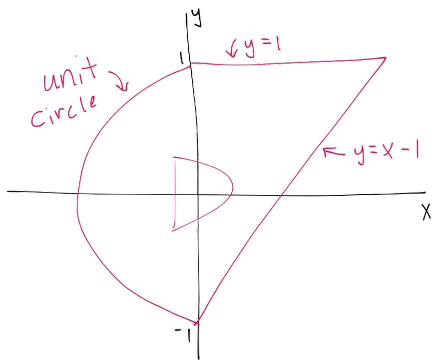
- Type I: regions where the boundaries on top and on the bottom can each be described using single equations, so integrating with respect to y first does not require splitting up
- Type II: regions where the boundaries on the left and on the right can each be described using single equations, so integrating with respect to x first does not require splitting up
- Type III: regions which are both of Type I and of Type II, so integrating in either order does not require splitting up.

Now, the exact words I used in class regarding this were "Forget this Type I and Type II nonsense!" Indeed, I don't think it's worth memorizing what Type I and Type II mean and falling back to those definitions every time you want to set up an integral—I think it's easier to get used to deciding how you should integrate based on a drawing of the region. In particular, look at the region and decide whether the left and right boundaries require single equations, or whether the top and bottom boundaries require single equations. This becomes even more true once we get to *triple integrals*, where there are more "types" to consider.

Example 4. For an unknown function f, let's set up iterated integrals giving the value of

$$\iint_D f(x,y)\, dA$$

where D is the following region, bounded by the left unit semi-circle, the line $y = x - 1$, and the line $y = 1$:



Since the left boundary (the circle) has one single equation and the right boundary (the line $y = x - 1$) does as well, integrating with respect to x first requires one iterated integral. Indeed, the bounds on y are -1 to 1 (i.e. smallest to largest values of y in this region) and at a fixed y, x starts along the circle and moves to the right to the line $x = y + 1$. The circle has equation $x^2 + y^2 = 1$, and solving for x gives $x = -\sqrt{1 - y^2}$ as the equation for the left half. (The positive square root would give the right half of the unit circle.) The integral is thus:

$$\iint_D f(x,y)\,dA = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{y+1} f(x,y)\,dx\,dy.$$

Now, the bottom boundary consists of two pieces (the circle and the line $y = x - 1$) and so does the top boundary (the circle and the line $y = 1$). Thus integrating with respect to y first requires that we split up the region, into say the piece to the left of the y-axis and the piece to the right:

$$\iint_D f(x,y)\,dA = \iint_{\text{left piece of } D} f(x,y)\,dA + \iint_{\text{right piece of } D} f(x,y)\,dA.$$

Over the left piece, the outer bounds on x are -1 to 0 and y moves from the bottom semicircle with equation $y = -\sqrt{1 - x^2}$ to the semicircle with equation $y = \sqrt{1 - x^2}$. Over the right piece, the outer bounds on x are 0 to 1 and y moves from the line $y = x - 1$ up to the line $y = 1$. Thus we get

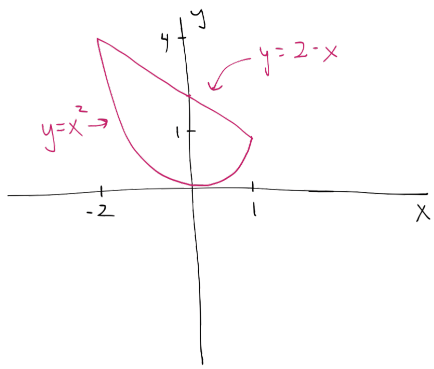
$$\iint_D f(x,y)\,dA = \int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx + \int_{0}^{1} \int_{x-1}^{1} f(x,y)\,dy\,dx$$

when integrating with respect to y first.

Lecture 4: Changing Order of Integration

Today we spoke about changing the order of integration in a double integral, which can be useful in situations where integrating with respect to a given order is impossible to carry out directly. The key is in using the bounds on a given iterated integral to determine the actual region of integration.

Warm-Up. We find the area of the region between $y = x^2$ and $y = 2 - x$, which looks like:



Calling this region D the area we're looking for is given by $\iint_D dA$. Note that since the rightmost boundary of D consists of two different curves (the line on top and the parabola on the bottom) integrating with respect to x first we will require that we split up the region into two pieces.

First let's integrate in the order $dy\,dx$. The minimum and maximum values of x in D are -2 and 1 and at a fixed x the value of y moves from the parabola $y = x^2$ on the bottom to the line $y = 2 - x$ on top. These give us the bounds we need, so

$$\begin{aligned}
\iint_D dA &= \int_{-2}^{1} \int_{x^2}^{2-x} dy\,dx \\
&= \int_{-2}^{1} y \Big|_{x^2}^{2-x} dx \\
&= \int_{-2}^{1} (2 - x - x^2)\,dx \\
&= \left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right) \Big|_{-2}^{1} \\
&= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) \\
&= \frac{9}{2}
\end{aligned}$$

is the area of D. (In class I think I said this was 3, which is incorrect.)

For practice, let's setup the integral in the other order as well. As said earlier, when integrating with respect to x first we have to split up our region at the line $y = 1$, namely where the rightmost boundary changes from the line $y = 2 - x$ to the parabola $y = x^2$. Over the top piece of D, the min and max values of y are 1 and 4 while x moves from $x = -\sqrt{y}$ on the left to $x = 2 - y$ on the right. (The $x = -\sqrt{y}$ comes from solving for x in $y = x^2$ and recognizing that the negative square root gives the left half of the parabola.) Over the bottom piece of D the min and max values of y are 0 and 1 while x moves from the left half of the parabola at $x = -\sqrt{y}$ to the right half at $x = \sqrt{y}$. Thus

$$\iint_D dA = \int_1^4 \int_{-\sqrt{y}}^{2-y} dx\,dy + \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy.$$

Computing this out gives the same area $\frac{9}{2} = 4.5$ as before.

Riemann sum approach. The question remains as to whether double integrals over non-rectangular regions can be given precise definitions in terms of Riemann sums as we had for double integrals over rectangles. The answer is yes, for a simple reason: we can just enclose our given region in a larger rectangle and extend the definition of our function to be 0 outside of the given region.

Namely, saw we wanted to define $\iint_D f\,dA$ where D is some non-rectangular region in the xy-plane. Pick any rectangle R which surrounds D and define an extended function f^{ext} by saying

$$f^{\text{ext}} = \begin{cases} f & \text{inside } D \\ 0 & \text{outside } D. \end{cases}$$

Then we can define $\iint_D f\,dA$ to be $\iint_R f^{\text{ext}}$, so that $\iint_D f\,dA$ can be defined using Riemann sums for the extended function f^{ext} over the rectangle R. However, this is never something we have to worry about: since f^{ext} is zero outside of D, the only possible nonzero contribution to the double integral can come from the values of f inside D, and as the book shows this has the practical effect that the double integral can be computed using an iterated integral where we use non-constant bounds to single-out D. So, as long as we can come up with such bounds, we don't have to worry about Riemann sums at all.

Probability example. Among all possible applications of double integrals, using them to compute probabilities is likely to be one of the most useful, especially in future economics/finance courses. So, let's work through one such example. Such an example will **NOT** be on any homework, quiz, nor exam since it requires material beyond the scope of this course, such as what it means to integrate between infinite bounds. Still, this is nice example to look at since it really illustrates a main use of double (and triple) integrals in general.

The setup is as follows: you and your friend are checking out at the grocery store, and you get into a checkout line which has an average checkout time of 10 minutes while your friend gets into a line which has an average checkout time of 5 minutes. The question we want to answer is: what is the probability that you will checkout before your friend? Letting x denote the time you checkout at and y the time your friend does, we want the probability that $x < y$; usually this probability is denoted $P(x < y)$.

But we don't yet have enough information to determine this—we need to know the so-called *probability density function* for this scenario, which is a function $\rho(x, y)$ which tells us the likelihood that you checkout at time x and your friend does at time y. We assume that the density function for a single person is given by

$$\rho(t) = \begin{cases} \frac{1}{\tau} e^{-t/\tau} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

where τ is the average checkout time for whichever line that single person is in. We interpret this function as saying roughly that the probability that this person checks out at time t is $\frac{1}{\tau} e^{-t/\tau}$ for $t \geq 0$ and 0 for $t < 0$, which makes sense since it is not possible to checkout at a time before you get in line at time $t = 0$.

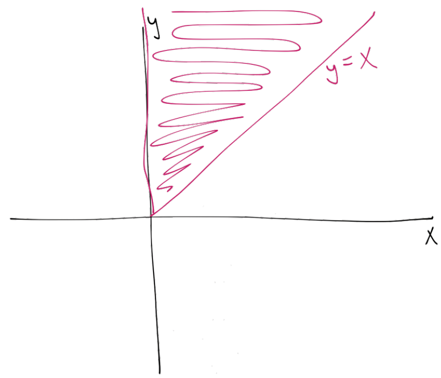
So, you have one density function $\rho_1(x)$ with $\tau = 10$ and your friend has one $\rho_2(y)$ with $\tau = 5$. The *combined* density function is then the product of these:

$$\rho(x, y) = \rho_1(x)\rho_2(y),$$

which is interpreted as the probabilistic fact that since your checkout time is independent of your friend's, the combined probability we want is the product of individual probabilities. The probability that $x < y$ is then obtained as the value of the double integral:

$$P(x < y) = \iint_{\text{region described by } x < y} \rho(x, y)\,dA.$$

First, since $\rho(x,y) = \rho_1(x)\rho_2(y)$ is zero for $x < 0$ or $y < 0$, the only possible nonzero contribution to this double integral comes when both x and y are positive, which means we are left integrating over the region:



Integrating in the order $dx\,dy$ we get that the outer bounds on y are 0 to ∞ since there is no restriction to high large y can get, and then the inner bounds on x are 0 to y:

$$P(x < y) = \int_0^\infty \int_0^y \rho_1(x)\rho_2(y)\,dx\,dy = \int_0^\infty \int_0^y \frac{1}{50}e^{-x/10}e^{-y/5}\,dx\,dy.$$

If you've seen the notion of an *improper integral* before, which in this case refers to the infinite bound on the outer integral, here is the resulting computation:

$$\begin{aligned} P(x < y) &= \int_0^\infty \int_0^y \frac{1}{50}e^{-x/10}e^{-y/5}\,dx\,dy \\ &= \int_0^\infty -\frac{1}{5}\left(e^{-3y/10} - e^{-y/5}\right)\,dy \\ &= \frac{1}{5}\left(-\frac{10}{3} + 5\right) \\ &= \frac{1}{3}. \end{aligned}$$

This is approximately 0.33, which means there is a 33% change that you will checkout before your friend. This makes sense: you're in a line with a longer average checkout time, so it is more likely than not that your friend will checkout first. Notice also that we can now give a reason as to why the probability density functions we used were reasonable: if you integrate the combined density $\rho(x,y)$ over the entire xy-plane, so

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \rho(x,y)\,dx\,dy,$$

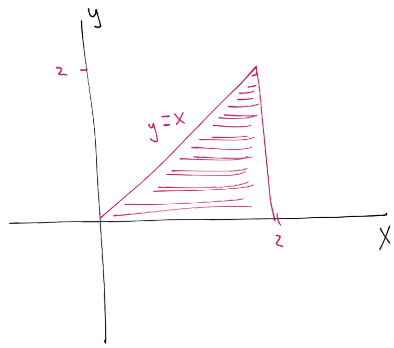
you will get a value of 1, which makes sense since the probability that you checkout after some finite amount of time and that your friend does as well is 100%. This is why we used the density functions we did, but there are others which would also work and which to use depends on the scenario you're interested in.

Changing order of integration. Let's now get back to actual course material. Say we wanted to compute

$$\int_0^2 \int_y^2 e^{x^2}\,dx\,dy.$$

The problem is that as written the inner integral is impossible to compute directly, since the antiderivative of e^{x^2} with respect to x CANNOT be expressed in terms of elementary functions, meaning polynomials, exponentials, logarithms, or trig functions. (To be clear: it's not that no one's ever found a way to do it, it's that it can be proved that it is not possible do so.) So, if we want to have any hope of actually computing this double integral, we must switch the order of integration. After doing so we will be integrating with respect to y on the inside, which we can do since then e^{x^2} is treated as a constant.

To switch the order we first need to determine the region of integration, which we find from the given bounds. The outer bounds on y say that our region lies fully within the strip bounded by the lines $y = 0$ and $y = 2$, and then the inner bounds say that $x = y$ forms the left boundary of our region and $x = 2$ the right boundary. Hence the region of integration is:



After changing the order of integration we get:

$$\begin{aligned}\int_0^2 \int_y^2 e^{x^2}\,dx\,dy &= \int_0^2 \int_0^x e^{x^2}\,dy\,dx \\ &= \int_0^2 xe^{x^2}\,dx \\ &= \frac{1}{2}e^{x^2}\Big|_0^2 \\ &= \frac{1}{2}(e^4 - 1).\end{aligned}$$

Now, be careful: changing the order of integration will not always result in an integral which is actually computable. Say instead we had started with

$$\int_0^2 \int_0^y e^{x^2}\,dx\,dy.$$

After changing the order we would get

$$\int_0^2 \int_x^2 e^{x^2}\,dy\,dx = \int_0^2 (2e^{x^2} - xe^{x^2})\,dx,$$

so while the inner integral was now computable the outer integral is not. So, whether or not changing the order will actually be useful also depends on the region of integration.

Important. When changing the order of integration in a given iterated integral, use the given bounds to determine the region of integration and then use the region to determine the new bounds after you've changed the order. This is useful when an iterated integral is not possible to compute directly in a given order.

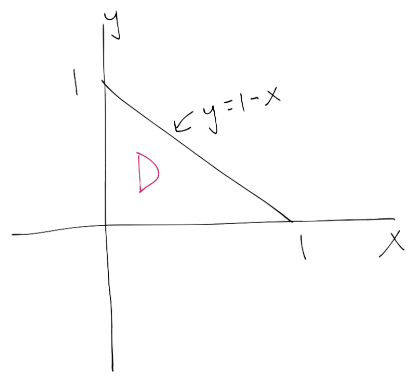
In general, the choice of a good order of integration will depend on both the region you're integrating over AND the function which you're integrating.

Another example. Here's another example. Say we want the value of

$$\int_0^1 \int_0^{1-x} \cos(1-y)^2 \, dy \, dx.$$

Again, it is not possible to compute this directly as is since the antiderivative of $\cos(1-y)^2$ with respect to y cannot be written down explicitly. So we change the order of integration.

The region of integration lies in the strip between $x = 0$ and $x = 1$, and is determined by y values starting at 0 and increasing to the line $y = 1 - x$, so this region is:



Changing the order gives

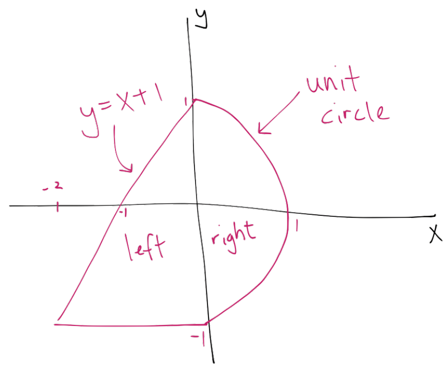
$$\begin{aligned}\int_0^1 \int_0^{1-x} \cos(1-y)^2 \, dy \, dx &= \int_0^1 \int_0^{1-y} \cos(1-y)^2 \, dx \, dy \\ &= \int_0^1 (1-y)\cos(1-y)^2 \, dy \\ &= -\frac{1}{2} \sin(1-y)^2 \Big|_0^1 \\ &= \frac{1}{2} \sin 1.\end{aligned}$$

Final example. Finally we consider

$$\int_{-2}^0 \int_{-1}^{x+1} f(x,y) \, dy \, dx + \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y) \, dy \, dx$$

for some unknown function $f(x,y)$. The point here is that changing the order of integration results in a single iterated integral instead of two as given.

The left piece of the region of integration comes from the bounds on the first iterated integral, and so occurs between $x = -2$ and $x = 0$ with y starting at the bottom on the line $y = -1$ and moving up to the line $y = x + 1$. The right piece of the region of integration comes from the second set of bounds, and is within the strip bounded by $x = 0$ and $x = 1$ with y going from the bottom half of the unit circle $y = -\sqrt{1-x^2}$ up to the top half $y = \sqrt{1-x^2}$. The entire region of integration is thus



Integrating with respect to $dx\,dy$, the min and max values of y are -1 and 1, and the values of x move from $x = y - 1$ on the left to $x = \sqrt{1-y^2}$ (the right half of the unit circle) on the right. Hence

$$\int_{-2}^{0}\int_{-1}^{x+1} f(x,y)\,dy\,dx + \int_{0}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx = \int_{-1}^{1}\int_{y-1}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

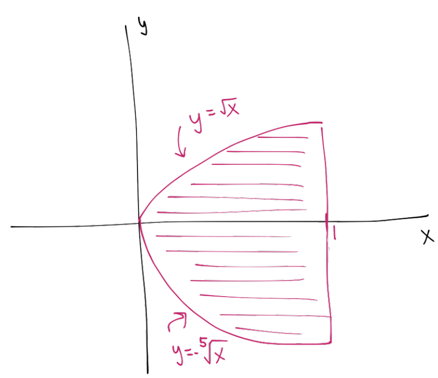
Lecture 5: Triple Integrals

Today we started talking about *triple integrals*, which are integrals of three-variable functions. As with double integrals, most of the work goes into setting up the bounds, which now come from 3-dimensional objects.

Warm-Up. We want to compute

$$\int_{0}^{1}\int_{-\sqrt[5]{x}}^{\sqrt{x}} \sin y^3\,dy\,dx.$$

This is not possible to do in the given order since $\sin \mathbf{y}^3$ does not have an antiderivative which can be expressed in an explicit way using basic functions, so we must change the order of integration. The region of integration is



since x has min and max values 0 and 1, and the bottom boundary is $y = -\sqrt[5]{x}$ while the top boundary is $y = \sqrt{x}$. Since the leftmost boundary consists of two different curves, we have to split up our region along the x-axis when writing the integral in the opposite order. We get

$$\int_0^1 \int_{y^2}^1 \sin y^3 \, dx \, dy + \int_{-1}^0 \int_{-y^5}^1 \sin y^3 \, dx \, dy$$

for the integrals over the top and bottom pieces of our region respectively.

Now we can compute the inner integrals:

$$\int_0^1 \int_{y^2}^1 \sin y^3 \, dx \, dy + \int_{-1}^0 \int_{-y^5}^1 \sin y^3 \, dx \, dy = \int_0^1 (\sin y^3 - y^2 \sin y^3) \, dy + \int_{-1}^0 (\sin y^3 + y^5 \sin y^3) \, dy.$$

To compute this, let's write this all as

$$\int_0^1 \sin y^3 \, dy - \int_0^1 y^2 \sin y^3 \, dy + \int_{-1}^0 \sin y^3 \, dy + \int_{-1}^0 y^5 \sin y^3 \, dy.$$

Now, the first and third integrals combine to give

$$\int_{-1}^1 \sin y^3 \, dy = 0$$

since $\sin y^3$ is odd with respect to y (meaning $\sin(-y)^3 = -\sin y^3$) and the interval $[-1, 1]$ is symmetric across the origin. So we're left with

$$-\int_0^1 y^2 \sin y^3 \, dy + \int_{-1}^0 y^5 \sin y^3 \, dy.$$

For the first integral we can use the substitution $u = y^3$, in which case we get

$$-\int_0^1 y^2 \sin y^3 \, dy = \frac{1}{3} \cos y^3 \Big|_0^1 = \frac{1}{3}(\cos 1 - 1).$$

The remaining integral above can be computed using a clever integration by parts:

$$\begin{aligned} u &= y^3 & v &= -\frac{1}{3} \cos y^3 \\ du &= 3y^2 \, dy & dv &= y^2 \sin y^3 \, dy \end{aligned}$$

which gives

$$\begin{aligned} \int_{-1}^0 y^5 \sin y^3 \, dy &= -\frac{1}{3} y^3 \cos y^3 \Big|_{-1}^0 + \int_{-1}^0 y^2 \cos y^3 \, dy \\ &= -\frac{1}{3} \cos(-1) + \frac{1}{3} \sin y^3 \Big|_{-1}^0 \\ &= -\frac{1}{3} \cos 1 + \frac{1}{3} \sin 1. \end{aligned}$$

Hence putting it all together we have

$$\int_0^1 \int_{-\sqrt[5]{x}}^{\sqrt{x}} \sin y^3 \, dy \, dx = \int_0^1 \int_{y^2}^1 \sin y^3 \, dx \, dy + \int_{-1}^0 \int_{-y^5}^1 \sin y^3 \, dx \, dy = \frac{1}{3} \sin 1 - \frac{1}{3}.$$

Rest assured that this problem is too long and too tricky for an exam, but on the other hand it does bring together different concepts we've been looking at.

Triple integrals. For a function f of three variables and a solid region E in $\mathbb{R}^3$, the *triple integral* of f over E is denoted by

$$\iiint_E f(x,y,z)\,dV$$

and is computed using iterated integrals. In this case, there are six total possible orders of integration. Such integrals can be defined precisely using Riemann sums, but we won't go into that at all. The new work goes into coming up with the bounds on these iterated integrals, which should describe the 3-dimensional region E.

Example 1. We compute the triple integral

$$\iiint_{[0,1]\times[0,2]\times[0,3]} 8xyz\,dV$$

where $[0,1]\times[0,2]\times[0,3]$ denotes the rectangular cube consisting of points whose (x,y,z) coordinates satisfy

$$0 \le x \le 1,\ 0 \le y \le 2,\ \text{and } 0 \le z \le 3.$$

As with double integrals over rectangles, integrating over a rectangular cube uses constant bounds on the inner, middle, and outer integrals, so we have:

$$\begin{aligned}\iiint_{[0,1]\times[0,2]\times[0,3]} 8xyz\,dV &= \int_0^3 \int_0^2 \int_0^1 8xyz\,dx\,dy\,dz \\ &= \int_0^3 \int_0^2 4x^2yz\Big|_0^1\,dy\,dz \\ &= \int_0^3 \int_0^2 4yz\,dy\,dz \\ &= \int_0^3 2y^2z\Big|_0^2\,dz \\ &= \int_0^3 8z\,dz \\ &= 4z^2\Big|_0^3 \\ &= 36.\end{aligned}$$

You can check that integrating in any other order (still using all constant bounds) will give the same result.

What is the interpretation of a triple integral? If single-variable integrals compute areas and double integrals compute volumes, what do triple integrals compute? The first not-so-satisfying answer is that they compute whatever the 4-dimensional analog of volume is; namely, the graph of $f(x,y,z)$ lives in $\mathbb{R}^4$ and the integral $\iiint_E f\,dV$ gives the "volume" of the region "between" this graph and the 3-dimensional solid E. Since we can't visualize 4-dimensional space, this doesn't help us much.

Instead, we give two other interpretations. First, in the special case where $f(x,y,z) = 1$ we have that

$$\iiint_E dV = \text{volume of } E,$$

analogously to how $\iint_D dA$ gives the area of D in the double integral case. So for instance,

$$\iiint_{\text{solid ball inside unit sphere}} dV = \frac{4}{3}\pi.$$

When f is not constant, we get other interpretations when viewing f as some type of "density" function. For instance, for a *mass density* function $\rho(x, y, z)$ (ρ is a common notation for density functions), we have

$$\iiint_E \rho(x, y, z)\, dV = \text{total mass of } E.$$

Indeed, a mass density function describes how much mass is concentrated at a point (x, y, z), and adding up all these individual masses as we do intuitively in a triple integral should give the total mass of E. For an electric *charge density* function ρ,

$$\iiint_E \rho(x, y, z)\, dV = \text{total electric charge of } E.$$

If we interpret $\rho(x, y, z)$ as a probability density, say describing the likelihood of finding an electron at a point (x, y, z), then the triple integrals gives the probability of finding the electron within E.

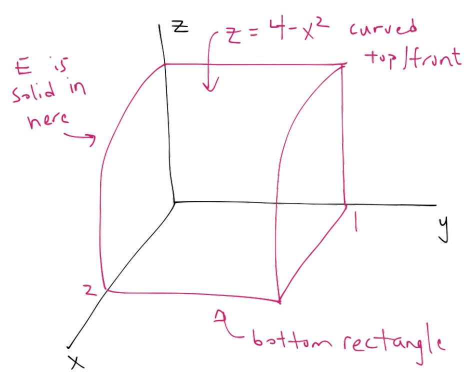
So, we get various interpretations for triple integrals based on various types of density functions. These interpretations also work for double integrals, and indeed lead to some of the most useful applications of double and triple integrals.

Important. For a solid region E in $\mathbb{R}^3$,

$$\iiint_E dV = \text{volume of } E.$$

Other common interpretations of $\iiint_E f(x, y, z)\, dV$ come from viewing f as some type of density function.

Example 2. Let us compute the triple integral of $f(x, y, z) = x + y$ over the region E drawn below:



So, E is bounded on top/front by the surface $z = 4 - x^2$, on the bottom by the rectangle $[0, 2] \times [0, 1]$ in the xy-plane, on the right by the plane $y = 1$, on the left by the xz-plane $y = 0$, and on the back by the yz-plane $x = 0$. We setup an iterated integral in the order $dz\, dx\, dy$.

The middle and outer bounds come from the projection of E onto the xy-plane (i.e. the plane determined by the middle and outer variables), which we'll call the *shadow* of E in the xy-plane.

In this case, collapsing E onto the xy-plane gives the rectangle $[0,2] \times [0,1]$, so the bounds on the outer and middle integrals are:

$$\iiint_E (x+y)\, dV = \int_0^1 \int_0^2 \int_?^? (x+y)\, dz\, dx\, dy.$$

Again, the middle and outer bounds simply describe the shadow of E in the plane determined by the middle and outer variables.

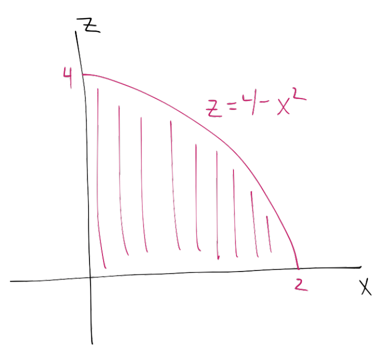
Now, at a fixed (x, y) in the xy-plane, the values of z in our region start on the xy-plane at $z = 0$ on the bottom and increase up to the surface $z = 4 - x^2$, so the inner bounds are 0 and $4 - x^2$. Thus overall we have

$$\iiint_E (x+y)\, dV = \int_0^1 \int_0^2 \int_0^{4-x^2} (x+y)\, dz\, dx\, dy.$$

Computing this gives:

$$\begin{aligned}
\iiint_E (x+y)\, dV &= \int_0^1 \int_0^2 \int_0^{4-x^2} (x+y)\, dz\, dx\, dy \\
&= \int_0^1 \int_0^2 (x+y)z\Big|_0^{4-x^2} dx\, dy \\
&= \int_0^1 \int_0^2 (x+y)(4-x^2)\, dx\, dy \\
&= \int_0^1 \int_0^2 (4x - x^3 + 4y - yx^2)\, dx\, dy \\
&= \int_0^1 \left(8 - 4 + 8y - \frac{8}{3}y\right) dy \\
&= \left(4y + 4y^2 - \frac{4}{3}y^2\right)\Big|_0^1 \\
&= \frac{20}{3}.
\end{aligned}$$

Now let's setup the bounds if integrating in the order $dy\, dz\, dx$. We start with the shadow in the xz-plane, which is the piece of the region lying in the xz-plane:



This gives the middle and outer bounds:

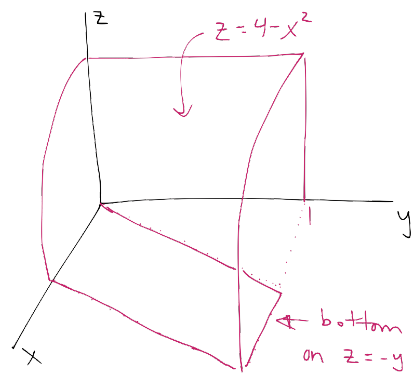
$$\iiint_E (x+y)\, dV = \int_0^2 \int_0^{4-x^2} \int_?^? (x+y)\, dy\, dz\, dx.$$

At any fixed (x, z) in the xz-plane, y starts on the left at $y = 0$ on the xz-plane and always increases to the right up until $y = 1$, so we have overall:

$$\iiint_E (x + y)\, dV = \int_0^2 \int_0^{4-x^2} \int_0^1 (x + y)\, dy\, dz\, dx.$$

You can check that computing this gives the same value as before.

Example 3. Finally, let us setup the same triple integral as in Example 2, only that we change the bottom of the region E to be on the plane $z = -y$ instead of on the xy-plane; so, imagine the same picture as in Example 2 only with the base slanted down so that it lies on $z = -y$:



Let's integrate in the order $dz\, dx\, dy$ again. The shadow of this region in the xy-plane is still the same rectangle $[0, 2] \times [0, 1]$ as before, but the difference is that at a fixed (x, y) the values of z start on the plane $z = -y$ and increase to the same $z = 4 - x^2$ as before. Hence we have

$$\iiint_E (x + y)\, dV = \int_0^1 \int_0^2 \int_{-y}^{4-x^2} (x + y)\, dz\, dx\, dy,$$

which I'll leave to you to compute.

Important. For a three-variable function f and solid region E in $\mathbb{R}^3$,

$$\iiint_E f(x, y, z)\, dV = \underbrace{\iint}_{\substack{\text{bounds describe}\\ \text{shadow of } E \text{ in}\\ \textit{mid-out} \text{ plane}}} \int_{\text{“lower” boundary of } in}^{\text{“upper” boundary of } in} f(x, y, z)\, d(in)\, d(mid)\, d(out)$$

In particular, the outer bounds are **ALWAYS** constant, the middle bounds can **ONLY** depend on the outer variable, and the inner bounds can **ONLY** depend on the middle and outer variables.

Lecture 6: More on Triple Integrals

Today we continued talking about triple integrals, and mainly looked at at examples which demonstrate the relation between the bounds of integration and the region of integration.

Warm-Up. We determine the value of

$$\iiint_E (e^{xz^2} y \cos xy + 3)\, dV$$

where E is the solid half-ball bounded by the front half of unit sphere and the yz-plane. The region E is symmetric across the xz-plane, and the function $e^{xz^2} y \cos xy$ is odd with respect to y, meaning that its value changes sign if you change the sign of y. Hence the contribution to the triple integral from a point (x, y, z) in the right half of E is negative the contribution from the corresponding point $(x, -y, z)$ in the left half, so the integral of $e^{xz^2 y \cos xy}$ over the left half is negative that over the right half. Thus

$$\iiint_E e^{xz^2} y \cos xy\, dV = 0.$$

On the other hand,

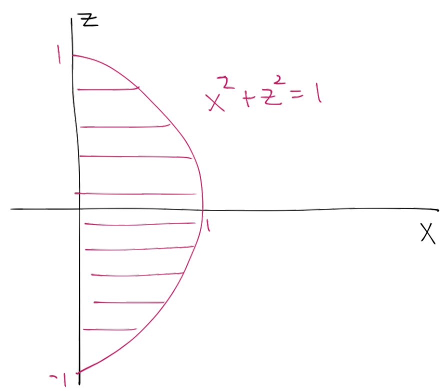
$$\iiint_E 3\, dV = 3 \iiint_E dV = 3\, vol(E) = 3 \cdot \frac{1}{2} \cdot \frac{4}{3}\pi = 2\pi.$$

Therefore

$$\iiint_E (e^{xz^2} y \cos xy + 3)\, dV = \iiint_E e^{xz^2} y \cos xy\, dV + \iiint_E 3\, dV = 0 + 2\pi = 2\pi$$

is the desired value.

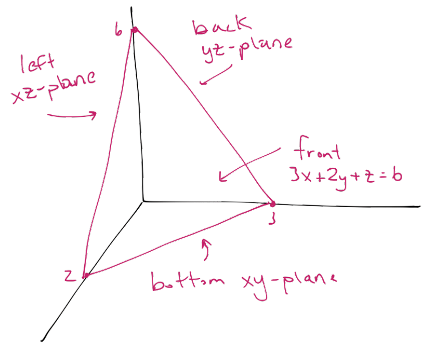
For practice, we'll set this up as an iterated integral anyway, say in the order $dy\, dx\, dz$. Collapsing E onto the xz-plane just gives the piece of E lying on the xz-plane, which looks like:



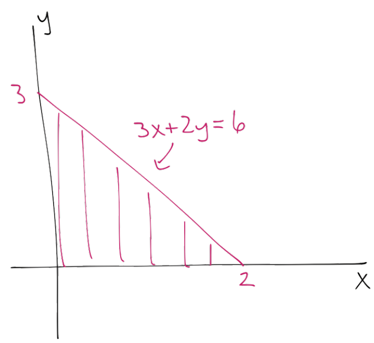
The equation for the boundary comes from setting $y = 0$ (since we're on the xz-plane) in the equation of the sphere $x^2 + y^2 + z^2 = 1$. Thus in this shadow region, z has min/max values 0 and 1 while x increases from $x = 0$ to $x = \sqrt{1 - z^2}$. Going back to our solid E, the leftmost boundary is given by the left half of the unit sphere while the rightmost boundary is the right half of the unit circle, so the inner bounds on y should come from the equations describing these halves. The left half is $y = -\sqrt{1 - x^2 - z^2}$ and the right half is $y = \sqrt{1 - x^2 - Z^2}$, so our final iterated integral is

$$\iiint_E (e^{xz^2} y \cos xy + 3)\, dV = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} (e^{xz^2} y \cos xy + 3)\, dy\, dx\, dz.$$

Example 1. We determine the volume of the solid E in the first octant by the plane $3x + 2y + z = 6$ and coordinate planes. The plane $3x + 2y + z = 6$ intersects the coordinate axes at $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 6)$, so the piece of this plane in the first octant looks like a triangle with these vertices. The solid region in question thus looks like:



We will setup and compute an iterated integral with respect to the order $dz\,dy\,dx$. The shadow of our solid in the xy-plane looks like a triangular region:

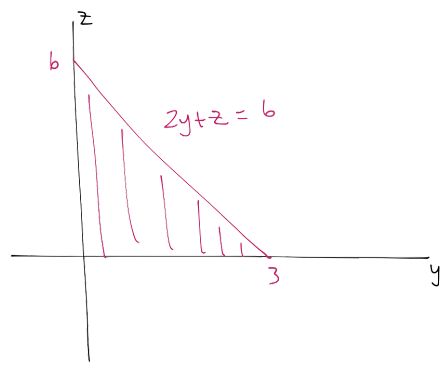


where the line giving the slanted boundary is obtained by finding where $3x + 2y + z = 6$ intersects the xy-plane. Hence x has min/max values 0 and 2 while y goes from 0 to the line $y = 3 - \frac{3}{2}x$. The bottom boundary of the solid is given by the xy-plane $z = 0$ and the top boundary is given by $z = 6 - 3x - 2y$, so the volume we want is:

$$\begin{aligned}
\iiint_E dV &= \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} dz\,dy\,dx \\
&= \int_0^2 \int_0^{3-\frac{3}{2}x} (6 - 3x - 2y)\,dy\,dx \\
&= \int_0^2 \left(6y - 3xy - y^2\right)\Big|_0^{3-\frac{3}{2}x} dx \\
&= \int_0^2 \left(9 - 9x + \frac{9}{4}x^2\right) dx \\
&= \left(9x - \frac{9}{2}x^2 + \frac{3}{4}x^3\right)\Big|_0^2 \\
&= 6.
\end{aligned}$$

Note that, using what we know about double integrals, the double integral obtained in the second step after the inner integration can be interpreted as the volume of the solid under the graph of $f(x, y) = 6 - 3x - 2y$ and above the region in the xy-plane described by the given xy-bounds. But this solid is precisely the solid E we're interested in here, so it makes sense that we end up with this double integral after the first integration; indeed, the fact that double integrals can be used to compute volume can be seen as a special case of the more general fact that triple integrals of the function 1 also compute volume.

For practice, let's setup the iterated integral in the order $dx\,dz\,dy$. The shadow of E in the yz-plane is the triangular region which looks like:



So, y goes from 0 to 3 and z from 0 to $6-2y$. The back side of E is the yz-plane at $x=0$ and the front side is the plane $x = 2 - \frac{2}{3}y - \frac{1}{3}z$, so the volume is:

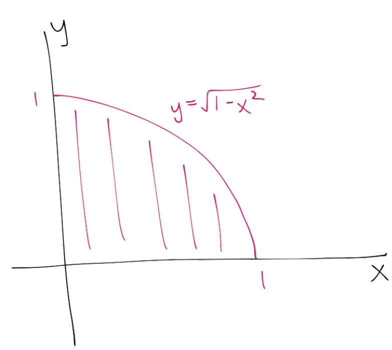
$$\iiint_E dV = \int_0^3 \int_0^{6-2y} \int_0^{2-\frac{2}{3}y-\frac{1}{3}z} dx\,dz\,dy.$$

Computing this gives the same value of 6 as before.

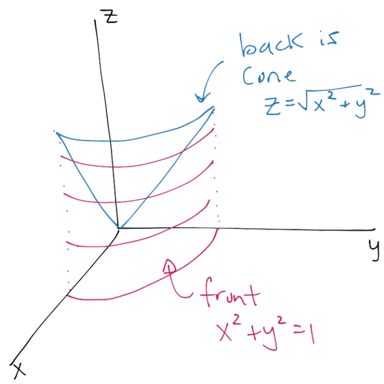
Example 2. We describe the region of integration for the iterated triple integral:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} f(x,y,z)\,dz\,dy\,dx.$$

First, the shadow region in the xy-plane has x-values between 0 and 1 with y going from $y=0$ on the x-axis to the top unit semicircle $y=\sqrt{1-x^2}$, so this shadow looks like:



Now, at a fixed (x,y) in the xy-plane, the z-values in our solid start on the xy-plane $z=0$ and go up to the cone $z=\sqrt{x^2+y^2}$. Thus the region of integration is the region in the first octant below the cone $z=\sqrt{x^2+y^2}$ and above the quarter unit disk in the first quadrant of the xy-plane:

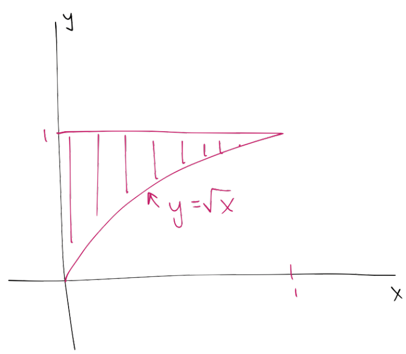


Another way to describe this is as the solid in the first octant outside the cone $z = \sqrt{x^2 + y^2}$ and within the cylinder $x^2 + y^2 = 1$.

Example 3. Finally, we sketch the region of integration for the iterated triple integral:

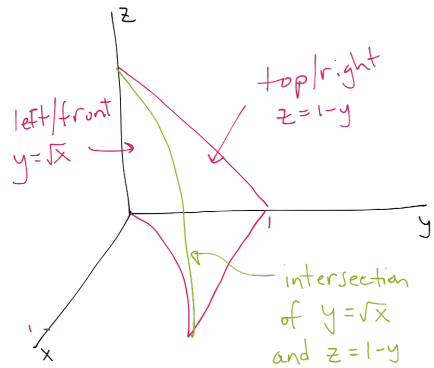
$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx.$$

The shadow in the xy-plane looks like



Now, the values of z start at $z = 0$ on the xy-plane (so the shadow drawn above is literally the bottom boundary of the entire 3-dimensional region) and move up to the plane $z = 1 - y$, which gives the top boundary of the region of integration.

So we want the region under this plane and above the shadow drawn before. To draw this precisely we must determine where the surface $y = \sqrt{x}$ (giving the leftmost boundary of our region) intersects the plane $z = 1 - y$. On the yz-plane this intersection occurs at $(0, 0, 1)$ while at the frontmost value of x at $x = 1$ this intersection occurs at $(1, 1, 0)$. (Imaging sliding the line $z = 1 - y$ in the yz-plane out in the positive x-direction, but only focusing on what's happening above the curve $y = \sqrt{x}$ in the xy-plane.) This intersection is thus a curve starting at $(0, 0, 1)$ and ending at $(1, 1, 0)$, so our entire region of integration looks like:



The left/front surface is $y = \sqrt{x}$, the top/right surface is $z = 1 - y$, the back is on the yz-plane, the bottom on the xy-plane, and the green curve is the intersection of $y = \sqrt{x}$ with $z = 1 - y$.

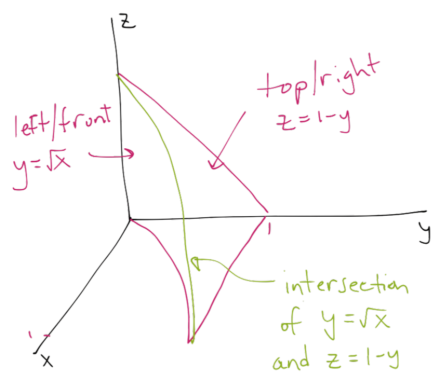
Lecture 7: Yet More on Triple Integrals

Today we continued with more examples of setting up triple integrals, and in particular looked at some instances where we have to split our region of integration up into pieces. Hopefully you'll agree that this is some super fun stuff! (Not sarcasm)

Warm-Up. Recall the triple integral we finished with last time:

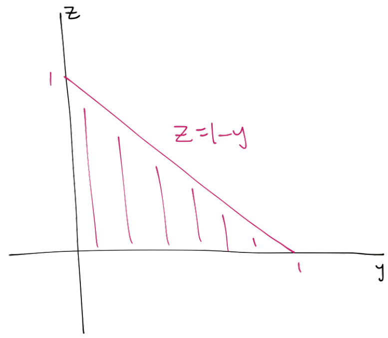
$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx,$$

whose region of integration we found looked like:



Now we rewrite this integral in the orders $dx\, dz\, dy$ and $dy\, dx\, dz$.

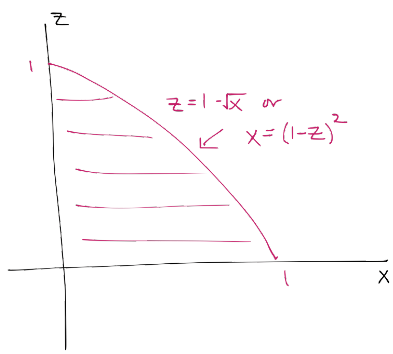
In the first case, the shadow in the yz-plane is simply the triangle forming the back side of our solid:



So, y goes from 0 to 1 and z from $z = 0$ to $z = 1 - y$. At a fixed (y, z) in the yz-plane, x starts on the yz-plane at $x = 0$ (the back boundary of our solid) and comes out forward as far as the surface $y = \sqrt{x}$ (or $x = y^2$, the front boundary), so our integral is:

$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\, dx\, dz\, dy.$$

Now, the shadow in the xz-plane is the piece of the xz-plane lying directly to the left of our solid, and looks like



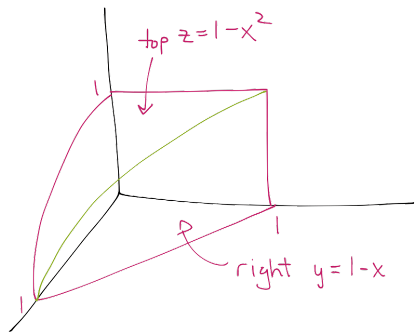
The equation for the boundary curve, which is the curve lying directly to the left of the green intersection curve in the 3d picture, is obtained by eliminating y in the equations of the surfaces $y = \sqrt{x}$ and $z = 1 - y$ forming this intersection; the idea is that a point on this intersection satisfies both of these equations simultaneously, and so its projection to the xz-plane is determined by the relation between the x- and z-coordinates of points on the intersection curve. Thus in the shadow z goes from 0 to 1 while x goes from $x = 0$ to $x = (1 - z)^2$. Going back to our 3d region, the leftmost boundary on y comes from the surface $y = \sqrt{x}$ while the rightmost boundary is the plane $z = 1 - y$, so y goes from $\sqrt{x}$ to $1 - z$. Thus the integral is:

$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dx\, dz.$$

Example 1. We rewrite

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z)\, dz\, dy\, dx$$

with respect to the orders $dy\,dx\,dz$ and $dx\,dz\,dy$. First we determine the region of integration. This has shadow in the xy-plane given by the triangular region in the first quadrant bounded by the line $y = 1 - x$. Since our z values start on the xy-plane at $z = 0$, this triangular region is the bottom piece of the region of integration. The top part of this region is given by the surface $z = 1 - x^2$, but we're only interested in the piece of this surface lying above the triangular region in the xy-plane. Thus the region of integration is:

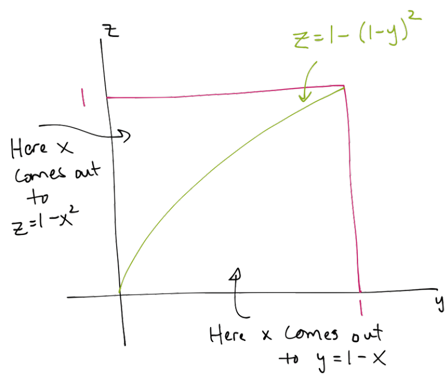


where the green curve is the intersection of the surfaces $y = 1 - x$ and $z = 1 - x^2$.

The shadow in the xz-plane is the left side of entire solid region, which is in the first quadrant of the xz-plane and lying below $z = 1 - x^2$. Hence z goes from 0 to 1 and x from $x = 0$ to $x = \sqrt{1 - z}$. The left boundary of the solid region is on the xz-plane at $y = 0$ and the right boundary is the plane $y = 1 - x$, so the triple integral is

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x, y, z)\, dy\, dx\, dz.$$

Now, the shadow of this region in the yz-plane is the square $[0, 1] \times [0, 1]$, which would suggest constant bounds on the middle and outer integrals when integrating in the order $dx\,dz\,dy$. However, note that the frontmost boundary on x actually consists of two pieces: the curved surface $z = 1 - x^2$ for points in the upper-left portion of our solid and the plane $y = 1 - x$ for points in the lower-right portion of our solid. Thus, when integrating with x on the inside we have to split up our region of integration. Going back to the shadow rectangle in the yz-plane, if we push the green intersection curve back we get:

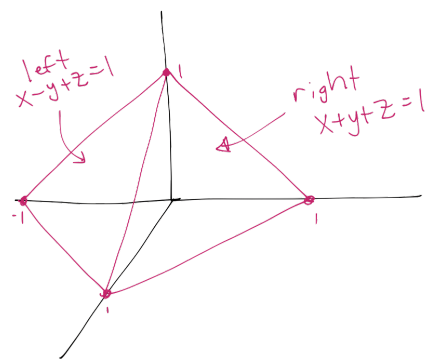


where the equation for this projected curve comes from eliminating x in $z = 1 - x^2$ and $y = 1 - x$. For points in this square above this curve x comes out as far as $z = 1 - x^2$ while for points below this curve x comes out as far as $y = 1 - x$, so we use this curve to split up our entire region of integration. Writing integrals over the piece above the intersection and below the intersection separately gives:

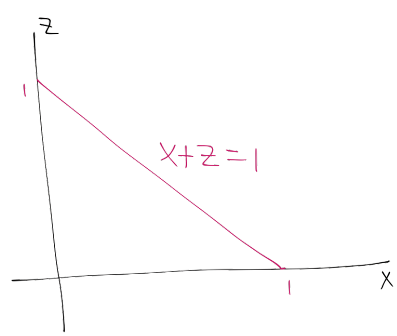
$$\int_0^1 \int_{1-(1-y)^2}^1 \int_0^{\sqrt{1-z}} f(x, y, z)\, dx\, dz\, dy + \int_0^1 \int_0^{1-(1-y)^2} \int_0^{1-y} f(x, y, z)\, dx\, dz\, dy$$

as the expression for our triple integral in the order $dx\, dz\, dy$.

Example 2. We setup the triple integral for a function f in the order $dy\, dz\, dx$ over the region bounded by the xy-plane, the yz-plane, and the planes $x + y + z = 1$ and $x - y + z = 1$. This region looks like:



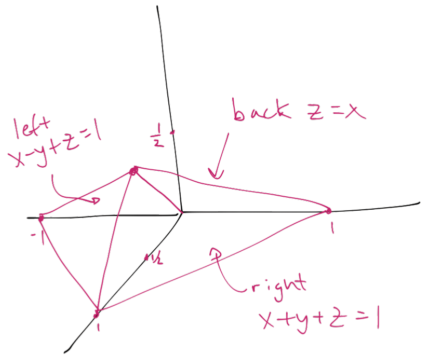
where the triangles forming the sides come from determining where the given planes intersect the axes. The shadow in the xz-plane is the piece of our region in the xz-plane and looks like:



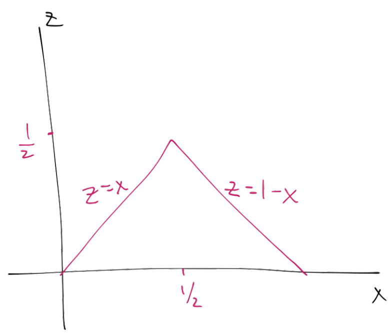
The equation for the boundary comes from setting $y = 0$ in the equation of either of the planes whose intersection gives this boundary. The leftmost boundary of our region is the plane $x - y + z = 1$ and the rightmost boundary is the plane $x + y + z = 1$, so our triple integral is:

$$\int_0^1 \int_0^{1-x} \int_{x+z-1}^{1-x-z} f(x, y, z)\, dy\, dz\, dx.$$

Now we make one modification and include the plane $z = x$ as one of the planes bounding our region of integration. Thus the back side is no longer on the yz-plane but is slanted down to align with the $z = x$ plane:



The shadow in the xz-plane now looks like:



and thus when describing this shadow in the order $dz\,dx$ we must split it up into the left and right pieces. Over each of these, the left and right boundaries on y are the same planes as before, so the inner bounds on either piece are the same as before. Thus we get the expression:

$$\int_0^{1/2}\int_0^x\int_{1-x-z}^{x+z-1} f(x,y,z)\,dy\,dz\,dx+\int_{1/2}^1\int_0^{1-x}\int_{1-x-z}^{x+z-1} f(x,y,z)\,dy\,dz\,dx$$

as the expression for the triple integration of this new region when integrating in the order $dy\,dz\,dx$. The point is that we still have to split up our region, but for a different reason than when integrating with respect to x on the inside in Example 1: there it was because the inner bounds on x required splitting, while here it is the shadow in the xz-plane which requires splitting. Note that when integrating in the order $dy\,dx\,dz$ instead there is no need to split up the integrals anymore.

Lecture 8: Integrals in Polar Coordinates

Today we spoke about rewriting double integrals in terms of polar coordinates, which is a special case of a more general change of variables method which we will look at next time. There are many examples of double integrals which are impossible to compute directly in rectangular coordinates but which become very straightforward after converting to polar coordinates.

Warm-Up. One more probability example! (Again, not something which you're expected to be familiar with.) Suppose we have three light bulbs: a red one which has an average lifetime of 800 hours, a green one with an average lifetime of 700 hours, and a blue one with an average lifetime of 600 hours. If each bulb has a probability density function of the form

$$f(t) = \begin{cases} \frac{1}{\tau} e^{-t/\tau} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases}$$

where t is the time which the bulb burns out at and τ is its average lifetime, what is the probability that the red bulb burns out first, then the green one, and finally the blue?

Letting x, y, z denote the times which the red, green, and blue bulbs burn out respectively, we are looking for the probability that $x < y < z$, or to be more precise that

$$0 \leq x < y < z$$

since a bulb can't burn out before the time we install it. The individual density functions are:

$$f_r(x) = \begin{cases} \frac{1}{800} e^{-x/800} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases} \quad f_g(y) = \begin{cases} \frac{1}{700} e^{-y/700} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0, \end{cases} \quad f_b(z) = \begin{cases} \frac{1}{600} e^{-z/600} & \text{if } z \geq 0 \\ 0 & \text{if } z < 0, \end{cases}$$

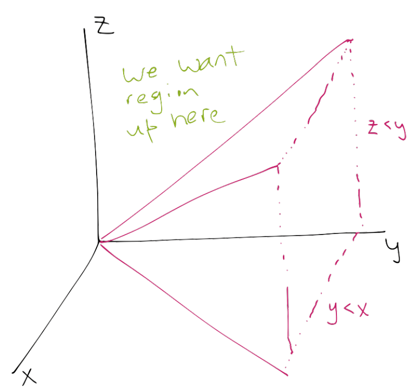
so since the times at which the bulbs burn out are independent of each other, the joint density function is

$$\rho(x, y, z) = f_r(x) f_g(y) f_b(z).$$

The probability we want is thus

$$P(0 \leq x < y < z) = P(0 \leq x < y < z) = \iiint_{0 \leq x < y < z} \rho(x, y, z)\, dV.$$

To determine the bounds, we first consider the region in the first octant described by the inequalities $x < y$ and $z < y$:



The region we are actually integrating over lies directly above this since we are considering $z > y$. With respect to the order $dz\, dy\, dx$, the shadow in the xy-plane is the base (where points satisfy $0 \leq x < y$) of the region drawn above, so x goes from 0 to ∞ and y from $y = x$ to ∞. Then, in our region of integration, z starts on the plane $z = y$ and increases unboundedly, so the bounds on z are y to ∞. Our integral is thus:

$$\iiint_{0 \leq x < y < z} \rho(x, y, z)\, dV = \int_0^\infty \int_x^\infty \int_y^\infty \frac{1}{800 \cdot 700 \cdot 600} e^{-x/800} e^{-y/700} e^{-z/600}\, dz\, dy\, dx$$

$$= \int_0^\infty \int_x^\infty \frac{1}{800 \cdot 700} e^{-x/800} e^{-y/700 - y/600}\, dy\, dx$$
$$= \int_0^\infty \frac{600}{1300 \cdot 800} e^{-x/800 - x/700 - x/600}\, dx$$
$$= \frac{600 \cdot 700 \cdot 600}{1300 \cdot 1460000}.$$

This is approximately 0.13277, meaning there is about a 13.3% chance that the red bulb burns out first, then the green, then the blue.

This makes sense: the red bulb has a longer average lifetime than the green bulb, so the red bulb burning out before the green one is unlikely, and the green bulb has a longer average lifetime than the blue one, so the green one burning out first is also unlikely, and hence the even we're interested in is made up of two unlikely events and so should be even more unlikely itself.

Example 1. Say we want to compute

$$\iint_D e^{x^2+y^2}\, dA$$

where D is unit disk. The iterated integral in the order $dx\, dy$ is

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy.$$

This is impossible to compute directly, and so is the iterated integral written in the opposite order. So, we need something new in order to be able to compute this. The presence of the $x^2 + y^2$ term and the fact that we are integrating over a disk suggests that this integral might be simpler to work with in polar coordinates.

Converting to polar coordinates. Recall the conversions between rectangular and polar coordinates:

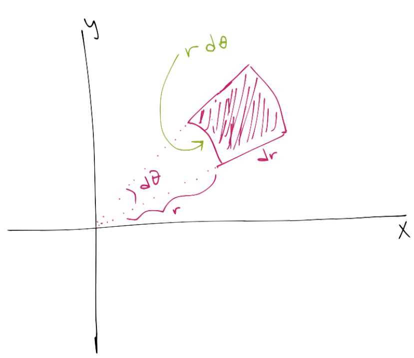
$$x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2.$$

In polar coordinates, the function from Example 1 is simply e^{r^2} and the region of integration is described by the bounds

$$0 \le \theta \le 2\pi \text{ and } 0 \le r \le 1.$$

The only other we need then to be able to rewrite the given integral in polar coordinates is to determine what happens to dA after switching to polar coordinates.

Recall that dA is supposed to represent the area of an "infinitesimal" rectangle, which in rectangular coordinates has sides of lengths dx and dy, giving $dx\, dy$ or $dy\, dx$ as the area dA. To determine the expression for dA in polar coordinates, consider the following picture:



The shaded region is the analog of an "infinitesimal" rectangle, and this has sides of lengths dr and $r\,d\theta$. Thus, this "rectangle" has area

$$dA = (dr)(r\,d\theta) = r\,dr\,d\theta.$$

Hence when converting to polar coordinates, dA becomes $r\,dr\,d\theta$ or $r\,d\theta\,dr$ depending on the order of integration.

Back to Example 1. Converting the double integral from Example 1 thus gives:

$$\int_0^{2\pi}\int_0^1 e^{r^2} r\,dr\,d\theta,$$

which we can now compute:

$$\begin{aligned}\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy &= \int_0^{2\pi}\int_0^1 re^{r^2}\,dr\,d\theta\\ &= \int_0^{2\pi} \frac{1}{2}e^{r^2}\Big|_0^1\,d\theta\\ &= \int_0^{2\pi}\frac{1}{2}(e-1)\,d\theta\\ &= \pi(e-1).\end{aligned}$$

Now that the extra r coming from the dA term is what makes this integral computable since now re^{r^2} has an antiderivative with respect to r which we can actually write down.

Important. When converting a double integral to polar coordinates,

$$dA = r\,dr\ d\theta \text{ or } r\,d\theta\,dr$$

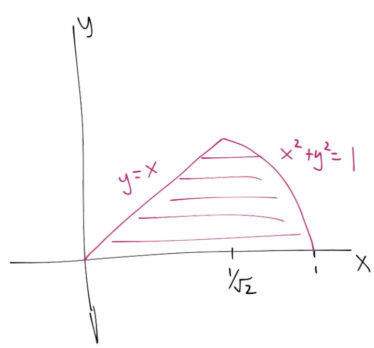
depending on the order we want to integrate in. (Normally we'll have $d\theta$ on the outside.)

Example 2. We determine the value of

$$\int_0^{1/\sqrt{2}}\int_y^{\sqrt{1-y^2}} dx\,dy.$$

This is possible to compute directly, only that a trig substitution is necessary for the outer integral. Instead, converting to polar coordinates gives a simpler computation.

The given bounds describe the following region:



In polar coordinates, θ moves from 0 along the positive x-axis to $\pi/4$ along the line $y = x$ and r from 0 at the origin out to the unit circle at $r = 1$. Thus we get:

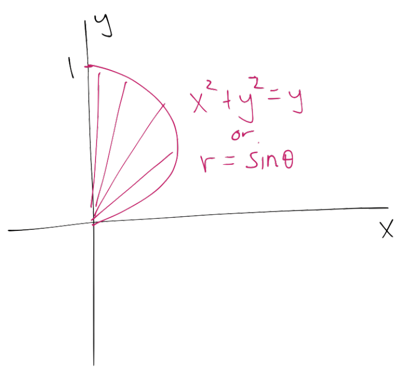
$$\begin{aligned}\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} dx\, dy &= \int_0^{\pi/4} \int_0^1 r\, dr\, d\theta \\ &= \int_0^{\pi/4} \frac{1}{2}\, d\theta \\ &= \frac{\pi}{8}.\end{aligned}$$

This makes sense: since we are integrating the function 1, the double integral should give the area of the region drawn above, which is an eighth of the unit disk which has area π.

Example 3. We determine the value of

$$\int_0^1 \int_0^{\sqrt{y-y^2}} \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy,$$

which again would be difficult to compute directly. The region of integration looks like:



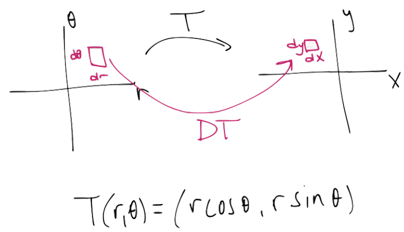
where $x = \sqrt{y - y^2}$ is the right half of the circle whose polar equation is $r = \sin\theta$. (Indeed, squaring both sides gives $x^2 = y - y^2$, so $x^2 + y^2 = x$ and hence $r^2 = r \sin\theta$.) Thus, in polar coordinates θ increases from 0 to $\pi/2$ and r moves from the origin at $r = 0$ out to the circle $r = \sin\theta$, so:

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{y-y^2}} \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy &= \int_0^{\pi/2} \int_0^{\sin\theta} \frac{1}{r} r\, dr\, d\theta \\ &= \int_0^{\pi/2} \sin\theta\, d\theta \\ &= \end{aligned}$$

Another approach. Finally, let's see another way to derive

$$dA = r\, dr\, d\theta,$$

which will generalize to other types of changes of variables. View the process of replacing x and y with polar coordinates as a transformation from the $r\theta$-plane to the xy-plane:



Consider an infinitesimal rectangle in the $r\theta$-plane, which is transformed into an infinitesimal rectangle in the xy-plane. The area of the latter rectangle is precisely $dA = dx\,dy$, and we want to relate this area to that of the first rectangle, which has area $dr\,d\theta$.

Here is the key point: the *Jacobian* of T is what tells us how to convert infinitesimal lengths in the $r\theta$-plane into infinitesimal lengths in the xy-plane. This is a property of Jacobians which we didn't look at last quarter, but finally gives us a geometric interpretation of general Jacobians. So, DT describes the *linear transformation* sending our first small rectangle to our second, and (thinking back to the fall quarter) it is the (absolute value of the) *determinant* of DT which tells how areas are altered under the linear transformation DT! Thus

$$dA = |\det DT|(\text{old area}) = |\det DT|\,dr\,d\theta.$$

We have

$$DT = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$

whose determinant is r. In double integrals relative to polar coordinates we will always restrict ourselves to values of $r \geq 0$, so $|\det DT| = |r| = r$. Thus, we see that the factor r obtained when converting dA simply comes from the geometric interpretation of a determinant as an *expansion factor*!

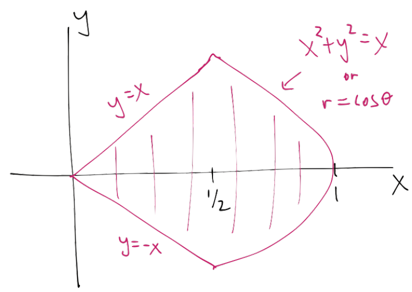
Lecture 9: Change of Variables

Today we looked at general changes of variables, building on what we did with polar coordinates last time. The key point is that the conversion of dA involves a "Jacobian expansion factor", analogous to the r factor in dA obtained when converting to polar coordinates.

Warm-Up 1. We determine the value of

$$\int_0^{1/2}\int_{-x}^{x} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx + \int_{1/2}^{1}\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx.$$

The combined region D described by both sets of bounds is:



In polar coordinates, this region has θ going between $-\pi/4$ (along the line $y=-x$) to $\pi/4$ (along the line $y=x$) and has r going from the origin at $r=0$ out to the circle $r=\cos\theta$, which is $y=\sqrt{x-x^2}$ in polar coordinates. Hence the given expression is equal to a single iterated integral in polar coordinates, which is:

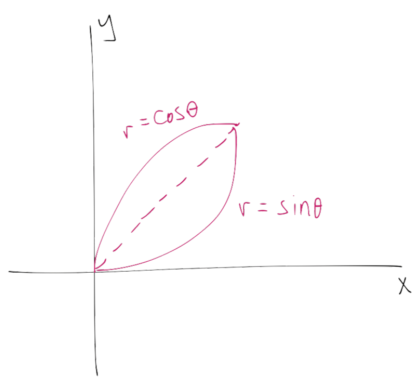
$$\begin{aligned}\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dA &= \int_0^{1/2}\int_{-x}^{x} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx + \int_{1/2}^{1}\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx \\ &= \int_{-\pi/4}^{\pi/4}\int_0^{\cos\theta} \frac{1}{r} r\,dr\,d\theta \\ &= \int_{-\pi/4}^{\pi/4} \cos\theta\,d\theta \\ &= \sqrt{2}.\end{aligned}$$

Note that this would be fairly difficult to compute directly in rectangular coordinates.

Warm-Up 2. We compute

$$\iint_D \theta\,dA$$

where D is the region:



The "outer boundary" of D, i.e. the upper bound on r, consists of two different curves: the circle $r=\cos\theta$ over the top half and the circle $r=\sin\theta$ over the bottom. Thus we must split up our polar integral into two pieces:

$$\iint_D \theta\,dA = \iint_{\text{top of } D} \theta\,dA + \iint_{\text{bottom of } D} \theta\,dA$$

$$= \int_0^{\pi/4} \int_0^{\sin\theta} \theta r\, dr\, d\theta + \int_{\pi/4}^{\pi/2} \int_0^{\cos\theta} \theta r\, dr\, d\theta$$
$$= \int_0^{\pi/4} \frac{1}{2}\theta \sin^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\theta\cos^2\theta\, d\theta.$$

Each of these can be evaluated using integration by parts, where the antiderivatives of $\sin^2\theta$ and $\cos^2\theta$ are found using the trig identities:

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \text{ and } \cos^2\theta = \frac{1}{2}(1+\cos 2\theta).$$

(You wouldn't be required to remember these identities on an exam.) We get:

$$\iint_D \theta\, dA = \int_0^{\pi/4} \frac{1}{2}\theta\sin^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\theta\cos^2\theta\, d\theta$$
$$= \frac{1}{4}\left(\frac{1}{2}\theta^2 - \frac{1}{2}\theta\sin 2\theta - \frac{1}{4}\cos 2\theta\right)\Big|_0^{\pi/4} + \frac{1}{4}\left(\frac{1}{2}\theta^2 + \frac{1}{2}\theta\sin 2\theta + \frac{1}{4}\cos 2\theta\right)\Big|_{\pi/4}^{\pi/2}$$
$$= \frac{1}{4}\left(\frac{\pi^2}{32} - \frac{\pi}{8} + \frac{1}{4}\right) + \frac{1}{4}\left(\frac{\pi^2}{8} - \frac{1}{4} - \frac{\pi^2}{32} - \frac{\pi}{8}\right)$$

and I'll omit simplifying any further. (Yes, this is too much work for an exam.)

Fun comptuation. Here is a fun application of what we've been doing. We want to determine the value of

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx,$$

which cannot be computed directly. The function $f(x) = e^{-x^2}$ is called a *Gaussian* function, and its graph models "bell curves" in statistics. Knowing the area under this graph is important in applications of probability and statistics.

Denote the value we want by I. The trick is that we can use polar coordinates to compute I^2:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy.$$

Here, we've written the second I term using y as the variable of integration, and then since the first integral is a constant with respect to y we can bring it inside the second integral, which is how we get the last expression. Now this last expression is computable using polar coordinates:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy = \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\, dr\, d\theta$$
$$= \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^{\infty}\, d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}\, d\theta$$
$$= \pi.$$

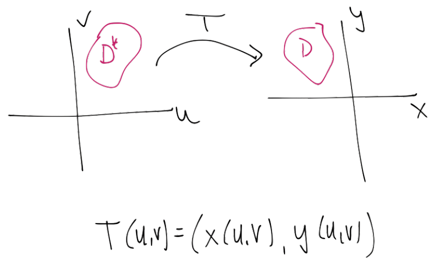
Hence $I^2 = \pi$, and thus $I = \sqrt{\pi}$:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

This is definitely the easiest way I know of to compute this integral.

Change of variables. The derivation of $dA = r\,dr\,d\theta$ at the end of last time in terms of Jacobian determinants suggests what to do when making a more general change of variables. The setup is as follows.

Introducing new variables u and v, we view a change of variables as a transformation from the uv-plane to the xy-plane:



Here D^* is the region in the uv-plane which is sent to the region D in the xy-plane; we also express this by writing $D = T(D^*)$. As with polar coordinates, the Jacobian of T tells us how infinitesimal lengths in uv-space are transformed into infinitesimal lengths in xy-space, and the absolute value of its determinant gives us the expansion factor relating infinitesimal areas in the uv-plane to those in the xy-plane:

$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

The expression $\frac{\partial(x,y)}{\partial(u,v)}$ is common notation for the determinant of the Jacobian of T:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det DT = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

Note that some people (such as our book) use the term "Jacobian" to mean this determinant, whereas we've used it to mean the matrix which this is the determinant of.

Analog in single-variable calculus. The same exact thing happens when making a substitution in single-variable integrals, only we don't normally phrase it in terms of Jacobians. For instance, suppose we want to compute:

$$\int_0^4 xe^{x^2}\,dx.$$

We make the substitution $x = \sqrt{u}$ (or $u = x^2$), which we view as a transformation from the u-line to the x-line. The Jacobian of this transformation is the 1×1 matrix

$$\left(\tfrac{dx}{du}\right) = \left(\tfrac{1}{2\sqrt{u}}\right),$$

whose determinant is $1/(2\sqrt{u})$. The region of integration in the original integral, which is the interval $[0,4]$, corresponds to the interval $[0,16]$ in u-space, so our integral becomes:

$$\int_0^4 xe^{x^2}\,dx = \int_0^{16} \sqrt{u}e^u \underbrace{\frac{1}{2\sqrt{u}}}_{\text{Jacobian factor}} du = \int_0^{16} \frac{1}{2}e^u\,du,$$

just as you would expect when making a u-substitution.

Important. Under a change of variables $(u, v) \mapsto (x(u, v), y(u, v))$,

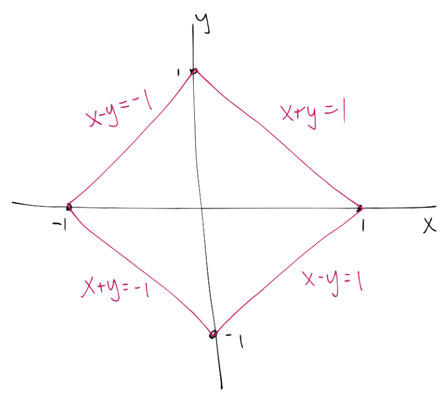
$$\iint_D f(x, y)\, dA = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

where D^* is the region in the uv-plane corresponding to D.

Example 1. Let's compute

$$\iint_D (x^2 - y^2)\, dA$$

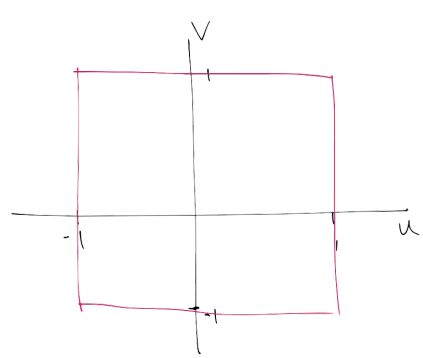
where D diamond-shaped region with vertices $(1, 0), (0, 1), (-1, 0)$, and $(0, -1)$:



Note that to do this directly as is would require that we split up the region into pieces. Based on the equations for the various boundaries, we introduce the variables:

$$u = x + y \text{ and } v = x - y.$$

The boundaries above then become $u = \pm 1$ and $v = \pm 1$, so the region D^* in the uv-plane corresponding to D is a square:



The integrand in the origin integral becomes

$$x^2 - y^2 = (x - y)(x + y) = vu.$$

To determine the expression for dA we need the Jacobian factor $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. Solving for x and y in terms of u and v gives

$$x = \frac{1}{2}(u+v) \text{ and } y = \frac{1}{2}(u-v),$$

so

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}.$$

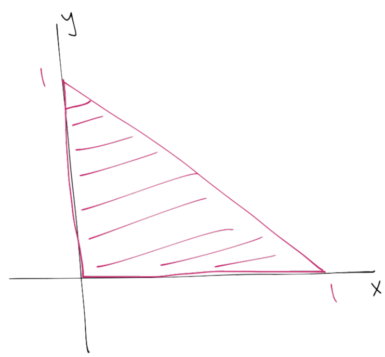
Hence the original integral is

$$\begin{aligned}\iint_D (x^2-y^2)\,dA &= \iint_{D^*} vu\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\\ &= \int_{-1}^{1}\int_{-1}^{1} vu \underbrace{\frac{1}{2}}_{\text{Jacobian}} du\,dv\\ &= \int_{-1}^{1} \frac{1}{4}vu^2\Big|_{-1}^{1} dv\\ &= 0.\end{aligned}$$

Example 2. Next we compute

$$\iint_D \cos\left(\frac{x-y}{x+y}\right) dA$$

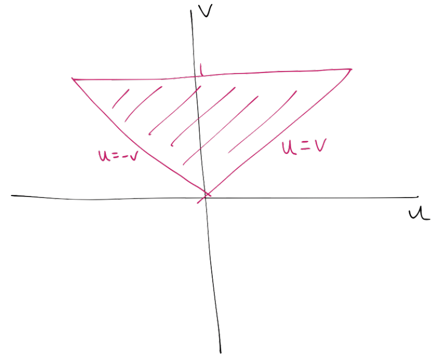
where D is the triangular region:



Based on the diagonal boundary of D and the expression for the integrand, we make the change of variables:

$$u = x-y \text{ and } v = x+y.$$

Under this change, the boundary $x+y=1$ of D becomes $v=1$, the boundary $y=0$ on the x-axis becomes $u=v$ since setting $y=0$ in the expression for u and v gives $u=x=v$, and the boundary $x=0$ on the y-axis becomes $u=-v$ since setting $x=0$ gives $u=-y=-v$. Thus the region D^* in the uv-plane corresponding to D is the triangular region:



The integrand becomes $\cos \frac{u}{v}$ so all that remains is to determine the Jacobian factor.

We can proceed as before by solving for x and y in terms of u and v, or we can do the following, which for more general changes of variables will likely be the easier method. We want $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$: the key fact is that

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

where the Jacobian on the right is the one for expressing u and v in terms of x and y; since we are introducing u and v by defining them in terms of x and y, computing this Jacobian does not require the intermediate step of solving for some variables in terms of the others. The point is if T is the transformation sending (u,v) to (x,y), then the transformation sending (x,y) to (u,v) is the *inverse* of T:

$$T : (u,v) \mapsto (x,y) \text{ and } T^{-1} : (x,y) \mapsto (u,v).$$

According to the chain rule for Jacobians, the Jacobian matrix of T^{-1} is the inverse of the Jacobian matrix for T:

$$D(T^{-1}) = (DT)^{-1},$$

so the determinants of each are reciprocals of each other, giving

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

as claimed.

So, using $u = x - y$ and $v = x + y$, we have

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \left|\det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right|^{-1} = \frac{1}{2}.$$

Thus after converting our integral becomes:

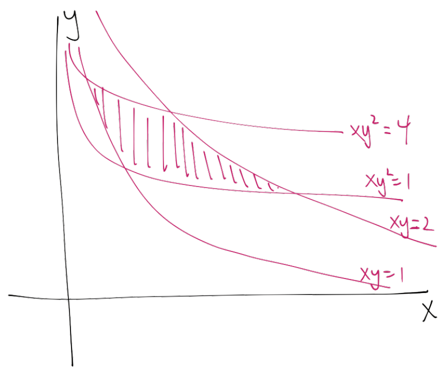
$$\begin{aligned} \iint_D \cos\left(\frac{x-y}{x+y}\right) dA &= \iint_{D^*} \cos\left(\frac{u}{v}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \\ &= \int_0^1 \int_{-v}^{v} \frac{1}{2} \cos\frac{u}{v}\, du\, dv \\ &= \int_0^1 \frac{1}{2} v \sin\frac{u}{v}\Big|_{-v}^{v}\, dv \end{aligned}$$

$$= \int_0^1 v \sin 1 \, dv$$
$$= \frac{1}{2} \sin 1.$$

Example 3. Finally we compute

$$\iint_D xy^2 \, dA$$

where D is the region in the first quadrant bounded by $xy = 1, xy = 4, xy^2 = 1$, and $xy^2 = 4$:



Based on the boundaries of D, we use the change of variables

$$u = xy \text{ and } v = xy^2.$$

As before, instead of having to solve for x and y in terms of u and v, we can find the Jacobian we need via:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \left|\det\begin{pmatrix} y & x \\ y^2 & 2xy \end{pmatrix}\right|^{-1} = \frac{1}{xy^2} = \frac{1}{v}.$$

The bounds on D become constant bounds in terms of u and v, so we get

$$\iint_D xy^2 \, dA = \int_1^4 \int_1^4 v \underbrace{\frac{1}{v}}_{\text{Jacobian}} \, du \, dv = \int_1^4 \int_1^4 du \, dv = 9.$$

Important. When determining the Jacobian factor $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$, if you have defined u and v in terms of x and y, use

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

instead of having to solve for x and y in terms of u and v.

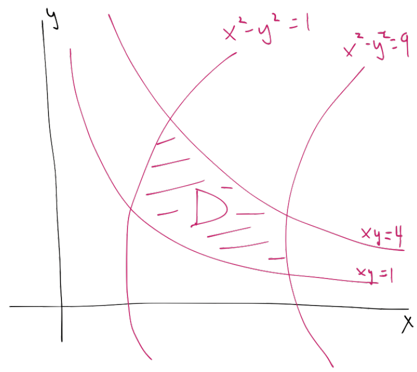
Lecture 10: Cylindrical and Spherical Integrals

Today we finished talking about the concept of making a change of variables in an integral, focusing on rewriting triple integrals in terms of cylindrical and spherical coordinates. The key, as usual, is determining what happens to dV and understanding how to describe bounds on regions in terms of these new coordinates.

Warm-Up. Let's compute

$$\iint_D (x^2+y^2)e^{x^2-y^2}\,dA$$

where D is the region in the first quadrant bounded by the hyperbolas $x^2-y^2=1, x^2-y^2=9, xy=1$, and $xy=4$:



Based on the region we make the change of variables

$$u = xy \text{ and } v = x^2-y^2,$$

so that the bounds become constant in terms of u and v. Then

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\det\begin{pmatrix} y & x \\ 2x & -2y\end{pmatrix}\right| = |-2y^2-2x^2| = 2(x^2+y^2),$$

so

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \frac{1}{2(x^2+y^2)}.$$

Note that this will simplify with the x^2+y^2 part of the integrand, which is good since solving for x^2+y^2 explicitly in terms of u and v will not be easy here.

We get

$$\begin{aligned}\iint_D (x^2+y^2)e^{x^2-y^2}\,dA &= \int_1^9\int_1^4 (x^2+y^2)e^v \underbrace{\frac{1}{2(x^2+y^2)}}_{\text{Jacobian}}\,du\,dv \\ &= \int_1^9\int_1^4 \frac{1}{2}e^v\,du\,dv \\ &= \int_1^9 \frac{3}{2}e^v\,dv \\ &= \frac{3}{2}(e^9-e).\end{aligned}$$

Converting to cylindrical and spherical coordinates. Everything we've done so far works for triple integrals as well, where the Jacobian factor is now determined by the determinant of a 3×3 matrix. In particular, we can write triple integrals in terms of cylindrical and spherical coordinates, which are the only types of 3-dimensional changes of variables we'll actually consider.

In cylindrical coordinates, we have

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

so the Jacobian factor is

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \left|\det\begin{pmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\right| = |r| = r.$$

Thus $dV = r\,dz\,dr\,d\theta$. (This makes sense, since only x and y are affected when converting to cylindrical coordinates and $dx\,dy = r\,dr\,d\theta$ in polar coordinates.)

In spherical coordinates, we have

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$$

so the Jacobian factor is

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \left|\det\begin{pmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{pmatrix}\right| = |\rho^2\sin\phi| = \rho^2\sin\phi.$$

(Of course, I omitted the computation of this determinant, but it is straightforward if a bit tedious—you should all do it at least once in your lives! You can also find the computation in the solution to Problem 5 here: `http://math.northwestern.edu/~scanez/courses/math290/winter13/handouts/final-practice-solns.pdf`.) Note that from now we will always restrict the values of ϕ to run between 0 and π, and for such values $\sin\phi \geq 0$, which is why $|\sin\phi| = \sin\phi$. Thus $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

Important. When converting triple integrals to cylindrical or spherical coordinates:

$$dV = r\,dz\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Just to emphasize: do NOT forget the Jacobian factors!

Example 1. Let's derive the formula for the volume of the region B enclosed by a sphere of radius R. In cylindrical coordinates with respect to the order $dz\,dr\,d\theta$, the middle and outer bounds can be determined from the shadow in the xy-plane—this is a disk of radius R, so θ makes one full revolution from 0 to 2π and r moves from the origin at $r = 0$ out to the boundary circle at $r = R$. Now, the equation of the sphere in cylindrical coordinates is

$$r^2 + z^2 = R^2.$$

With z on the inside, z starts at the bottom half of the sphere at $z = -\sqrt{R^2 - r^2}$ and moves to the top half at $z = \sqrt{R^2 - r^2}$. Thus in cylindrical coordinates the volume is given by

$$\iiint_B dV = \int_0^{2\pi}\int_0^R\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\,dz\,dr\,d\theta.$$

This is computable, but instead we'll do the actual computation in a bit using spherical coordinates.

For practice, let's also write this integral in the order $dr\,dz\,d\theta$. The outer bounds on θ stay the same. The middle z bounds come from the min/max values of z over B, which are $z = -R$ at the south pole and $z = R$ at the north pole. Finally, r (the distance from the z-axis) starts at $r = 0$ on the z-axis and moves away from the z-axis until it hits the sphere at $r = \sqrt{R^2 - z^2}$. Thus

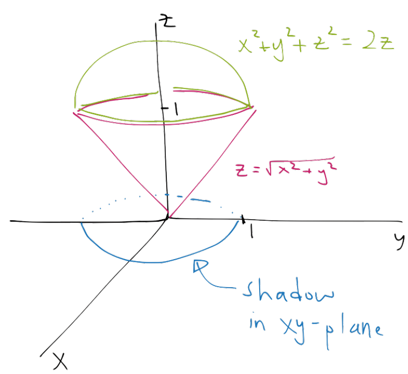
$$\iiint_B dV = \int_0^{2\pi} \int_{-R}^{R} \int_0^{\sqrt{R^2-z^2}} r\, dr\, dz\, d\theta.$$

Now we express and compute the volume in terms of spherical coordinates in the order $d\rho, d\phi, d\theta$. The bounds on θ are the same as in cylindrical coordinates, to get the entire sphere ϕ has to cover all possible values from 0 to π, and in any direction ρ starts at the origin at $\rho = 0$ and extends away from the origin until it hits the sphere at $\rho = R$. Thus:

$$\begin{aligned}\iiint_B dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{R^3}{3} \sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} \frac{2R^3}{3}\, d\theta \\ &= \frac{4}{3}\pi R^3.\end{aligned}$$

Now how straightforward this was to compute in spherical coordinates as opposed to cylindrical or, even worse, rectangular coordinates.

Example 2. We determine the integral of $f(x, y, z) = xz$ over the "ice cream cone" shaped region E bounded by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2z$ (which is the same as $x^2 + y^2 + (z - 1)^2 = 1$):



First we set this up in cylindrical coordinates in the order $dz\, dr\, d\theta$. The shadow of this region in the xy-plane is the disk with boundary circle $x^2 + y^2 = 1$, which is the circle lying directly below the intersection of the cone and the sphere; this intersection occurs at $z = 1$ (by using the cone and sphere equations to solve for z), so the equation of the shadow circle comes from setting $z = 1$ in either the cone or sphere equations. Thus to describe this shadow disk θ makes one full revolution and r goes from 0 to 1. Now, in the given region z starts along the bottom cone and moves up to the top sphere, so z goes from $z = \sqrt{x^2 + y^2} = r$ to $z = 1 + \sqrt{1 - x^2 - y^2} = 1 + \sqrt{1 - r^2}$. Thus

$$\iiint_E xz\, dV = \int_0^{2\pi} \int_0^{1} \int_r^{1+\sqrt{1-r^2}} zr^2 \cos\theta\, dz\, dr\, d\theta.$$

Note that one of the r factors comes from $x = r\cos\theta$ and the other from $dV = r\,dz\,dr\,d\theta$. Computing this directly is possible if a bit tricky (you'll have to integrate $\sqrt{1-r^2}$ with respect to r at some point), but note that since there are no θ's in the inner and middle bounds, the result of the inner and middle integrations will be some constant M times $\cos\theta$, so this integral becomes:

$$\int_0^{2\pi}\int_0^1\int_r^{1+\sqrt{1-r^2}} zr^2\cos\theta\,dz\,dr\,d\theta = \int_0^{2\pi} M\cos\theta\,d\theta = M\sin\theta\Big|_0^{2\pi} = 0.$$

This makes sense: $f(x,y,z) = xz$ is odd with respect to x and the region E is symmetric across the yz-plane, so the integral should indeed be zero.

For practice, say we wanted to set this up in the order $dr\,dz\,d\theta$ instead. Now the "outer" boundary of r depends on where we are: over the "scoop" part of the region r extends to the sphere while over the "cone" part r extends to the cone. So in this order we have to split up the region at $z = 1$ where the cone and sphere intersect. We get

$$\int_0^{2\pi}\int_0^1\int_0^z zr^2\cos\theta\,dr\,dz\,d\theta + \int_0^{2\pi}\int_1^2\int_0^{1-\sqrt{z-1}^2} zr^2\cos\theta\,dr\,dz\,d\theta,$$

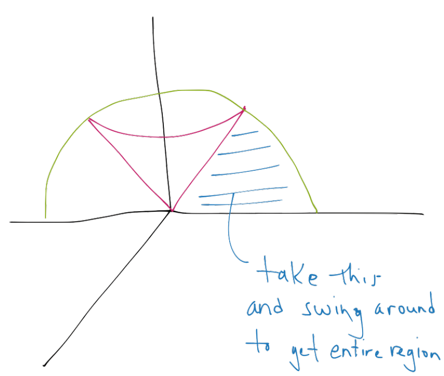
where the first term is the integral over the "cone" part and the second the integral over the "scoop" part.

Finally let's set this up in spherical coordinates as well, in the order $d\rho\,d\phi\,d\theta$. The bounds on theta are the same as before. Over our region, ϕ starts along the positive z-axis at $\phi = 0$ and swings down until it is aligned with the cone, which has equation $\phi = \frac{\pi}{4}$ in spherical coordinates. Thus the bounds on ϕ are 0 to $\pi/4$. Now, in any fixed ϕ-direction, ρ starts at the origin at $\rho = 0$ and extends out to the sphere, which has equation $\rho = 2\cos\phi$ in spherical coordinates. (Indeed, the sphere is $\rho^2 = 2z$, so substitute in $z = \rho\cos\phi$ and canceling a ρ.) Thus we have:

$$\iiint_E xz\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} \rho^4\sin^2\phi\cos\theta\cos\phi\,d\rho\,d\phi\,d\theta,$$

where one of the ρ factors comes from x, one from z, and two from dV, and one of the $\sin\phi$ factors comes from x and one from dV. This is computable, although you'll have to use some trig identities to make it work out. Using symmetry is definitely the quickest way of determining the value of this integral.

Example 3. Finally we determine the volume of the solid E above the xy-plane, outside the cone $z = \sqrt{x^2+y^2}$, and inside the unit sphere:



In spherical coordinates, θ goes from 0 to 2π, ϕ from $\pi/4$ along the cone down to $\pi/2$ on the xy-plane, and ρ from 0 at the origin out to 1 on the unit sphere. Thus:

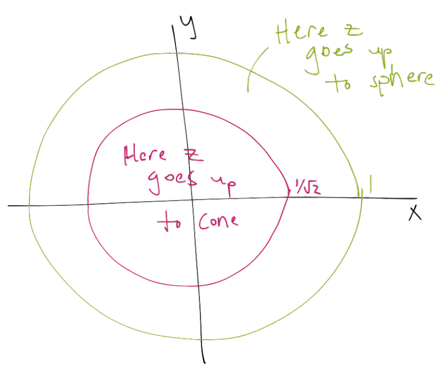
$$\begin{aligned}\iiint_E dV &= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{1}{3}\sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} \frac{1}{3\sqrt{2}}\, d\theta \\ &= \frac{\pi\sqrt{2}}{3}.\end{aligned}$$

In cylindrical coordinates with respect to $dr\,dz\,d\theta$, θ again makes one full revolution, z goes from 0 to its max value over the entire region, which occurs at $z = 1/\sqrt{2}$ where the cone and sphere intersect, and r (measured moving away from the z-axis) starts along the cone $r = z$ and increases out to the sphere at $r = \sqrt{1 - z^2}$. Thus

$$\begin{aligned}\iiint_E dV &= \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_z^{\sqrt{1-z^2}} r\, dr\, dz\, d\theta \\ &= \int_0^{2\pi}\int_0^{1/\sqrt{2}} \frac{1}{2}(1 - z^2 - z^2)\, dz\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left(z - \frac{2}{3}z^3\right)\Big|_0^{1/\sqrt{2}}\, d\theta \\ &= \pi\left(\frac{1}{\sqrt{2}} - \frac{2}{6\sqrt{2}}\right)\end{aligned}$$

which simplifies to the value found when using spherical coordinates.

Finally, let's also set this integral up in the order $dz\,dr\,d\theta$. The issue here is that the "top" surface of our region, and hence the upper bound on z, is the cone for part of the region but the sphere over the rest. Thus we must split up our region according to the intersection of the cone and the sphere. This intersection occurs at a distance of $1/\sqrt{2}$ away from the z-axis, so the shadow of our entire region in the xy-plane looks like:



Hence we get:

$$\int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_0^{r} r\, dz\, dr\, d\theta + \int_0^{2\pi}\int_{1/\sqrt{2}}^{1}\int_0^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta,$$

where the first piece is the integral over the "inner disk" of the shadow while the second is the integral over the "outer ring" of the shadow.

Lecture 11: Back to Parametric Curves

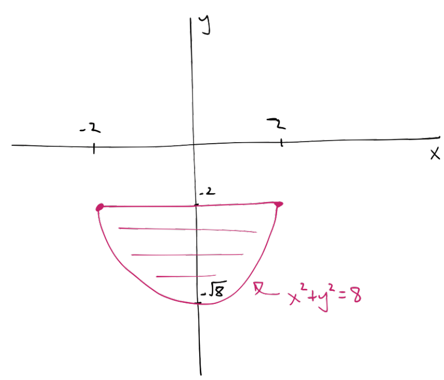
Today we took a brief detour from integration, going back to review some things about curves. But have no fear, we'll be back to considering new types of integrals soon enough.

Warm-Up. Note: There was a subtle error I made in class when setting up this integral in spherical coordinates, which one of your classmates pointed out in class (12pm section) and afterwards as well. Kudos to this student for not letting me get away with it! The solution below is the one we went through in class, which is thus incorrect. Still, the error is hard to see, and I think correcting it fully will overcomplicate things. So, I'll leave the solution as is, but I'll point out where the error is and how it could be fixed. As a result, setting up this specific integral in spherical coordinates would be too difficult for an exam; the integral in cylindrical coordinates is setup correctly.

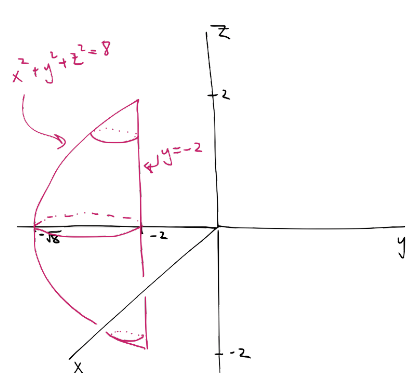
We rewrite the following triple integral in cylindrical and spherical coordinates:

$$\int_{-\sqrt{8}}^{-2}\int_{-\sqrt{8-y^2}}^{\sqrt{8-y^2}}\int_{-\sqrt{8-x^2-y^2}}^{\sqrt{8-x^2-y^2}} y\,dz\,dx\,dy.$$

The shadow in the xy-plane is



and the entire region of integration is



which is the piece of the sphere of radius $\sqrt{8}$ which lies to the left of the $y = -2$ plane.

In cylindrical coordinates, we first need bounds on θ which start along the leftmost point in the xy-shadow drawn above and swing over to the rightmost point. The left point has the same x and y coordinate, so $\theta = 5\pi/4$, and the right point has the same x and y coordinate except for the sign, so $\theta = 7\pi/4$. Next, in this shadow r starts along the line $y = -2$, which is $r \sin\theta = -2$ in polar coordinates, and moves away from the origin out to $r = \sqrt{8}$. Finally, in the entire region of integration z starts along the bottom half of the sphere $x^2 + y^2 + z^2 = 8$ and increases up to the top half, so we get

$$\int_{5\pi/4}^{7\pi/4} \int_{-2/\sin\theta}^{\sqrt{8}} \int_{-\sqrt{8-r^2}}^{\sqrt{8-r^2}} r^2 \sin\theta \, dz \, dr \, d\theta.$$

Note that for the values of θ we're considering, $\sin\theta < 0$ so $-2/\sin\theta$ is indeed ≥ 0, as bounds on r should always be in cylindrical coordinate integrals.

In spherical coordinates, the bounds on θ are the same as before. The topmost point of our solid is $(-2, 0, 2)$, which has $\phi = \pi/4$ in spherical coordinates, and the bottommost point is $(-2, 0, -2)$, which has $\phi = 3\pi/4$; these give the bounds on ϕ. (Determining the bounds on ϕ is the error I alluded to earlier—I'll talk about why and how to fix it at the end.) Finally, ρ starts along the plane $y = -2$, which is $\rho \sin\phi \sin\theta = -2$ in spherical coordinates, and moves out away from the origin to $\rho = \sqrt{8}$ on the sphere. Thus we get

$$\int_{5\pi/4}^{7\pi/4} \int_{\pi/4}^{3\pi/4} \int_{-2/\sin\phi\sin\theta}^{\sqrt{8}} \rho^3 \sin\phi^2 \sin\theta \, d\rho \, d\phi \, d\theta.$$

Note that for the values of θ and ϕ we consider, $\sin\phi \sin\theta > 0$ so $-2/\sin\phi\sin\theta \geq 0$, as bounds on ρ should be in spherical integrals.

Now to the error. The problem is that the bounds on ϕ only work for the part of the region which is on the yz-plane, but not elsewhere. Notice that as θ moves away from the $3\pi/2$ value which points us on the negative y-side of the yz-plane, the topmost point on our region has ϕ angle larger than $\pi/4$ and the bottommost point has ϕ less than $3\pi/4$ (i.e. both points are closer to the xy-plane than the points at $\theta = 3\pi/2$ were), where the bounds we setup would say that no matter what θ is, ϕ always starts at $\pi/4$ and swings down to $3\pi/4$. In fact, for the bounds we gave there are values where the plane $y = -2$ is *beyond* the sphere, which would say that ρ starts on the plane and moves *backwards* to the sphere.

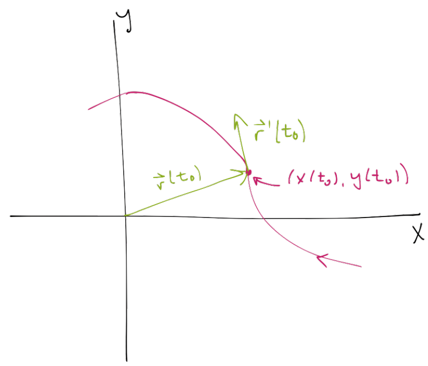
So, to do this correctly we would have to determine how the bounds on ϕ change as θ changes, so that the bounds on ϕ will depend on θ instead of being constant. This is hard to do: you have to write the equations for the plane $y = -2$ and for its intersection with the sphere (which is the circle $x^2 + z^2 = 8$ in the plane $y = -2$) in spherical coordinates, and solving for ϕ in terms of θ. This will involve some tricky trig computations, which is why it's not worth going through. My bad! Again, you should not expect that anything where bounds on ϕ or θ depend on each other will show up on an exam.

Curves and tangent vectors. Recall that curves in 2 and 3 dimensions can be described using *parametric equations*:

$$\mathbf{r}(t) = (x(t), y(t)) \text{ or } \mathbf{r}(t) = (x(t), y(t), z(t))$$

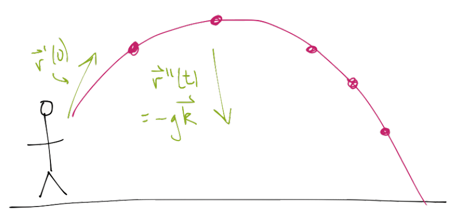
where $a \leq t \leq b$. To be clear, at a given "time" t_0 the vector $\mathbf{r}(t_0)$ extends from the origin to the point $(x(t_0), y(t_0), z(t_0))$ on the curve determined by the value of the parametric equations at time t_0; $\mathbf{r}(t_0)$ does not lie on the curve nor is it drawn as starting at the point $(x(t_0), y(t_0), z(t_0))$—the curve in question is traced out by the *endpoints* of the vectors $\mathbf{r}(t)$ as t varies.

When moving along the curve, the direction of travel is given by $\mathbf{r}'(t)$, which gives the *tangent vector* at points along the curve:



This tangent vector is also interpreted as the *velocity* vector when moving along the curve, and its derivative $\mathbf{r}''(t)$ is the *acceleration* vector.

Example. Here is a classic application, the type of which you would see in any introductory mechanics physics course. Suppose we are standing on the ground and throw a ball in the air with an initial velocity $\mathbf{r}'(0) = (1, 1)$, meaning the ball is thrown at a 45° angle. We want to determine parametric equations of the path the ball will follow, assuming that the only thing affecting the ball's movement is the downward force of gravity:



The acceleration due to gravity is $-g\mathbf{j}$ where g is about 9.8 meters/seconds^2. (It is not important that you know this value—this is not a physics course!) Thus the path we want should satisfy

$$\mathbf{r}''(t) = -g\mathbf{j}, \mathbf{r}'(0) = (1, 1), \text{ and } \mathbf{r}(0) = (0, 0),$$

the last condition since we are assuming that we we are standing is the origin. The goal is then to determine $\mathbf{r}(t)$ from this information. Taking the antiderivative of $\mathbf{r}''(t)$ gives

$$\mathbf{r}'(t) = a\mathbf{i} + (-gt + b)\mathbf{j}$$

for some constants a and b. Since

$$\mathbf{r}'(0) = a\mathbf{i} + b\mathbf{j}$$

should be $(1, 1) = \mathbf{i} + \mathbf{j}$, we must have $a = 1$ and $b = 1$. Thus our velocity vector is

$$\mathbf{r}'(t) = \mathbf{i} + (-gt + 1)\mathbf{j}.$$

Taking another antiderivative gives

$$\mathbf{r}(t) = (t+m)\mathbf{i} + \left(-\frac{1}{2}gt^2 + t+\right)\mathbf{j}$$

for some constants m and n. Since $\mathbf{r}(0) = (0,0)$, we get that $m = n = 0$ so our path is given by

$$\mathbf{r}(t) = t\mathbf{i} + \left(-\frac{1}{2}gt^2 + t\right)\mathbf{j}.$$

Hence the path the ball will follow has parametric equations

$$x = t,\ y = -\frac{1}{2}gt^2 + t,\ 0 \leq t \leq \frac{2}{g}$$

where the upper bound on t comes from determine when the ball will hit the ground, which happens when $y = 0$. Note that these equations makes sense, since intuitively the ball should indeed follow some parabola-like path.

Arclength. The *arclength* (or simply *length*) of a curve given by parametric equations

$$\mathbf{r}(t), a \leq t \leq b$$

is given by

$$\text{length} = \int_a^b \|\mathbf{r}'(t)\|\ dt.$$

Intuitively, $\|\mathbf{r}'(t)\|$ gives the "length" of the small "infinitesimal" piece of the curve at time t, and adding up all these small lengths as the integral does should give the total length of the curve.

For instance, the unit circle has parametric equations $\mathbf{r}(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$, in which case $\|\mathbf{r}'(t)\| = 1$ for all t. Thus the unit circle has length

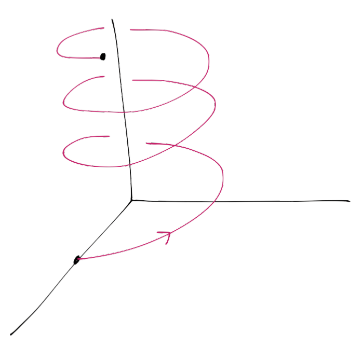
$$\int_0^{2\pi} \|\mathbf{r}'(t)\|\ dt = \int_0^{2\pi} dt = 2\pi,$$

as you would expect.

Final example. We determine the length of the curve

$$\mathbf{r}(t) = (\cos 3t, \sin 3t, 2t^{3/2}),\ 0 \leq t \leq 2\pi.$$

To get a sense for what length we're actually computing, these parametric equations describe a *helix* starting at the origin and ending at $(1, 0, 2(2\pi)^{3/2})$



Indeed, the x and y equations describe a circle in the xy-plane, which is the "shadow" of the entire 3-dimensional curve. As x and y move around the z-axis in this circular direction, the z coordinate increases, so the curve moves up.

We have

$$\mathbf{r}'(t) = (-3\sin 3t, 3\cos 3t, 3\sqrt{t}),$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 9t} = 3\sqrt{1+t}$$

and thus

$$\int_0^{2\pi} \|\mathbf{r}'(t)\| \, dt = \int_0^{2\pi} 3\sqrt{t+1}\, dt = 2(t+1)^{3/2}\Big|_0^{2\pi} = 2(2\pi+1)^{3/2}$$

is the length of the helix.

Important. A curve with parametric equations $\mathbf{r}(t)$ for $a \leq t \leq b$ has tangent/velocity vectors given by $\mathbf{r}'(t)$, length given by

$$\int_a^b \|\mathbf{r}'(t)\| \, dt,$$

and acceleration vectors given by $\mathbf{r}''(t)$.

Where this will lead. Why did we come back to parametric curves after talking about integration? The answer is that curves will play an essential role in the types of integral we will soon be looking at. To get a sense as to why, consider first the case of a single-variable integral:

$$\int_a^b f(x)\, dx.$$

Here we are integrating f over the interval $[a, b]$, which we can view as a curve on the x-axis. With this interpretation it is then natural to consider integrating functions over more general curves in 2 and 3 dimensions.

Given a two-variable function $f(x, y)$, we will discuss what it means to integrate it over a curve C in the xy-plane; this will give rise to what's called the *line integral* of f over C, which is denoted by

$$\int_C f(x, y)\, ds.$$

The "ds" refers to an infinitesimal piece of arclength, generalizing the idea that dx represents an infinitesimal length on the x-axis. Just as single-variable integrals compute areas, such line integrals also have a nice interpretation in terms of areas.

So, we will need to be able to talk about equations for general types of curves, which is why we're filling in some gaps now. This will then lead directly to the notion of integrating a "vector field" along a curve, which is what we really care about.

Lecture 12: Vector Fields

Today we started talking about vector fields, which will be our main object of study the rest of the quarter. Vector fields are used to model tons of different phenomena in many different fields (different use of the word field here than in "vector field"!), leading to many applications of the things we will look at.

Speed. One quick thing to note about curves. Given a curve $\mathbf{r}(t), a \le t \le b$, we can define the function

$$s(t) = \int_a^t \left\|\mathbf{r}'(\tau)\right\| \, d\tau$$

which gives the length of the curve up to time t, i.e. the distance traveled along the curve at time t. The Fundamental Theorem of Calculus says that

$$\frac{ds}{dt} = \left\|\mathbf{r}'(t)\right\|,$$

so we find that the length $\|\mathbf{r}'(t)\|$ of the velocity vector at time t is precisely the *speed* (since speed is indeed the rate of change of distance with respect to time) at time t. This is a basic fact learned in a physics course, but the justification as to why is often glossed over.

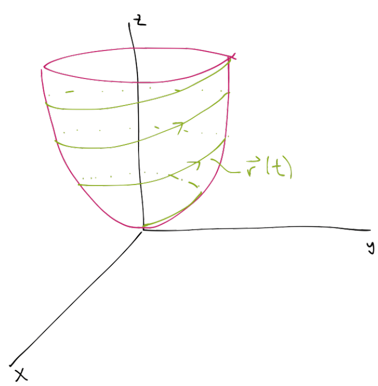
Warm-Up 1. We find parametric equations for the tangent line to

$$\mathbf{r}(t) = (t\cos t, t\sin t, t^2)$$

at $t = \pi$. First, to have a visualization for what we're trying to do, let's determine what the curve looks like. The x and y equations describe some circular motion, only with increasing radius as we go around. As this is happening in the x and y directions, we move up due to the $z = t^2$ equation. Thus, we get some kind of spiral moving upwards; to be precise, note that the given parametric equations satisfy

$$x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z,$$

so the curve in question lies on the paraboloid $z = x^2 + y^2$:



To describe the required tangent line we need a point on the line and a direction vector for the line. The point

$$\mathbf{r}(\pi) = (-\pi, 0, \pi^2)$$

is where we're at on the line, and the direction of the tangent line at this point is given by the tangent vector at this point:

$$\mathbf{r}'(t) = (-t\sin t + \cos t, t\cos t + \sin t, 2t), \text{ so } \mathbf{r}'(\pi) = (1, \pi, 2\pi).$$

Thus the tangent line has parametric equations:

$$x = -\pi + t, \; y = \pi t, \; z = \pi^2 + 2\pi t.$$

(Remember from last quarter that the coefficients of t come from the direction vector.)

Warm-Up 2. If you draw a circle $\mathbf{r}(t)$, you may notice that at any point along the circle the position vector $\mathbf{r}(t)$ is orthogonal to the tangent vector $\mathbf{r}'(t)$:

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0.$$

The same is true for any curve on a sphere in $\mathbb{R}^3$, and we now show that circles in $\mathbb{R}^2$ or spheres in $\mathbb{R}^3$ are the only curves with this property. To be precise, the claim (say in $\mathbb{R}^3$) is that: if $\mathbf{r}(t)$ is a curve with the property that $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for all t, then $\mathbf{r}(t)$ is a curve which completely lies on a sphere.

The key is the following "product rule" for dot products:

$$\frac{d}{dt}[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] = \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t).$$

In particular,

$$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

Since $\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|^2$, this says that

$$\frac{d}{dt}\|\mathbf{r}(t)\|^2 = 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

since $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. Thus $\|\mathbf{r}(t)\|^2$ must be constant (since it has zero derivative), so $\|\mathbf{r}(t)\| = k$ is constant is well. Hence since $\mathbf{r}(t)$ always has the same length k for any t, $\mathbf{r}(t)$ lies on the sphere of radius k.

Vector fields. A *vector field* is an assignment of a vector to each point of our space, whether is be the $\mathbb{R}^2$ or $\mathbb{R}^3$. To be precise, a vector field on $\mathbb{R}^2$ is a function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and a vector field on $\mathbb{R}^3$ is a function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$.

So, a vector field on $\mathbb{R}^2$ has an equation of the form:

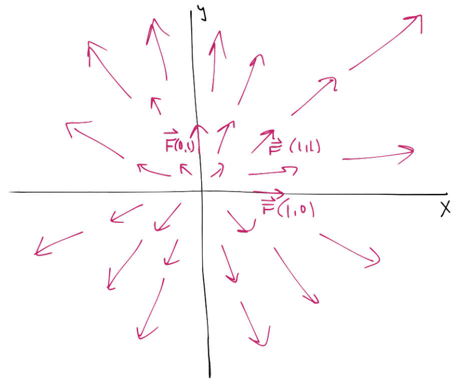
$$\mathbf{F}(x, y) = (P(x, y), Q(x, y)) \text{ or } P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

where P and Q themselves are functions. Given a point (x, y), this gives a vector $(P(x, y), Q(x, y))$ which we draw as starting at the point (x, y) itself. Drawing these all thus gives a collection of vectors covering the entire xy-plane.

Example 1. Let's sketch the vector field $\mathbf{F}(x, y) = (x, y)$. For instance, at $(1, 0), (0, 1)$, and $(1, 1)$ we respectively draw the vectors

$$\mathbf{F}(1, 0) = (1, 0),\ \mathbf{F}(0, 1) = (0, 1),\ \mathbf{F}(1, 1) = (1, 1).$$

In this case, since the vector at the point (x, y) is the vector (x, y) itself, we get vectors which all point "radially" away from the origin, so our vector field looks like:



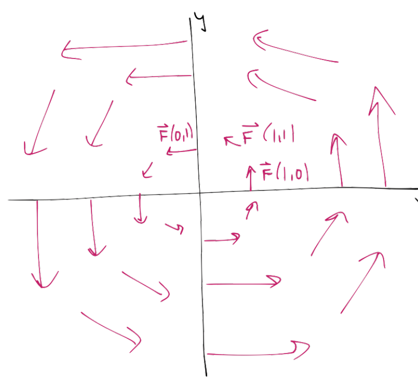
Note that as we move further away from the origin our vectors get longer. Similarly, in 3-dimensions the vector field $\mathbf{F}(x, y, z) = (x, y, z)$ would also consist of vectors pointing directly away from the origin.

The pictures of these specific vector fields are ones which should be ingrained in your memory.

Example 2. Here is another vector field whose sketch should be ingrained in your minds: $\mathbf{F}(x, y) = (-y, x)$. At the points $(1, 0), (0, 1)$, and $(1, 1)$ we get

$$\mathbf{F}(1, 0) = (0, 1),\ \mathbf{F}(0, 1) = (-1, 0),\ \mathbf{F}(1, 1) = (-1, 1).$$

As you draw more and more of these vectors you should start to recognize a pattern:



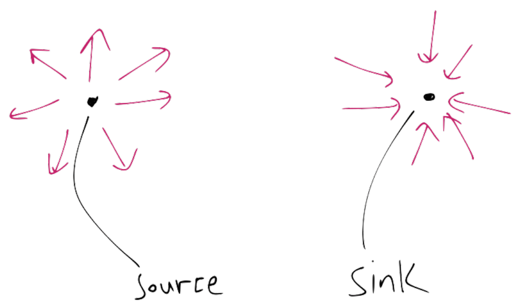
so this vector field consists of vectors which move counterclockwise around the origin, again getting longer as we move away from the origin. The vector field

$$\mathbf{G}(x, y) = \left(-\frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right)$$

also consists of vectors moving counterclockwise around the origin, only in this case all vectors have length 1.

Here is one way to be sure that this is the picture we get: at any point (x, y) the vector $(-y, x)$ from this vector field is orthogonal to the vector (x, y) from the radial vector field in Example 1, and drawing vectors orthogonal to each vector in the picture for Example 1 gives vectors which look like the ones for $\mathbf{F}(x, y) = (-y, x)$.

Intuition. What good are vector fields for? The answer is that they are used to models "flows", such as the flow of water on the surface of a like. (Indeed, we simply view each vector as describing the motion of water at the given point.) For instance, throwing a rock into the lake produces "ripples" of water moving away from a point—we call such a point a "source". Instead, if our field described the flow of water in a bathtub, at the drain we would have vectors moving in towards a point—we call such a point a "sink":



Even though thinking of vectors fields as describing the flow of water will give the best geometric intuition, there are other types of flows we can consider: the flow of a population across different regions of a country, the flow of money across different sectors of an economy, the flow of electrons in a battery, etc. The great thing is that all of these seemingly different types of "flow" are modeled by the same mathematical object: vector fields.

Important. Vector fields give a way of assigning a vector to each point of space, which we visualize as a collection of a bunch of vectors covering the xy-plane in the 2-dimensional case or $\mathbb{R}^3$ in the 3-dimensional case. Geometrically, we will often think of a vector field as describing the flow of a liquid.

Hairy Ball Theorem. Here is a fun fact about vector fields, purely to illustrate an interesting way in which they show up. (So, this is outside the scope of our course.) Imagine a ball covered with hair, which we want to attempt to comb. The fact is that this is not possible to do, i.e. "you cannot comb a hairy ball." Indeed, you can start combing the hairs along the equator in one direction, then move a little off the equator, and so on—the problem is that once you get to the hair on the edge of the ball, combing it will mess up some of the hair you had previously combed. Combing this will then cause some other previously-combed hair to pop back up, and so on. The only way in which combing a hairy ball would actually work is if you *removed* some of the hair.

What does this have to do with mathematics? Imagine that the ball was the surface of a sphere, and that each hair was represented by a vector. Combing a piece of hair amounts to making this vector be *tangent* to the sphere. Thus, in asking to comb all the hair we are asking to give a vector field on the sphere which is tangent to the sphere at each point. Since combing the ball is only possible if we remove some hair, having a vector field be tangent to the sphere is only possible if at some points we have no vector at all, or in other words have the zero vector there. This is the Hairy Ball Theorem (which is no joke the true name this fact is known as):

> Given any tangent vector field on a sphere, there must be at least one point on the sphere where the vector field gives is zero.

Economic application. And finally, to justify that the Hairy Ball Theorem isn't just some mathematical curiosity, here is a concrete application. Say we have three products, with prices p_1, p_2, and p_3 which we can put into a *price vector*

$$\mathbf{p} = (p_1, p_2, p_3).$$

Corresponding to each price there is a demand, and in particular an *excess demand vector*:

$$\mathbf{d} = (d_1, d_2, d_3).$$

Ideally, we would like to be in "equilibrium", meaning that we want prices which result in the perfect amount of demand, so no excess demand. (In other words, prices which result in demand equalling supply.) There is an economic principle (look up Walras' Law to learn more) which implies that the excess demand vector is actually orthogonal to the price vector:

$$\mathbf{p} \cdot \mathbf{d} = 0.$$

Viewing our price vectors as being on a sphere, the excess demand vectors then form a tangent vector field on that sphere. Thus, according to the Hairy Ball Theorem, there is a point on the sphere at which this vector field is zero, or in other words a collection of prices (p_1, p_2, p_3) which result in zero excess demand. Huzzah!

Lecture 13: Divergence and Curl

Today we spoke about the notions of the *divergence* and *curl* of a vector field. These will play crucial roles in some theorems we will soon look at, and have nice geometric interpretations emphasizing the ways in which we think of vector fields as modeling "flows".

Warm-Up. We will sketch the vector field

$$\mathbf{F}(x, y) = (ye^{xy}, xe^{xy} + y).$$

This is not easy to do as is, since drawing a bunch of vectors arising from this field does not give a recognizable pattern. Instead, the point here is that this field is actually the *gradient* of a function! Gradient vector fields arise in many applications, and determining whether or not a vector field is the gradient of a function is an important concept we'll come back to later.

We want to function a function $f(x, y)$ such that $\nabla f = \mathbf{F}$, meaning

$$(f_x, f_y) = (ye^{xy}, xe^{xy} + y).$$

Thus we need a function which satisfies

$$f_x = ye^{xy} \text{ and } f_y = xe^{xy} + y.$$

Taking the antiderivative of the first expression with respect to x gives

$$f(x, y) = e^{xy} + V(y)$$

where $V(y)$ is some expression only involving y. (This term is needed since it represents a "constant" with respect to x.) But taking the derivative of this with respect to y gives

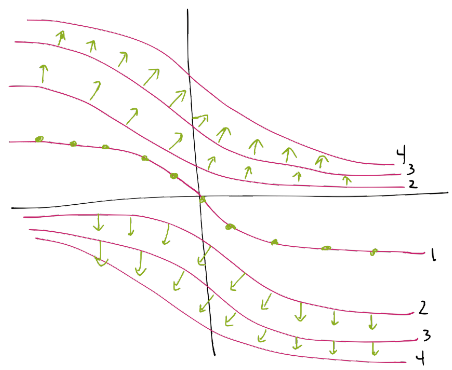
$$f_y = xe^{xy} + V_y.$$

This should be equal to $xe^{xy} + y$ for the function we're looking for, so we need $V_y = y$. Thus one possibility is $V(y) = \frac{1}{2}y^2$, meaning that

$$f(x, y) = e^{xy} + \frac{1}{2}y^2$$

is a function such that $\mathbf{F} = \nabla f$.

Recall from last quarter that at any point in the xy-plane, ∇f points in the direction in which f increases most rapidly and is orthogonal to the level curve of f through that point. Thus, by drawing these level curves and vectors orthogonal to them we can get a picture of the vector field $\mathbf{F} = \nabla f$. We get something like:



Again, note that each vector is orthogonal to the corresponding level curve and that each points in the direction in which f increases most rapidly. (I determined the shape of these level curves using a computer, so this is not something you'd be expected to be able to do unless the level curves were simpler.) The zero vectors along the level curve at $z = 1$ reflect the fact that 1 is a minimum value of this function.

Divergence. The *divergence* of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the function $\operatorname{div}\mathbf{F}$ defined by

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Introducing the notation

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

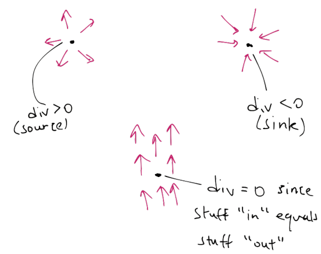
which we think of as a "vector" even though it's really not (its entries are not functions but rather "operations"), we can recall the formula for divergence using the notation

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}.$$

Indeed, we think of the right side as a usual dot product, except that instead of "multiplying" the entries of ∇ with those of $\mathbf{F}$, we *apply* the entries of ∇ *to* those of $\mathbf{F}$:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (P, Q, R) = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Geometrically, $\operatorname{div}\mathbf{F}$ measures the net tendency of $\mathbf{F}$ to flow either "away" or "towards" a given point: $\operatorname{div}\mathbf{F}(p) > 0$ if the "outward" flow of $\mathbf{F}$ through p outweighs the "inward" flow, $\operatorname{div}\mathbf{F}(p) < 0$ if the inward flow of $\mathbf{F}$ through p outweighs the outer flow, and $\operatorname{div}\mathbf{F}(p) = 0$ if there is no "net" flow of $\mathbf{F}$ through p:



The reason as to why the divergence of a field has this geometric property depends on what's called *Gauss's Theorem*, which we'll talk about at the end of the quarter.

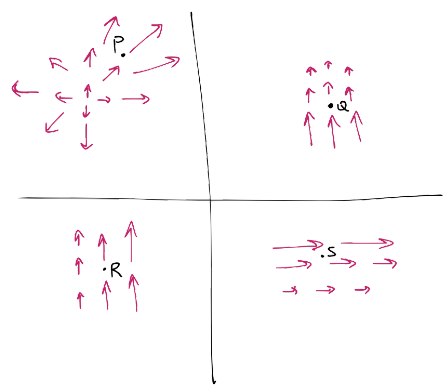
Example 1. Suppose that $\mathbf{F}$ is the vector field

$$\mathbf{F} = (xy\sin z + y)\mathbf{i} + (y - xe^z)\mathbf{j} + xyz\mathbf{k}.$$

Then

$$\operatorname{div}\mathbf{F} = \frac{\partial(xy\sin z + y)}{\partial x} + \frac{\partial(y - xe^z)}{\partial y} + \frac{\partial(xyz)}{\partial z} = y\sin z + 1 + xy.$$

Example 2. Consider the 2-dimensional vector field $\mathbf{G}$ drawn below:



At the point P, the vectors flowing "away" from P are longer than those flowing "towards" P. This means that there is a greater outward flow and inward flow at P, so $\operatorname{div}\mathbf{F}(P) > 0$. (Indeed, this piece of the vector field looks similar to the vector field $x\mathbf{i} + y\mathbf{j}$, which has positive divergence 2 everwhere.) At Q, the inward flow is stronger than the outward flow, so $\operatorname{div}\mathbf{F}(Q) < 0$. At R, the lower right vector flowing "in" has the same length as the upper right vector flowing "out", so the effect of these two cancel out. Indeed, any vector flowing "towards" R has a corresponding one of the same length flowing out, so $\operatorname{div}\mathbf{F}(R) = 0$.

Curl. The *curl* of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the vector field $\operatorname{curl}\mathbf{F}$ defined by

$$\operatorname{curl}\mathbf{F} = (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_y - P_x)\mathbf{k}.$$

Now, do not waste your time memorizing this formula, but rather note the following: using the same $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ notation as before, the formula for the curl comes from the cross product-like expression:

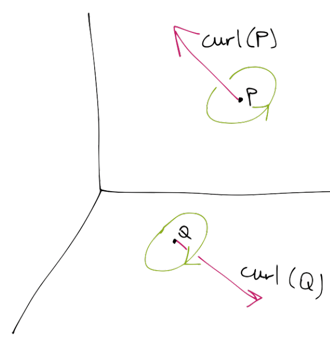
$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

where we compute this as we would a usual cross product, only we *apply* the entries in the second row *to* those in the third. Again, this is not really a cross product since the entries in the second row are operations, but thinking about it as a cross product anyway gives a nice way of remembering the curl formula. Thus

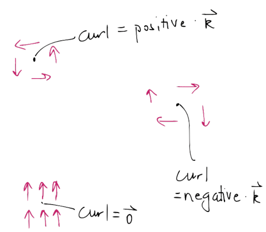
$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$$

is another common notation we'll use for curl.

Geometrically, $\operatorname{curl}\mathbf{F}$ measures the net "circulation" of $\mathbf{F}$ around different points: $\operatorname{curl}\mathbf{F}(p)$ gives a vector whose direction gives the "axis" of circulation of $\mathbf{F}$ at p and whose length gives the strength of circulation:



The precise direction of $\operatorname{curl}\mathbf{F}(p)$ is determined by the right-hand rule—curling your fingers in the direction of the circulation should result in your thumb pointing in the direction of the curl. For a two-dimensional vector field $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$, $\operatorname{curl}\mathbf{F}$ is always a multiple of $\mathbf{k}$: it is a positive multiple of $\mathbf{k}$ if $\mathbf{F}$ circulates "counterclockwise" around p, it is a negative multiple of $\mathbf{k}$ is $\mathbf{F}$ circulates "clockwise" around p, and the curl at p is $\mathbf{0}$ if there is no net circulation around p:

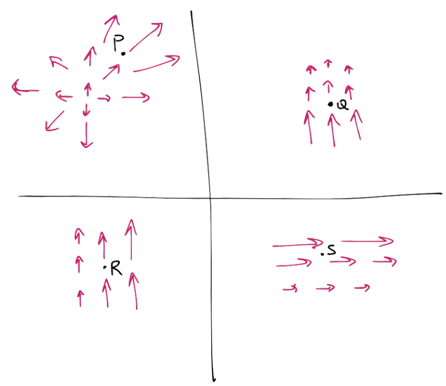


The reason as to why the curl of a field has these geometric properties depends on what's called *Stokes' Theorem*, which we'll talk about in a few weeks.

Back to Example 1. The curl of the vector field from Example 1 is

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy\sin z + y & y - xe^z & xyz \end{vmatrix} = (xz - xe^z)\mathbf{i} + (xy\cos z - yz)\mathbf{k} + (-e^z - x\sin z - 1)\mathbf{k}.$$

Back to Example 2. Recall the field $\mathbf{G}$ from Example 2:



Imagine putting a little plank of wood at each point. At R, the upward flow along the right edge of the plane is stronger than the upward flow along the left edge, so the plane will turn counterclockwise overall and hence $\operatorname{curl}\mathbf{G}(R)$ is a positive multiple of $\mathbf{k}$. At S with a vertical plank, the rightward flow above S is stronger than the rightward flow below S, so the plank will turn clockwise and thus $\operatorname{curl}\mathbf{G}(S)$ is a negative multiple of $\mathbf{k}$. At Q with a horizontal plank, the upward flow along the left side is the same as the upward flow along the right, so the plank of wood won't turn at all and hence $\operatorname{curl}\mathbf{G}(Q) = \mathbf{0}$.

Important. The divergence and curl of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ are given by the formulas

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z \text{ and } \operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_y - P_x)\mathbf{k}.$$

In particular, $\operatorname{div}\mathbf{F}$ is a **function** and $\operatorname{curl}\mathbf{F}$ is a **vector field**. Geometrically, $\operatorname{div}\mathbf{F}$ measures the inward/outward flow of $\mathbf{F}$ at a point and $\operatorname{curl}\mathbf{F}$ measures the circulation of $\mathbf{F}$ at a point.

Lecture 14: Scalar Line Integrals

Today we started talking about line integrals, the first new type of integral we will focus on these final few weeks. There are two types of line integrals: those which arise when integrating a function along a a curve and those where we integrate a vector field along a curve.

Warm-Up. Say that $\mathbf{F}$ and $\mathbf{G}$ are vector fields and that f is a function. Which of the following expressions makes sense?

$$\operatorname{curl}(\operatorname{div}\mathbf{F}),\ \operatorname{curl}(\nabla f),\ \operatorname{div}(\mathbf{F}\cdot\mathbf{G}),\ \operatorname{curl}(\operatorname{div} f),\ \operatorname{div}(\operatorname{curl}\mathbf{G}),\ \operatorname{div}(\nabla f)$$

The first expression $\operatorname{curl}(\operatorname{div}\mathbf{F})$ makes no sense: curl is something we apply to vector fields, but $\operatorname{div}\mathbf{F}$ is a function. The third expression also makes no sense: $\mathbf{F}\cdot\mathbf{G}$ gives a function and we cannot take the divergence of a function. Similarly, the fourth expression does not make sense since $\operatorname{div} f$ is undefined, so we can certainly not take its curl.

The remaining expressions make sense: in the second, ∇f is a vector field and we can thus take its curl; in the fifth, $\operatorname{curl}\mathbf{G}$ is a vector field and we can thus take its divergence; and in the last ∇f is a vector field and we can thus take its divergence.

Two important identities. In fact, not only do the second and fifth expressions above make sense, but they always give the same result no matter the function f nor field $\mathbf{G}$. Indeed, for any function $f(x, y, z)$ we have:

$$\operatorname{curl}(\nabla f) = \operatorname{curl}(f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) = (f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{xy}) = (0, 0, 0)$$

by Clairaut's Theorem! Similarly, for any vector field $\mathbf{G} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ we have

$$\operatorname{div}(\operatorname{curl}\mathbf{G}) = \operatorname{div}(R_y - Q_z, P_z - R_x, Q_x - P_y) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

again by Clairaut's Theorem. Thus, starting with any function, taking gradient and then curl always gives the zero vector field, and starting with any vector field, taking curl and then divergence always gives the zero function.

Important. For any function f, $\operatorname{curl}(\nabla f) = \mathbf{0}$; for any vector field $\mathbf{F}$, $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$.

Scalar line integrals. Given a function $f(x, y)$ and a curve C in the xy-plane with parametric equations $\mathbf{x}(t), a \leq t \leq b$, the *(scalar) line integral* of f over C is:

$$\int_C f(x, y)\, ds = \int_a^b f(\mathbf{x}(t)) \left\|\mathbf{x}'(t)\right\|\, dt.$$

Here, $f(\mathbf{x}(t))$ means the expression obtained when substituting the parametric equations for x and y into f and $\|\mathbf{x}'(t)\|$ denotes the length of the tangent vector $\mathbf{x}'(t)$. In other words, "ds" represents an infinitesimal piece of arclength:

$$ds = \left\|\mathbf{x}'(t)\right\|\, dt$$

along C. (So, $\|\mathbf{x}'(t)\|$ should be thought of as a "Jacobian" factor telling us how lengths dt in the interval $[a, b]$ are "expanded" to lengths ds along C.) The same definition works when f is a three-variable function and C a 3-dimensional curve.

In the two-variable case, $\int_C f(x, y)\, ds$ gives the area of the "fence" extending from C in the xy-plane to the surface which is the graph of f in $\mathbb{R}^3$: Indeed, if ds is a little piece of arclength, then $f(x, y)\, ds$ represents the area of a little piece of this fence, and adding all these little areas together gives the total area of the fence. In the three-variable caes, $\int_C f(x, y, z)\, ds$ would have a similar interpretation in terms of a 4-dimensional "area", which we can't picture. Better interpretations here come from thinking of $f(x, y, z)$ as some type of density function.

Example 1. We compute the line integral of $f(x, y) = 8x$ along the piece C of the parabola $y = x^2$ between $x = 1$ and $x = 2$. Parametric equations for this curve are given by:

$$\mathbf{x}(t) = (t, t^2),\ 1 \leq t \leq 2.$$

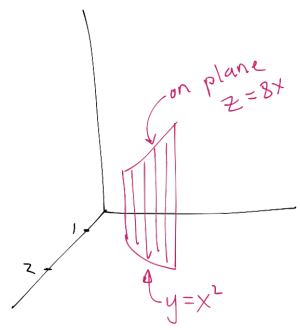
Indeed, once we have a candidate for the x parametric equation the y equation is determined by the fact that $(x(t), y(t))$ should lie on the curve $y = x^2$; instead we could have used something like $x = t^3$, in which case $y = x^2 = t^6$ and t runs between 1 and $\sqrt[3]{2}$ in order to have x run between 1 and 2. This is a general strategy for finding parametric equations for a curve with given Cartesian equation: determine a possible equation for one variable and use the Cartesian equation of the curve to determine the parametric equation for the other variable.

We have:

$$\begin{aligned}
\int_C 8x\, ds &= \int_1^2 f(\mathbf{x}(t)) \left\|\mathbf{x}'(t)\right\|\, dt \\
&= \int_1^2 f(t, t^2) \left\|(1, 2t)\right\|\, dt \\
&= \int_1^2 8t\sqrt{1 + 4t^2}\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}(1+4t^2)^{3/2}\Big|_1^2 \\
&= \frac{2}{3}(17^{3/2} - 5^{3/2}).
\end{aligned}
$$

Geometrically, this gives the area of the fence stretching from C in the xy-plane up to the graph of $f(x, y) = 8x$:



Example 2. We compute the line integral of $f(x, y, z) = xy$ over the helix C with parametric equations

$$\mathbf{x}(t) = (\cos t, \sin t, t),\ \ 0 \le t \le 4\pi.$$

Since

$$\mathbf{x}'(t) = (-\sin t, \cos t, 1) \text{ and } \left\|\mathbf{x}'(t)\right\| = \sqrt{2},$$

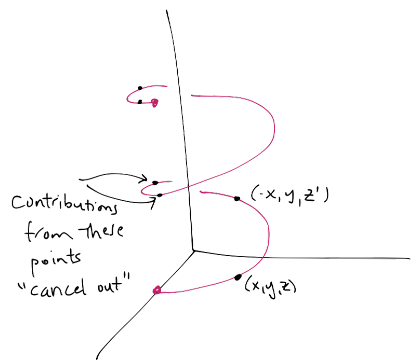
we have

$$
\begin{aligned}
\int_C xy\, ds &= \int_0^{4\pi} f(\cos t, \sin t, t)\left\|\mathbf{x}'(t)\right\|\, dt \\
&= \int_0^{4\pi} \sqrt{2}\cos t \sin t\, dt \\
&= \frac{\sqrt{2}}{2}\sin^2 t\Big|_0^{4\pi} \\
&= 0.
\end{aligned}
$$

Now, we could have determined this without doing any computation using symmetry! Indeed, given any point (x, y, z) on our helix, we can always find another point $(-x, y, z')$ with the same y-coordinate but opposite x-coordinate—the z-coordinates are different but that doesn't matter since our function $f(x, y, z) = xy$ does not depend on z! Evaluating the function at these two points gives exactly the opposite values:

$$f(-x, y, z') = -xy = -f(x, y, z),$$

so the contribution to the integral from one of these points cancels with the contribution from the other. Since we can always pair-off points like this along our curve, the entire line integral should indeed be zero.



Note that this would not work if the function were $f(x, y, z) = xyz$ instead since now the z-coordinate of a point *does* matter: evaluating at (x, y, z) versus $(-x, y, z')$ does NOT give opposite function values since z' is different than z. Also, the above reasoning would not work if our bounds were $0 \leq t \leq 4\pi + \frac{\pi}{2}$ since in this case we get a "quarter more" of the helix on top, giving points along this quarter for which there is no "opposite" point behind the yz-plane with which to "cancel out" its contribution to the integral. So, in this example, both the form of the function $f(x, y, z) = xy$ and the shape of the curve as a helix with precisely two revolutions were important in the use of symmetry.

Here is one possible interpretation of this computation. Suppose that the helix represented a piece of wire and that $f(x, y, z) = xy$ described the "electric charge density" on this wire, so $f(x, y, z)$ essentially gives the electric charge on the wire at the point (x, y, z). Then the line integral is adding together all these individual charges, giving the *total* electric charge of the wire, which is zero in this case. If instead we had a function describing mass density, then the line integral would give the total mass of the wire, and so on for other density functions.

Important. Given a two- or three-variable function f and respectively a curve C in the xy-plane or in $\mathbb{R}^3$ with parametric equations $\mathbf{x}(t), a \leq t \leq b$, the line integral of f over C is

$$\int_C f\, ds = \int_a^b f(\mathbf{x}(t)) \left\|\mathbf{x}'(t)\right\| dt.$$

(Think of ds as $ds = \|\mathbf{x}'(t)\|\, dt$.) In the two-variable case, the line integral gives the area of a "fence", and other interpretations (in either the two- or three-variable case) come from density functions.

Example 3. For any curve C, the line integral of the constant function 1 gives the arclength of the curve:

$$\int_C ds = \text{ length of } C.$$

This is similar to how for double integrals, integrating 1 gives area, and for triple integrals, integrating 1 gives volume.

Example 4. For a function $f(x, y) = f(x)$ which doesn't depend on y and a curve C which is just a line segment $[a, b]$ on the x-axis, the line integral of f over C becomes simply:

$$\int_C f\, ds = \int_a^b f(x)\, dx,$$

showing that good ole' fashioned single-variable integrals can be viewed as particular types of scalar line integrals, namely those where the curve we are integrating over lies completely on the x-axis.

Lecture 15: Vector Line Integrals

Today we spoke about the second type of line integral we care about, which arises when integrating a *vector field* along a curve. Such line integrals have a nice geometric interpretation, which hints at their possible applications.

Warm-Up 1. We compute the scalar line integral

$$\int_C (z^2 - 12y + z)\, ds$$

where C is the curve with parametric equations $\mathbf{x}(t) = (t^3, 3t^2, 6t), -1 \le t \le 2$. We have

$$\mathbf{x}'(t) = (3t^2, 6t, 6), \text{ so } \left\|\mathbf{x}'(t)\right\| = \sqrt{9t^4 + 36t^2 + 36} = \sqrt{(3t^2+6)^2}.$$

Thus with $f(x, y, z) = z^2 - 12y + z$, we get

$$\begin{aligned}\int_C f(x,y,z)\, ds &= \int_{-1}^{2} f(\mathbf{x}(t)) \left\|\mathbf{x}'(t)\right\|\, dt \\ &= \int_{-1}^{2} (36t^2 - 36t^2 + 6t)\sqrt{(3t^2+6)^2}\, dt \\ &= \int_{-1}^{2} 6t(3t^2+6)\, dt \\ &= \frac{1}{2}(3t^2+6)^2\Big|_{-1}^{2} \\ &= \frac{1}{2}(18^2 - 9^2) \\ &= \frac{243}{2}.\end{aligned}$$

Warm-Up 2. We determine the value of

$$\int_C (x^2y^3 \sin z + 1)\, ds$$

where C consists of the unit circle in the $y = -1$ plane together with the line segment from $(0, -1, 1)$ to $(0, 1, 1)$. The circle is symmetric across the xy-plane, meaning that if a point $(x, -1, z)$ is on the circle so is the point $(x, -1, -z)$ with opposite z coordinate. Since the function $f(x, y, z) = x^2y^3 \sin z$ is odd with respect to z:

$$f(x, y, -z) = x^2y^3 \sin(-z) = -x^2y^3 \sin z = -f(x, y, z),$$

the contribution to the integral of f from $(x, -1, z)$ on the circle cancels with the contribution from $(x, -1, -z)$, so the integral of f over the circle is zero:

$$\int_{\text{circle}} x^2y^3 \sin z\, ds = 0.$$

Similarly, the line segment is symmetric across the xz-plane, so if $(0, y, 1)$ is on this segment so is $(0, -y, 1)$; since $f(x, y, z) = x^2y^3 \sin z$ is odd with respect to y, the contribution from these two points also cancel out, so

$$\int_{\text{line segment}} x^2y^3 \sin z\, ds = 0.$$

Hence we get

$$\int_C (x^2y^3 sinz + 1)\, ds = \int_C x^2y^3 \sin z\, ds + \int_C ds = 0 + \int_C ds.$$

Since integrating the constant function 1 gives arclength, we have

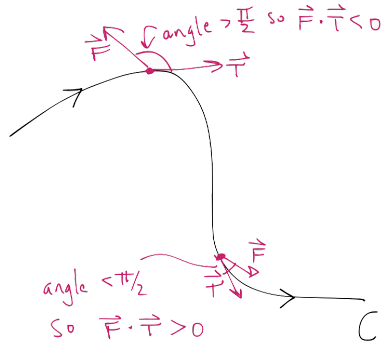
$$\int_C ds = \text{ length of circle } + \text{ length of line segment } = 2\pi + 2.$$

Thus overall we have

$$\int_C (x^2y^3 \sin z + 1)\, ds = 2\pi + 2$$

as the desired value.

Vector line integrals. Given a vector field **F** and an *oriented* curve (meaning a curve with a specific direction chosen), we define the *(vector) line integral* of **F** over C as follows. At each point along the curve, we look at the value of the vector field **F** at that point and the unit tangent vector **T** at that point:



The dot product $\mathbf{F} \cdot \mathbf{T}$ gives us a way to measure the extent to which **F** and **T** point in the same or opposite "general" direction; in particular, $\mathbf{F} \cdot \mathbf{T} > 0$ if the angle between **F** and **T** is less than $\pi/2$ and $\mathbf{F} \cdot \mathbf{T} < 0$ if the angle is greater than $\pi/2$. Adding up all these dot product quantities along the curve then gives a "net" measure of the extent to which the curve moves "with" the flow of **F** or "against" the flow of **F**, and this is how we define the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C (\mathbf{F} \cdot \mathbf{T})\, ds.$$

See Example 2 below for pictures of what these line integrals look like geometrically. The notation $\mathbf{F} \cdot d\mathbf{s}$ is meant to suggest the idea of taking "vector field dot tangent vector". Note that changing the orientation of C changes the sign of the tangent vector, so we get that

$$\int_{-C} \mathbf{F} \cdot d\, s = -\int_C \mathbf{F} \cdot d\, s$$

where $-C$ denotes C with the opposite orientation.

One final comment: why do we require a *unit* tangent vector? The point is that the line integral (as a measure how whether we're going with or against the flow) should only depend on the vector field and the direction of travel along the curve, but NOT on the speed (i.e. length of the tangent vector) at which we're moving. To get rid of this dependence then, we make all tangent vectors we use have length 1. We'll use this general formula to do some computations in the next Example, but soon we'll see a better way of computing line integrals in general.

Example 1. We compute the line integral of $\mathbf{F} = (x^2 + y)\mathbf{i} - (x+1)y\mathbf{j}$ over the curve consisting of the line segment on the x-axis from $(0,0)$ to $(2,0)$ followed by the vertical line segment from $(2,0)$ to $(2,2)$. Along the horizontal segment C_1, the unit tangent vector at any point is given by $\mathbf{T} = \mathbf{i}$, so "vector field dot unit tangent vector" along this segment is

$$\mathbf{F} \cdot \mathbf{T} = \mathbf{F} \cdot \mathbf{i} = x^2 + y = x^2$$

where in the last step we use the fact that $y = 0$ along this horizontal segment. On this horizontal segment, x moves from 0 to 2 and the infinitesimal arclength term ds is simply dx (since there is no change in y), so we get

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} (\mathbf{F} \cdot \mathbf{T})\, ds = \int_0^2 x^2\, dx = \frac{8}{3}.$$

Note that the sign of this value makes sense: along the x-axis we have $\mathbf{F}(x,0) = x^2\mathbf{i}$, meaning that the vector field always points to the right and hence always in the same direction as the curve C_1, so the line integral should be positive.

Along the vertical segment C_2, the unit tangent vector is $\mathbf{T} = \mathbf{j}$ so

$$\mathbf{F} \cdot \mathbf{T} = \mathbf{F} \cdot \mathbf{j} = -(x+1)y = -3y$$

since $x = 2$ all along C_2. The arclength term ds is dy and y moves from 0 to 2, so

$$\int_{C_2} \mathbf{F} \cdot\, ds = \int_{C_2} (\mathbf{F} \cdot \mathbf{T})\, ds = \int_0^2 -3y\, dy = -6.$$

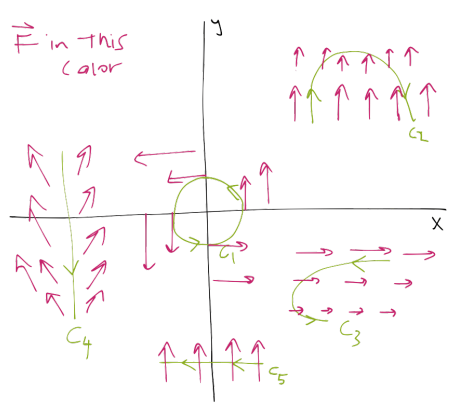
Along C_2 we have $\mathbf{F}(2,y) = (4+y)\mathbf{i} - 3y\mathbf{j}$, so since y is positive along C_2 the vector field always points downward and thus against the direction of C_2. Thus it makes sense that this line integral is negative.

Overall we thus get (where $C_1 + C_2$ denotes the curve consisting of the horizontal piece C_1 together with the vertical piece C_2):

$$\int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} = \frac{8}{3} - 6 = -\frac{10}{3}.$$

Thus, overall we tend to move more against the flow of the field $\mathbf{F}$ than with the flow.

Example 2. Consider the vector field $\mathbf{F}$ and curves drawn below:



The curve C_1 is always moving in the same direction as the field, so $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} > 0$; to be precise, at any point the angle between the vector field and tangent vector is less than $\pi/2$, so the dot product of $\mathbf{F}$ and $\mathbf{T}$ at any point is positive and integrating all positive values gives something positive.

Over the left half C_2 we go with the flow of $\mathbf{F}$ but over the right half we go against the flow. Thus the line integral over the left half is positive and the integral over the right half is negative, and these two quantities exactly cancel each other out since the vectors along the left are the same length as those along the right. Hence $\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = 0$.

Over the top half of C_2 we have a negative line integral and over the bottom a positive line integral. The negative part outweighs the positive part since the vectors along the top are longer than those along the bottom, so $\int_{C_3} \mathbf{F} \cdot d\mathbf{s} < 0$.

Along C_4 we are always going against the field, so $\int_{C_4} \mathbf{F} \cdot d\mathbf{s} < 0$. Finally, along C_5 we go neither against nor with the field at any point, so $\int_{C_5} \mathbf{F} \cdot d\mathbf{s} = 0$. Indeed, this this case $\mathbf{F}$ and $\mathbf{T}$ are always perpendicular, so their dot product is always zero, and integrating 0 gives 0.

Example 3. We determine the value of the line integral of $\mathbf{F} = -\frac{y \sin x}{x^2}\mathbf{i} + \frac{\cos x}{2x}\mathbf{j}$ over the piece C of the parabola $y = x^2$ from $(\pi/2, \pi^2/4)$ to $(5\pi/4, 25\pi^2/16)$. The problem is that in this case unit tangent vectors along the curve are not as easy to find as in Example 1. To find these we must use parametric equations for the curve, say for instance

$$\mathbf{x}(t) = (t, t^2), \ \frac{\pi}{2} \leq t \leq \frac{5\pi}{4}.$$

Then $\mathbf{x}'(t)$ gives the tangent vector at a point, so $\mathbf{T} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$ gives the unit tangent vector. But notice the amazing thing which happens when we plug this into our line integral definition (recalling that $ds = \|\mathbf{x}'(t)\| \ ds$):

$$\int_C (\mathbf{F} \cdot \mathbf{T}) \, ds = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} \, \|\mathbf{x}'(t)\| \, dt = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt,$$

so we end up with an integral involving simply $\mathbf{x}'(t)$ instead of the unit tangent vector! The point is that when using this formula we never have to worry about whether or not our tangent vectors have length 1 since it doesn't matter, and this gives the most general way possible of computing line integrals. (To be clear, the notation $\mathbf{F}(\mathbf{x}(t))$ means what you get when you plug in the parametric equations into the vector field.)

In our case we have $\mathbf{x}'(t) = (1, 2t)$, so

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_{\pi/2}^{5\pi/4} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt \\
&= \int_{\pi/2}^{5\pi/4} \left(-\frac{t^2 \sin t}{t^2}, \frac{\cos t}{2t} \right) \cdot (1, 2t) \, dt \\
&= \int_{\pi/2}^{5\pi/4} (-\sin t + \cos t) \, dt \\
&= (\cos t + \sin t) \Big|_{\pi/2}^{5\pi/4} \\
&= -\sqrt{2} - 1.
\end{aligned}$$

Thus overall when moving along C we work against the field more than we do with it. Traversing the parabola in the opposite direction would give:

$$\int_{-C} \mathbf{F} \cdot d\mathbf{s} = -\int_C \mathbf{F} \cdot d\mathbf{s} = \sqrt{2} + 1.$$

Important. For a vector field $\mathbf{F}$ and a curve C with parametric equations $\mathbf{x}(t), a \leq t \leq b$, the line integral of $\mathbf{F}$ over C is given by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

This is the best way to compute vector line integrals; the method used in Example 1 where we take $\mathbf{F} \cdot \mathbf{T}$ should **only** be used for curves which are just horizontal or vertical line segments, in which case unit tangent vectors are simple to describe.

The sign of $\int_C \mathbf{F} \cdot d\mathbf{s}$ tells us the extent to which C moves with or against the flow of $\mathbf{F}$: positive means more with the flow than against, and negative means more against the flow than with.

Example 4. Finally, we compute the line integral

$$\int_C yz\, dx + xz\, dy + xy\, dz$$

where C is the curve with parametric equations $\mathbf{x}(t) = (t, t^2, t^3),\ 0 \leq t \leq 2$. What on earth does this notation mean? It is just an alternate expression for the line integral of the vector field

$$\mathbf{F} = (yz, xz, xy)$$

over C. Indeed, imagine (dx, dy, dz) as being notation for the tangent vector, indicating that the tangent vector is obtained by taking the derivative dx of the x equation, the derivative dy of the y equation, and the derivative dz of the z equation; then

$$\mathbf{F} \cdot (\text{tangent vector}) = (yz, xz, xy) \cdot (dx, dy, dz) = yz\, dx + xz\, dy + xy\, dz,$$

which is what we're integrating.

In our case, with $x = t, y = t^2, z = t^3$, we have

$$dx = dt,\ dy = 2t\, dt,\ dz = 6t^2\, dt.$$

Thus

$$\begin{aligned} \int_C yz\, dx + xz\, dy + xy\, dz &= \int_0^2 [t^2 t^3(1) + tt^3(2t) + tt^2(3t^2)]\, dt \\ &= \int_0^2 6t^5\, dt \\ &= t^6 \Big|_0^2 \\ &= 64. \end{aligned}$$

Note how nice this notation for line integrals is: the $yz\, dx$ term tells us that we should integrate the product of the y equation, the z equation, and the derivative of the dx equation, which is what gives the $t^2 t^3(1) = t^5$ term in the integral above, and similarly the $xz\, dy$ term gives $tt^3(2t) = 2t^5$ and the $xy\, dz$ term gives $tt^2(3t^2) = 3t^5$.

Lecture 16: Green's Theorem

Today we spoke about Green's Theorem, which gives a seemingly surprising connection between line integrals and double integrals. (We'll see however that it's not actually so surprising and is in fact quite intuitive.) Practically, this gives us a way to compute line integrals which would be otherwise difficult to compute directly.

Warm-Up 1. We determine the value of

$$\int_C z\,dx + (1-y)\,dy + (z-x)\,dz$$

where C is intersection of the cylinder $x^2+z^2=1$ with the plane $y+z=1$ with counterclockwise orientation as viewed from the negative y-axis. Recall that this notation denotes nothing but the line integral of the vector field $\mathbf{F} = z\mathbf{i} + (1-y)\mathbf{j} + (z-x)\mathbf{k}$ over this curve.

First we need parametric equations for C. The equations for x, y, and z should satisfy the equations of the cylinder and plane simultaneously. Based on the cylinder, a good choice for x and z might be

$$x = \cos t \text{ and } z = \sin t \text{ for } 0 \leq t \leq 2\pi.$$

Then the equation of the plane demands that

$$y = 1 - z = 1 - \sin t,$$

so we get

$$\mathbf{x}(t) = (\cos t, 1-\sin t, \sin t),\ 0 \leq t \leq 2\pi$$

as parametric equations for C. We should make sure that these parametric equations actually give the orientation we want: the x and z equations describe a counterclockwise circle in the xz-plane, which is indeed what we get when collapsing our curve onto the xz-plane since when viewing our curve from the negative y-axis the positive x-axis would be on our right, so counterclockwise from this vantage point means counterclockwise in the xz-plane. (If our parametric equations gave the wrong orientation, all that would happen is our final value would differ by a sign.)

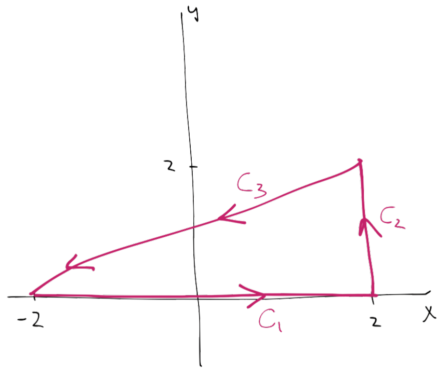
With these parametric equations we get

$$dx = -\sin t\,dt,\ dy = -\cos t\,dt,\ dz = \cos t\,dt,$$

so

$$\begin{aligned}
\int_C z\,dx + (1-y)\,dy + (z-x)\,dz &= \int_0^{2\pi} [\sin t(-\sin t) + \sin t(-\cos t) + (\sin t - \cos t)\cos t]\,dt \\
&= \int_0^{2\pi} -(\sin^2 t + \cos^2 t)\,dt \\
&= -\int_0^{2\pi} dt \\
&= -2\pi.
\end{aligned}$$

Warm-Up 2. We compute the line integral of $\mathbf{F} = (2x^2 - 3y^2)\,dx + (2x + 3y^2)\,dy$ over the curve C consisting of the following line segments:



Over C_1 we have $\mathbf{x}(t) = (t, 0),\ -2 \le t \le 2$ so

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{-2}^{2} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt = \int_{-2}^{2} (2t^2, 2t) \cdot (1, 0)\, dt = \int_{-2}^{2} 2t^2\, dt = \frac{32}{3}.$$

Over C_2 we have $\mathbf{x}(t) = (2, t),\ 0 \le t \le 2$ so

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = \int_{0}^{2} (8 - 3t^2, 4 + 3t^2) \cdot (0, 1)\, dt = \int_{0}^{2} (4 + 3t^2)\, dt = 16.$$

Note that instead of using parametric equations here, we can also use the fact that the unit tangent vector along C_1 is $\mathbf{T} = \mathbf{i}$ so $\mathbf{F} \cdot \mathbf{T} = 2x^2 - 3y^2 = 2x^2$ along C_1 and the unit tangent vector along C_2 is $\mathbf{T} = \mathbf{j}$ so $\mathbf{F} \cdot \mathbf{T} = 2x + 3y^2 = 4 + 3y^2$ along C_2; the the line integral over C_1 is simply the integral of $2x^2$ with respect to $ds = dx$ and the line integral over C_2 is the integral of $4 + 3y^2$ with respect to $ds = dy$. Again, ONLY use this approach if you fully understand it—using parametric equations is the safer bet.

Finally we need parametric equations for C_3. In general, parametric equations for the line segment from a point P to a point Q are given by

$$tQ + (1 - t)P \text{ for } 0 \le t \le 1.$$

Indeed, when $t = 0$ we get $0Q + (1-0)P = P$ so we start at P and when $t = 1$ we get $1Q + (1-1)P = Q$ so we end at Q. In our case, we start at $(2, 2)$ and end at $(-2, 0)$ so parametric equations for this line segment are:

$$\mathbf{x}(t) = t(-2, 0) + (1 - t)(2, 2) = (-2t, 0) + (2 - 2t, 2 - 2t) = (2 - 4t, 2 - 2t) \text{ for } 0 \le t \le 1.$$

Thus

$$\begin{aligned}
\int_{C_3} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt \\
&= \int_0^1 (2[2 - 4t]^2 - 3[2 - 2t]^2, 2[2 - 4t] + 3[2 - 2t]^2) \cdot (-4, -2)\, dt \\
&= \int_0^1 (32t^2 - 32t + 8 - 12t^2 + 24t - 12, 4 - 8t + 12t^2 - 24t + 12) \cdot (-4, -2)\, dt \\
&= \int_0^1 (-104t^2 + 96t - 16)\, dt
\end{aligned}$$

$$= -\frac{104}{3} + 32.$$

Hence all together we get

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \int_{C_3} \mathbf{F} \cdot d\mathbf{s} = \frac{32}{3} + 16 - \frac{104}{3} + 32 = 24.$$

Green's Theorem. The computation in Warm-Up 2 was quite tedious, but note what happens when we compute the *double integral* of the function $2+6y$ over the region D enclosed by the curve in this Warm-Up: the line segment from $(2,2)$ to $(-2,0)$ has equation $2y = x+2$, so integrating with respect to $dx\,dy$ gives:

$$\iint_D (2+6y)\,dA = \int_0^2 \int_{2y-2}^2 (2+6y)\,dx\,dy \int_0^2 (-12y^2 - 4y + 8)\,dy = -32 - 8 + 16 = 24.$$

We get the same value as in the above line integral computation! Crazy as it may seem, this is not a coincidence, but is in fact the statement of Green's Theorem:

> For a 2-dimensional vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ where P and Q have continuous partial derivatives throughout a region D in the xy-plane whose boundary ∂D consists of simple, closed curves, we have
>
> $$\int_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$
>
> when ∂D is oriented so that moving along it has the region D on the left-hand side.

A "simple, closed curve" is a curve which starts and ends at the same point and does not intersect itself. In the example of Warm-Up 2, $P = 2x^2 - 3y^2$ and $Q = 2x + 3y^2$ so

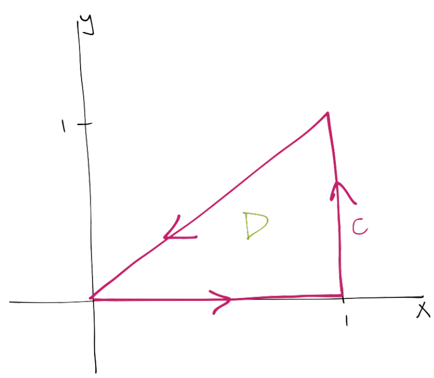
$$Q_x - P_y = 2 - (-6y) = 2 + 6y,$$

and Green's Theorem says that the line integral of $F = P\mathbf{i} + Q\mathbf{j}$ over ∂D is the same as the double integral of $2+6y$ over D, as we saw by computing both the line integral and double integral directly. The point is that Green's Theorem gives a way to compute line integrals by converting them into certain double integrals. I claim that Green's Theorem is actually (somewhat) clear if you have the right intuition in mind; we'll come back to this.

Example 1. We compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{F} = y(1 - e^{x^2})\mathbf{i} + x\mathbf{j}$ and C is the curve:



The field $\mathbf{F}$ is continuous everywhere, $P = y(1-e^{x^2})$ and $Q = x$ have continuous partials everywhere, and the curve C (which is simple and closed) is oriented so that the region D it encloses is on its the left. Thus Green's Theorem applies to give:

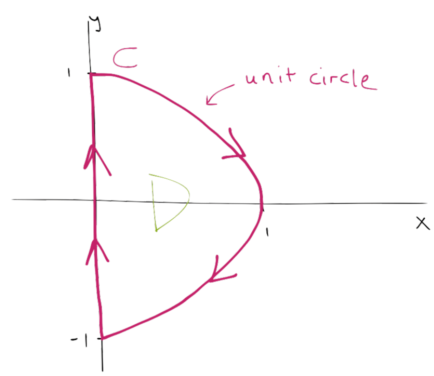
$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \iint_D (Q_x - P_y)\, dA \\
&= \iint_D [1 - (1 - e^{x^2})]\, dA \\
&= \int_0^1 \int_0^x e^{x^2}\, dy\, dx \\
&= \int_0^1 xe^{x^2}\, dA \\
&= \frac{1}{2}(e-1).
\end{aligned}$$

Note how much quicker this is as opposed to computing the line integral directly, in which case we'd have to integrate over the three segments making up $C = \partial D$ separately.

Example 2. We compute

$$\int_C -y\, dx + x\, dy$$

where C is the curve:



Computing this directly would not take too long, even though it requires integrating over the two pieces making up C separately. Green's Theorem just gives a way to do it in one shot. Note that in this case C has the wrong orientation for Green's Theorem: moving along the curve will have the region D it encloses on your *right*. But, all we have to do then is change the sign of the double integral in Green's Theorem. We get (with $P = -y$ and $Q = x$):

$$\begin{aligned}
\int_C -y\, dx + x\, dy &= -\iint_D (Q_x - P_y)\, dA \\
&= -\iint_D 2\, dA \\
&= -2(\text{area of } D) \\
&= -\pi.
\end{aligned}$$

Example 3. Finally, we consider

$$\int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$$

where C is the unit circle oriented counterclockwise. With D denoting the unit disk, C has the right orientation to be able to apply Green's Theorem. For the vector field in question,

$$P = \frac{-y}{x^2+y^2} \text{ and } Q = \frac{x}{x^2+y^2}.$$

Using the quotient rule, we get

$$Q_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \text{ and } P_y = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

so

$$Q_x - P_y = 0.$$

Thus Green's Theorem would seem to suggest that the line integral in question is equal to

$$\iint_D (Q_x - P_y)\,dA = \iint_D 0\,dA = 0.$$

However (!!!), this is nonsense: the vector field in question visually gives a counterclockwise flow around the origin, so the curve C moves in the same direction as this flow and hence the line integral we want should in fact be positive! In fact, using the parametric equations

$$\mathbf{x}(t) = (\cos t, \sin t),\ 0 \le t \le 2\pi$$

for C, we get

$$\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} -\frac{-\sin^2 t(-\sin t) + \cos t(\cos t)}{\sin^2 t + \cos^2 t}\,dt = \int_0^{2\pi} dt = 2\pi,$$

which is positive as expected.

Why is the actual value of this line integral 2π when Green's Theorem seems to say that it should be zero? The problem is that Green's Theorem is NOT applicable in this case: the field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is not continuous everywhere over the region enclosed by C since the denominators of P and Q are not defined at the origin. Thus, the line integral does not necessarily have to equal the double integral of $Q_x - P_y$ as given in Green's Theorem. Note that if instead C was a circle which did not enclose the origin, Green's Theorem would in fact be applicable and the line integral of $\mathbf{F}$ over such a circle would indeed be 0.

Important. When attempting to apply Green's Theorem, make sure that it is actually applicable. In particular, the components of your vector field should be continuous everywhere on the region enclosed by your curve, and the curve should have the orientation which would have the region enclosed by it on the left side; having the wrong orientation can be fixed by changing the sign of the double integral in Green's Theorem.

Intuition. Rather than go into full detail here as to why Green's Theorem makes sense, let me instead direct you to some notes I wrote up last year: `http://math.northwestern.edu/~scanez/courses/math290/spring13/handouts/greens-thrm.pdf`. The section titled "Intuition Behind

Green's Theorem" is the relevant portion. In particular, note that the thing we integrate in the double integral part of Green's Theorem

$$Q_x - P_y$$

is precisely the type of thing which shows up when computing the *curl* of $\mathbf{F}$! Indeed, for a 2-dimensional vector field $F = P\mathbf{i} + Q\mathbf{j}$, we have

$$\operatorname{curl} \mathbf{F} = (Q_x - P_y)\mathbf{k}$$

so that Green's Theorem can be rewritten as

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\operatorname{curl} \mathbf{F} \cdot \mathbf{k})\, dA.$$

The main point is that Green's Theorem says that how a field $\mathbf{F}$ circulates along the boundary of a region D (this circulation is measured by the line integral in question) is determined by how it circulates at various points throughout that region (which is measured by the double integral of $\operatorname{curl} \mathbf{F} \cdot \mathbf{k}$). This is something which we experience all the time: imagine you have a tub full of water and that you stick your finger in the middle and start making some circular motions—then eventually the water along the rim of the tub will begin to move in a circular manner as well! By moving your finger around in the middle you are affecting the value of $\iint_D (Q_x - P_y)\, dA$ and Green's Theorem says that the motion of the water along the rim as measured by $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ should change accordingly. We'll see instances of similar ideas later on when we talk about Stokes' Theorem and Gauss's Theorem.

Lecture 17: More on Green's Theorem, Conservative Fields

Today we continued talking about Green's Theorem, focusing on some non-obvious uses which illustrate the wide range of scenarios to which Green's Theorem can be applied. We also starting talking about conservative vector fields, which as we saw are incredibly simple to integrate along curves.

Warm-Up. We compute

$$\int_C (x^{x^x} - y^3)\, dx + (x^3 + y^{\cos y^y})\, dy$$

where C is the unit circle oriented counterclockwise. Note that computing this directly would be crazy impossible: parametric equations for C are easy enough to come up with, for instance $\mathbf{x}(t) = (\cos t, \sin t)$, but when we plug them in we get terms like

$$(\cos t)^{(\cos t)^{\cos t}} \text{ and } (\sin t)^{(\cos \sin t)^{\sin t}},$$

so we'd have no hope of computing the resulting integral with respect to t.

The curve C is simple and closed and bounds the unit disk D; also, C is oriented so that D is on its left side. The vector field

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = (x^{x^x} - y^3)\mathbf{i} + (x^3 + y^{\cos y^y})\mathbf{j}$$

is continuous and has continuous partials throughout D, so Green's Theorem is applicable. We get

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D (Q_x - P_y)\, dA$$

$$
\begin{aligned}
&= \iint_D (3x^2 + 3y^2)\, dA \\
\text{(convert to polar)} &= \int_0^{2\pi} \int_0^1 3r^3\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{3}{4}\, d\theta \\
&= \frac{3\pi}{2}.
\end{aligned}
$$

If instead we had the unit circle oriented clockwise, the value of the line integral would be $-\frac{3\pi}{2}$.

Non-obvious uses of Green's Theorem. In class we looked at two interesting and completely non-obvious uses of Green's Theorem, using the ideas of "closing off a curve" and "replacing a closed curve". These uses are both described on the same set of notes I linked to last time at `http://math.northwestern.edu/~scanez/courses/math290/spring13/handouts/greens-thrm.pdf`. So, I won't reproduce all of that here, but instead encourage you to check out those notes.

Conservative vector fields. A vector field $\mathbf{F}$ is *conservative* if it is the gradient of some function: i.e. $\mathbf{F} = \nabla f$ for some function f. The function f is called a *potential function* for the field $\mathbf{F}$. Determining whether or not a vector field is conservative can be very useful, since as we'll see line integrals involving conservative fields are incredibly easy to compute.

Example 1. We check that the field $\mathbf{F}(x, y) = (e^y + y^2 + 1, xe^y + 2xy + \cos y)$ is conservative. This means we need to find a function $f(x, y)$ such that $\nabla f = (f_x, f_y) = \mathbf{F}$, so this function should satisfy

$$f_x = e^y + y^2 + 1 \text{ and } f_y = xe^y + 2xy + \cos y.$$

Taking antiderivatives of the first equation with respect to x gives

$$f(x, y) = xe^y + y^2x + x + V(y)$$

where $V(y)$ denotes some expression involving only y—such an expression is a "constant" with respect to x. Differentiating this "candidate" potential function with respect to y gives

$$f_y = xe^y + 2xy + V_y,$$

so in order to satisfy our second requirement that

$$f_y = xe^y + 2xy + \cos y,$$

we should have $V_y = \cos y$ so that $V(y) = \sin y$. Thus

$$f(x, y) = xe^y + 2xy + \sin y$$

is one possible function with gradient $\mathbf{F}$, or in other words one possible potential function for $\mathbf{F}$.

Fundamental Theorem of Line Integrals. Here is why we care about conservative vector fields: for a curve C and a conservative field ∇f (where the potential f is defined everywhere on a region containing C), we have

$$\int_C \nabla f \cdot d\mathbf{s} = f(\text{end point of } C) - f(\text{start point of } C).$$

So, to compute the line integral of a conservative vector field all we need to do is find a potential function, evaluate this potential function at the endpoints of the curve, and subtract those values! This is exactly analogous to the single-variable Fundamental Theorem of Calculus which says that:

$$\int_a^b f'(x)\,dx = f(b) - f(a),$$

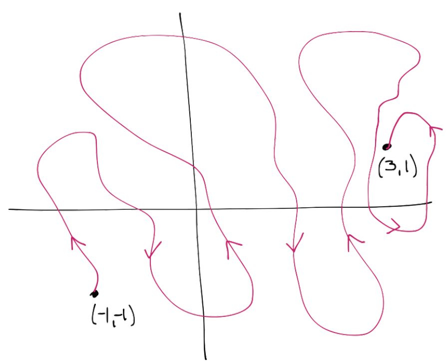
or in other words we find a "potential" f for $f'(x)$, evaluate at the endpoints of the "curve" which is the interval $[a, b]$, and subtract. (In fact, if you consider a vector field of the form $\mathbf{F} = f'(x)\mathbf{i}$ with potential $f(x)$ and a curve C which is a line segment on the x-axis, then the usual Fundamental Theorem of Calculus is indeed just a special case of this theorem.)

Note that the book doesn't give this result a name and just refers to it as Theorem 3.3 in Section 6.3. However, I think this fact is important enough to deserve its own name, and my use of "Fundamental Theorem" is meant to emphasize the relation between this and the usual Fundamental Theorem of Calculus.

Back to Example 1. Using the same vector field $\mathbf{F}$ as in Example 1, we compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

where C is the curve



Since $\mathbf{F} = \nabla f$ for $f(x, y) = xe^y + 2xy + \sin y$, the Fundamental Theorem of Line Integrals gives:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_C \nabla f \cdot d\mathbf{s} \\
&= f(\text{end point of } C) - f(\text{start point of } C) \\
&= f(3, 1) - f(-1, -1) \\
&= (3e + 6 + \sin 1) - (-e^{-1} + 2 + \sin(-1)) \\
&= 3e + \frac{1}{e} + 8 + 2\sin 1,
\end{aligned}$$

where in the last step we use $\sin(-1) = -\sin 1$.

Important. For a conservative vector field $\mathbf{F} = \nabla f$, computing line integrals amounts to evaluating the potential f at the endpoints of the curve and subtracting the resulting values. To find a potential, use the process of taking "partial anti-derivatives" as in Example 1.

Lecture 18: More on Conservative Fields

Today we spoke more about conservative vector fields, moving beyond the definition and looking at their important properties. The end result is that there are various ways of characterizing what it means for a field to be conservative.

Warm-Up 1. We compute the line integral of

$$\mathbf{F} = (e^{yz} + z)\mathbf{i} + (xze^{yz} + 2yz)\mathbf{j} + (xye^{yz} + y^2 + x)\mathbf{k}$$

over the helix $\mathbf{x}(t) = (\cos t, \sin t, t), 0 \leq t \leq 5\pi/2$. This field is conservative, and we show this by finding a potential function.

This potential $f(x, y, z)$ must satisfy

$$f_x = e^{yz} + z,\ f_y = xze^{yz} + 2yz,\ f_z = xye^{yz} + y^2 + x.$$

Anti-differentiating the first equation with respect to x shows that the candidate potential function must look like:

$$f(x, y, z) = xe^{yz} + xz + V(y, z)$$

for some function $V(y, z)$ depending only on y and z. The derivative of this candidate with respect to y is:

$$f_y = xze^{yz} + V_y,$$

which must equal $xze^{yz} + 2yz$ according to the second required equation our potential must satisfy. Thus $V_y = 2yz$ so $V(y, z) = y^2z + V(z)$ where $V(z)$ is some function depending only on z. Thus so far our candidate potential function looks like:

$$f(x, y, z) = xe^{yz} + xz + y^2z + V(z).$$

Finally, from this we get $f_z = xye^{yz} + x + y^2 + V_z$, which by the final requirement on f must equal $xye^{yz} + y^2 + x$. Thus $V_z = 0$, so we can take $V(z) = 0$. (Actually, $V(z)$ can be any constant, but we only need one potential and $V(z) = 0$ is the simplest choice.)

Hence we find that

$$f(x, y, z) = xe^{yz} + xz + y^2z$$

is a potential function for $\mathbf{F}$, meaning that $\mathbf{F} = \nabla f$, so $\mathbf{F}$ is conservative. Thus letting C denote the helix, we have:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \nabla f \cdot d\mathbf{s} &= f(\text{end point of } C) - f(\text{start point of } C)\\ &= f(0, 1, 5\pi/2) - f(1, 0, 0)\\ &= \left(0 + 0 + \frac{5\pi}{2}\right) - (1 + 0 + 0)\\ &= \frac{5\pi}{2} - 1.\end{aligned}$$

I dare you to try to compute this directly using the parametric equations alone!

Warm-Up 2. Now we take the field

$$\mathbf{G} = (e^{yz} + z)\mathbf{i} + (xze^{yz} + 2yz)\mathbf{j} + (xye^{yz} + y^2 + x + xy)\mathbf{k},$$

which is the same as the field before only with an extra xy term in the $\mathbf{k}$-component. As before, we can try to find a potential for $\mathbf{G}$ by finding a function $g(x, y, z)$ with the components of $\mathbf{G}$ as its partial derivatives. After the second step in finding a potential we're at the same place as before where we find that g must look like

$$g(x, y, z) = xe^{yz} + xz + y^2z + V(z).$$

However, now the derivative of this with respect to z, which is $g_z = xye^{yz} + x + y^2 + V_z$, would have to equal

$$xye^{yz} + y^2 + x + xy.$$

Thus we would need $V_z = xy$, but this is not possible since $V(z)$ is supposed to depend only on z! The conclusion is that there is no possible function g such that $\nabla g = \mathbf{G}$, so $\mathbf{G}$ is not conservative.

But all is not lost! The trouble is with the extra xy term we added in the $\mathbf{k}$-component, so imagine we separate this piece out and write $\mathbf{G}$ as:

$$\mathbf{G} = (e^{yz} + z, xze^{yz} + 2yz, xye^{yz} + y^2 + z) + (0, 0, xy).$$

The first part here *is* (!) conservative, in fact this first part is the field $\mathbf{F}$ from the first Warm-Up. Using the potential function we found for $\mathbf{F}$ before, this means we can express $\mathbf{G}$ as

$$\mathbf{G} = \nabla f + xy\mathbf{k},$$

or in another words as the sum of something conservative and something pretty simple. The line integral of $\mathbf{G}$ over the helix C thus splits up:

$$\int_C \mathbf{G} \cdot d\mathbf{s} = \int_C (\nabla f + xy\mathbf{k}) \cdot d\mathbf{s} = \int_C \nabla f \cdot d\mathbf{s} + \int_C xy\mathbf{k} \cdot d\mathbf{s}.$$

The first piece is what we computed in Warm-Up 1, and the second piece can be computed using the given parametric equations:

$$\int_C xy\mathbf{k} \cdot d\mathbf{s} = \int_0^{5\pi/2} (0, 0, \cos t \sin t) \cdot (-\sin t, \cos t, 1)\, dt = \int_0^{5\pi/2} \cos t \sin t\, dt = \frac{1}{2}.$$

Thus

$$\int_C \mathbf{G} \cdot d\mathbf{s} = \int_C \nabla f \cdot d\mathbf{s} + \int_C xy\mathbf{k} \cdot d\mathbf{s} = \frac{5\pi}{2} - 1 + \frac{1}{2}.$$

The moral is: when a field isn't conservative, maybe *part* of it is. Be on the lookout for this when computing an integral directly using parametric equations looks difficult and when Green's Theorem is not obviously applicable.

Properties of conservative fields. Suppose that $\mathbf{F}$ is conservative, so that $\mathbf{F} = \nabla f$ for some function f. If C is any closed curve, the end point equals the start point so

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \nabla f \cdot d\mathbf{s} = f(\text{end point}) - f(\text{start point}) = 0.$$

Thus, line integrals of conservative fields over closed curves are **always** zero.

Now suppose that C_1 and C_2 are two curves with the same end points and the same start points. Then

$$\int_{C_1} \nabla f \cdot d\mathbf{s} = f(\text{common end point}) - f(\text{common start point}) = \int_{C_2} \nabla f \cdot d\mathbf{s}.$$

Thus, line integrals of conservative fields are **path-independent**, meaning that a line integral only depends on the endpoints of a curve and not on the specific path used to connect those two endpoints.

Amazingly, both of these conditions are in fact *equivalent* to a field being conservative: if $\mathbf{F}$ is a field with the property that its line integral over *any* closed curve is zero, then $\mathbf{F}$ must be conservative; and if $\mathbf{F}$ is a field with the property that its line integrals are path-independent, then $\mathbf{F}$ must be conservative.

Important. The following conditions are equivalent ways of expressing what it means for a field $\mathbf{F}$ to be conservative:

- $\mathbf{F} = \nabla f$ for some function f,
- $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$ for any closed curve C,
- $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$ for any curves C_1 and C_2 which start at the same point and end at the same point.

Example. Consider the field $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$, with

$$P = \frac{-y}{x^2+y^2} \text{ and } Q = \frac{x}{x^2+y^2}.$$

This field is not conservative over $\mathbb{R}^2$ since as we've seen before its line integral over the unit circle is nonzero. (The unit circle is closed, so if the field was conservative this line integral would have to be zero.)

But note the following. Let f be the function $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$, defined for $x \neq 0$. Using the fact that the derivative of $\tan^{-1} u$ with respect to u is $\frac{1}{1+u^2}$, we compute:

$$f_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} \text{ and } f_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2},$$

so $f_x = P$ and $f_y = Q$, meaning that $\mathbf{F}$ is in fact the gradient of the function $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$!

What's going on here? On the one hand we just said that $\mathbf{F}$ is not conservative over $\mathbb{R}^2$ since its line integral over a closed curve is not always zero, but now we've expressed $\mathbf{F}$ as the gradient of a function, which would seem to suggest that it is conservative. The answer comes in the fact that the potential f is not defined on all of $\mathbb{R}^2$. It is definitely true that there is no function defined and continuous on all of $\mathbb{R}^2$ whose gradient is $\mathbf{F}$, so $\mathbf{F}$ for sure is not conservative on $\mathbb{R}^2$, but we could say instead that $\mathbf{F}$ *is* conservative over regions where f is actually defined, say for instance in the first quadrant, or the entire right-half plane. Saying that $\mathbf{F}$ is conservative over the first quadrant (but not all of $\mathbb{R}^2$) is consistent with the properties of conservative fields listed above: if C is a closed curve which is fully contained in the first quadrant, then in fact the line integral of $\mathbf{F}$ over C *is* always zero, a fact which we can also justify using Green's Theorem.

So, the point is that when talking about whether or not a field is conservative, we should technically mention the region it is or isn't conservative over: some vector fields which are not conservative over all of $\mathbb{R}^2$ or $\mathbb{R}^3$ could in fact be conservative over some smaller restricted regions in these spaces.

Curls. How can we tell whether or not a field will be conservative over smaller regions? For the answer we go back to the fact that line integrals of conservative fields over closed curves are always zero. Previously we saw this as a consequence of the Fundamental Theorem of Line Integrals, but we can also view this as a consequence of Green's Theorem: if C is a simple closed curve bounding a region D and $\mathbf{F} = \nabla f$ has continuous partials on all of D, then

$$\int_C \nabla f \cdot d\mathbf{s} = \iint_D (\operatorname{curl} \nabla f \cdot \mathbf{k})\, dA = \iint_D 0\, dA = 0$$

since $\operatorname{curl} \nabla f = 0$ for any function f.

So, the fact that conservative fields have zero curl is seen to be another important property of conservative vector fields. For instance, the vector field

$$\mathbf{G} = (x^x - y)\mathbf{i} + (\sin\sin y^{\sin y} - x)\mathbf{j} + (z^{z^{e^z}} + y)\mathbf{k}$$

has $\operatorname{curl} \mathbf{G} = \mathbf{i} \neq \mathbf{0}$, so it is definitely not conservative. Conversely we can ask: if a field has zero curl, must it be conservative? The answer is no: the field $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$ from the previous Example has zero curl (compute it!), and yet is not conservative over $\mathbb{R}^2$. But of course, this field is conservative over smaller regions inside $\mathbb{R}^2$, which can be viewed as one reason why it must have curl zero.

It turns out that any field with curl zero, even if not conservative over all of $\mathbb{R}^2$ or $\mathbb{R}^3$, will still be conservative at least over restricted regions. To be precise, we say that a region D in $\mathbb{R}^2$ or $\mathbb{R}^3$ is *simply-connected* if it is connected (meaning it consists of one piece), and has the property that any loop in D can be shrunk to a point. For instance, $\mathbb{R}^2$ and $\mathbb{R}^3$ themselves are simply connected, but $\mathbb{R}^2$ with the origin removed is not since a loop enclosing the origin cannot be shrunk to a point. (Visually, in 2-dimensions simply connected essentially means the region has no "holes", although this isn't true in 3-dimensions.) The fact is: if a field $\mathbf{F}$ is defined and continuous on a simply-connected region and has $\operatorname{curl} \mathbf{F} = \mathbf{0}$ throughout that region, then $\mathbf{F}$ must indeed be conservative over that region.

Important. If $\mathbf{F}$ is a field such that $\operatorname{curl} \mathbf{F} \neq 0$, then $\mathbf{F}$ is nowhere conservative. If $\mathbf{F}$ is a field with $\operatorname{curl} \mathbf{F} = \mathbf{0}$ over a simply-connected region, then $\mathbf{F}$ is conservative on that region.

Another example. The vector field

$$\mathbf{F} = x^{\cos e^x} \cos x\mathbf{i} + (\sin\sin y^{\sin y} - z)\mathbf{j} + (z^{\sin\cos z} - y)\mathbf{k}$$

has curl $\mathbf{0}$. (In particular, note that when computing curl you'll never have to take a derivative of the $\mathbf{i}$-component with respect to x, nor of the $\mathbf{j}$-component with respect to y, nor of the $\mathbf{k}$-component with respect to z, so the complicated terms in $\mathbf{F}$ above don't matter when computing its curl.) Since $\mathbf{F}$ is defined and continuous on all of $\mathbb{R}^3$, which is simply-connected, we conclude that $\mathbf{F}$ is conservative on $\mathbb{R}^3$.

So, we've shown that $\mathbf{F}$ is conservative without having to find a potential function for it. This is good, since it will not be possible to find an explicit potential for $\mathbf{F}$ due to the complicated terms which are impossible to integrate directly.

What does "conservative" mean? Finally, we get to the question: why do we call these things "conservative" vector fields? One answer comes from the fact we've seen that conservative fields are essentially (with some caveats) the same as fields with zero curl. Recalling the geometric interpretation of curl, this means that conservative fields are ones where there is no net circulation around any given point. Geometrically, at any point a conservative field causes neither a "counterclockwise" nor "clockwise" flow around that point, so any kind of flow in one direction around that point must be balanced out by an opposite flow in another direction.

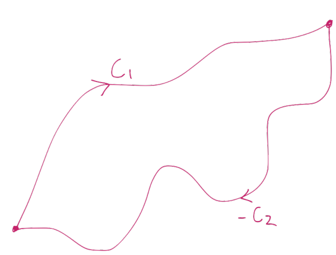
The real reason for the use of the term "conservative" comes from physics. In particular, the fact that line integrals of conservative fields over closed curves is zero suggests that something is being "conserved" as you move along that closed curve. When $\mathbf{F}$ is a conservative field describing some type of force (gravitational, electric, magnetic, and so on), the thing being "conserved" as you move along a closed curve is energy, so this fact about conservative fields becomes the so-called "Law of Conservation of Energy": in any system under the action of some force, there is no net gain nor net loss of energy. (Actually I'm simplifying this quite a bit, but it gives enough of the idea to indicate where the term "conservative field" comes from.) Incidentally, this is also where the term "potential function" comes from, where in physics we talk about the "potential energy" determined by a force.

Lecture 19: Parametric Surfaces

Today we spoke about surfaces, and in particular describing surfaces using parametric equations. This is all setup for our final topic: integration over surfaces.

Warm-Up. Suppose that $\mathbf{F}$ has the property that its integral over any closed curve is zero. We claim that $\mathbf{F}$ then has path-independent line integrals. This is in the book, but we'll reproduce the argument here. Of course, as we saw last time, both of these properties are equivalent to saying that $\mathbf{F}$ is conservative, so of course they should be equivalent to each other, but the point is to show this directly without appealing to the fact that $\mathbf{F}$ must be conservative.

Suppose that C_1 and C_2 are two curves which start at the same point and end at the same point. Then the curve $C_1 + (-C_2)$, which consists of C_1 followed by C_2 in the reserve orientation, is closed:



Thus

$$\int_{C_1+(-C_2)} \mathbf{F} \cdot d\mathbf{s} = 0.$$

But

$$\int_{C_1+(-C_2)} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{s},$$

so

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = -\int_{-C_2} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s},$$

meaning that $\mathbf{F}$ has path-independent line integrals.

The converse is also true: if $\mathbf{F}$ has path-independent line integrals, then its integral over any closed curve is zero. To see this we essentially run through the above argument in reverse: take any closed curve and break it up into two pieces C_1 and C_2, and then use path-independence and the fact that C_1 and $-C_2$ start at the same point and end at the same point to relate the line integrals over C_1 and C_2 to each other. Again, you can check the details in the book, or try to figure it out on your own!

Surfaces. Just as curves can be described using parametric equations, so too can surfaces. The main difference is that parametric equations for surfaces involve two parameters:

$$\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$$

and that the "bounds" on s and t are described using a region D in the st-plane: as (s,t) varies through all points in the region D, the given equations for $x = x(s,t), y = y(s,t)$ and $z = z(s,t)$ trace out the surface in question.

Example 1. We describe the surface given by the parametric equations:

$$\mathbf{X}(s,t) = (s, \cos t, \sin t),\ 0 \le s \le 1,\ 0 \le t \le 2\pi.$$

First, note that y and z equations (together with the bounds on t) describes a unit circle in the y and z directions, so that $s = 0$ we get a unit circle in the yz-plane. But we also get a circle at any fixed value of $x = s$, so overall we get the piece of the cylinder $y^2 + z^2 = 1$ sitting between $x = 0$ and $x = 1$. The t-parameter is describes what happens in the "circular" direction and s what happens in the x-direction.

By contrast, if instead we had the single-parameter equations

$$\mathbf{x}(t) = (t, \cos t, \sin t),$$

we would only get a curve on this cylinder; namely, a helix wrapping around the cylinder. In this case, what happens in the x-direction is not independent of what happens in the y, z-directions as it is for the parametric equations above. To get a surface, we always need two independent parameters.

Example 2. We find parametric equations for the piece of the cone $z = \sqrt{x^2 + y^2}$ below $z = 1$. Now, taking x and y to be the parameters themselves we could use

$$\mathbf{X}(x,y) = (x, y, \sqrt{x^2 + y^2}),\ (x,y) \text{ in the unit disk},$$

where the equation for z comes from the equation of the cone and the bounds on (x,y) come from looking at how x and y throughout our surface, which in this case is determined by the shadow in the xy-plane. This are perfectly good equations, but they might be hard to work with because of the square root term; for instance, taking derivatives might get messy.

So instead we find some better parametric equations. In cylindrical coordinates the cone is $z = r$, so using r and θ as parameters we get:

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, r),\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

These parametric equations are nicer than the ones before since there are no square roots terms, so computing derivatives of these will lead to simpler expressions.

Example 3. Finally, we find parametric equations for a sphere of radius R centered at the origin. The sphere has equation $x^2 + y^2 + z^2 = R^2$, so we could again use x and y as parameters to get:

$$\mathbf{X}(x,y) = (x, y, \pm\sqrt{R^2 - x^2 - y^2}),\ (x,y) \text{ in the unit disk}.$$

But again this is not so nice, both because of the square root term and because we actually need different sets of equations for the top and bottom halves of the sphere, depending on whether we take the positive or negative square root in the z-equation. Instead we could try cylindrical coordinates and use:

$$\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, \pm\sqrt{R^2 - r^2}),\ 0 \le \theta \le 2\pi,\ 0 \le r \le R,$$

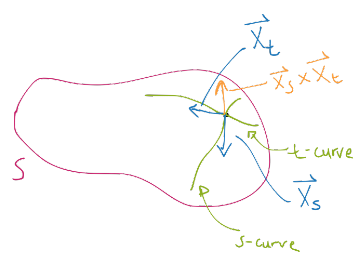
but we still have the same issues as before.

Instead we use spherical coordinates. Note that on our sphere, $\rho = R$ so we indeed only have two parameters: ϕ and θ. We get

$$\mathbf{X}(\phi, \theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi),\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$$

This set of parametric equations are much nicer, since we no longer have to worry about square roots and we've described both the top and bottom halves of the sphere all in one shot.

Normal vectors. Say we have a surface with parametric equations $\mathbf{X}(s, t)$. Holding t fixed and varying s gives a curve on our surface since now we only have one parameter s; let's call this an s-curve. Similarly, holding s fixed and varying t gives a curve we'll call a t-curve. Using what we know about parametric equations for curves, differentiating $\mathbf{X}$ with respect to s gives a vector tangent to the s-curve and differentiating with respect to t gives a vector tangent to the t-curve; we'll call these derivatives $\mathbf{X}_s$ and $\mathbf{X}_t$:



Then the cross product $\mathbf{X}_s \times \mathbf{X}_t$ is perpendicular to both tangent vectors, so this gives a vector *normal* to the surface at any point. This is one of the main uses of parametric equations: to find explicit expressions for normal vectors at any point. We say that a surface is *smooth* at a point if the normal vector $\mathbf{X}_s \times \mathbf{X}_t$ is nonzero at that point.

Back to Example 2. We use the parametric equations

$$\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, r)$$

for the cone $z = \sqrt{x^2 + y^2}$. Say we are at the point $(0, 1, 1)$ on the right edge of the cone, which occurs at $r = 1$ and $\theta = \pi/2$. The r-curve at this point has parametric equations:

$$\mathbf{x}(r) = (r\cos\pi/2, r\sin\pi/2, r) = (0, r, r),$$

which as r varies gives the straight line forming the right edge of the cone. The r-tangent vector at an arbitrary point is

$$\mathbf{X}_r = (\cos\theta, \sin\theta, 1),$$

which at $(0, 1, 1)$ is $(0, 1, 1)$. This is indeed tangent to the line formed by the r-curve at $(0, 1, 1)$.

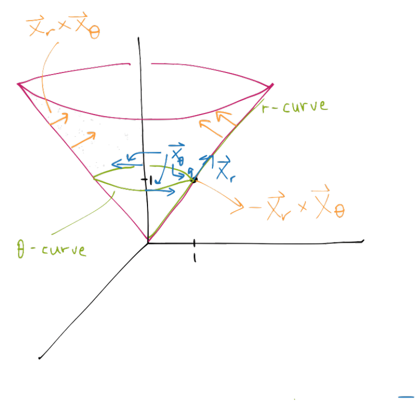
The θ-curve at this point has parametric equations

$$\mathbf{x}(\theta) = (\cos\theta, \sin\theta, 1),$$

which is a unit circle in the $z = 1$-plane. The θ-tangent vector at a point is

$$\mathbf{X}_\theta = (-r\sin\theta, r\cos\theta, 0),$$

which at $(0, 1, 1)$ is $(-1, 0, 0)$. This is indeed tangent to the unit circle at $z = 1$; in particular, notice that the z-component of $\mathbf{X}_\theta$ is zero, meaning that the θ-tangent vector is always horizontal, as the tangent vector to the horizontal circle should be.



Finally, we compute the normal vector $\mathbf{X}_r \times \mathbf{X}_\theta$:

$$\mathbf{X}_r \times \mathbf{X}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta, -r\sin\theta, r).$$

At $(0, 1, 1)$ (so with $r = 1$ and $\theta = \pi/2$), this normal vector is $(0, -1, 1)$, which gives the vector normal to the cone pointing inward. At a general point, $\mathbf{X}_r \times \mathbf{X}_\theta = (-r\cos\theta, -r\sin\theta, r)$ always gives the inward pointing normal vector. Note that at $r = 0$ this normal vector is zero, which makes sense since the cone is not smooth at the origin.

Back to Example 3. Using the parametric equations

$$\mathbf{X}(\phi, \theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$$

for the sphere of radius R centered at the origin, we compute:

$$\mathbf{X}_\phi = (R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi) \text{ and } \mathbf{X}_\theta = (-R\sin\phi\sin\theta, R\sin\phi\cos\theta, 0),$$

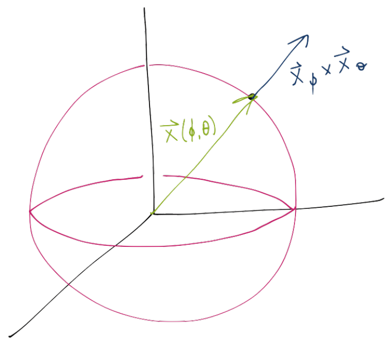
so

$$\mathbf{X}_\phi \times X_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix} = (R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\phi.)$$

Note that we can factor a $R\sin\phi$ term out of each component, so that we can write this normal vector as:

$$\mathbf{X}_\phi \times \mathbf{X}_\theta = R\sin\phi(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (R\sin\phi)\mathbf{X}(\phi, \theta),$$

so that the normal vector at a point is a multiple of the vector $\mathbf{X}(\phi, \theta)$—this makes sense geometrically! The vector $\mathbf{X}(\phi, \theta)$ at a point (x, y, z) is the vector going from the origin to that point, and the normal vector at (x, y, z) is indeed a multiple of this vector:



Given our restriction on ϕ, namely that $0 \leq \phi \leq \pi$, the term $R \sin \phi$ is always positive, so $\mathbf{X}_\phi \times X_\theta$ in this case always points in the same direction as (x, y, z), meaning away from the origin. Thus these normals vectors are the "outward" pointing ones everywhere along the sphere.

Orientation. In the above cone and sphere examples we see there are always two choices of normal vectors we can take: the "inward" pointing ones or the "outward" pointing ones. Picking the direction of the normal vectors we use is what it means to choose an *orientation* of a surface. For instance, for the cone or sphere we can talk about the *inward* orientation and the *outward* orientation.

A similar thing is true for other surfaces, only that we can't always describe the orientation using the words "inward" and "outward". For instance, for a horizontal plane we can have either normal vectors pointing up or normal vectors pointing down: in this case we can talk about the *upward* orientation of the *downward* orientation, where the terms "outward"/"inward" wouldn't make sense for a plane. When specifying an orientation we should always be sure that our description is unambiguous.

If the cross product $\mathbf{X}_s \times \mathbf{X}_t$ ends up giving us the wrong orientation, all we have to do is multiply through by -1. For instance, on the cone of Example 2, $-(\mathbf{X}_r \times X_\theta) = (r \cos \theta, r \sin \theta, r)$ would give the "outward" orientation.

Important. Surfaces are described using parametric equations of the form

$$\mathbf{X}(s, t) = (x(s, t), y(s, t), z(s, t)), \text{ for } (s, t) \text{ in a region } D \text{ of the } st\text{-plane.}$$

At the point determined by a specific (s, t), the cross product $\mathbf{X}_s \times \mathbf{X}_t$ gives a normal vector to the surface at that point, where $\mathbf{X}_s$ and $\mathbf{X}_t$ denote what you get when you differentiate the parametric equations with respect to s or t respectively. The collection of normal vectors $\mathbf{X}_s \times \mathbf{X}_t$ gives one orientation on the surface, and $-(\mathbf{X}_s \times \mathbf{X}_t) = \mathbf{X}_t \times \mathbf{X}_s$ gives the other orientation.

Do all surfaces have an orientation? We didn't touch on this in class, but it is not true that all surfaces actually have orientations, meaning that it might not always be possible to make a *consistent* choice of normal vectors throughout the entire surface. This won't be an issue for us since all surfaces we look at won't run into this problem, but in more advanced uses of surfaces this is an important consideration; the book goes into this a bit more. We might come back to this at some point just to give an example of a "non-orientable" surface, with the so-called *Möbius strip* being the most famous example.

Lecture 20: Scalar Surface Integrals

Today we started talking about *scalar surface integrals*, where we integrate a function over a general surface in $\mathbb{R}^3$. This includes double integrals as a special case, where in that case the "surface" we integrate over is just a flat surface in the xy-plane.

Warm-Up. We find an equation for the tangent plane to the surface with parametric equations

$$\mathbf{X}(s,t) = ((1+\cos t)\cos s, (1+\cos t)\sin s, \sin t),\ 0 \le s \le 2\pi,\ 0 \le t \le 2\pi$$

at the point $\left((1+\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}, (1+\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Note that this point is indeed on our surface, since it is the point we get when $s = t = \pi/4$.

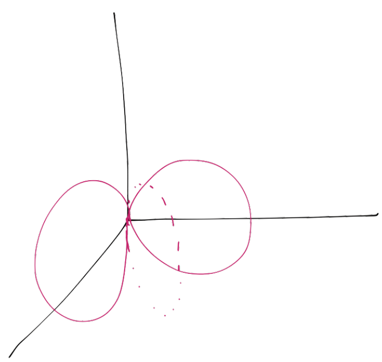
First, let's get a feel for what this surface actually looks like. At a fixed $t = k$, the s-curve is:

$$\mathbf{x}(s) = ((1+\cos k)\cos s, (1+\cos k)\sin s, \sin k).$$

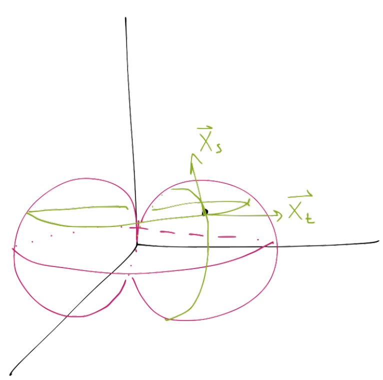
This describes a circle of radius $1+\cos k$ at a height of $z = \sin k$. Geometrically, this says that the s-curves are the horizontal circles we get when we cut through our surface with a horizontal plane at some height. Now, at $s = 0$ we get a t-curve of the form

$$(1+\cos t, 0, \sin t),$$

which describes a circle of radius 1 in the xz-plane centered at $(1,0,0)$. At other fixed s, the $\cos s$ and $\sin s$ terms in the parametric equations for the surface also give a circle, only one which has been rotated around the z-axis by an angle s. Thus the t-curves are all vertical circles, so we get a picture like:



Picturing the surface obtained by putting together all the s-curves or all the t-curves gives a torus, or in other words something which looks like the surface of a donut:



Technically, this would be called a *pinched torus* since there is no actual "donut hole" in the middle, but rather the top and bottom of the torus are being pinched in towards the origin. The point we're at lies in the first octant on this torus.

Now, we have:

$$\mathbf{X}_s = (-(1+\cos t)\sin s, (1+\cos t)\cos s, 0) \text{ and } \mathbf{X}_t = (-\sin t\cos s, -\sin t\sin s, \cos t),$$

so normal vectors to the torus are given by

$$\mathbf{X}_s \times \mathbf{X}_t = ((1+\cos t)\cos s\cos t, (1+\cos t)\sin s\cos t, (1+\cos t)\sin t).$$

At $s = t = \pi/4$ this gives:

$$\left((1+\frac{1}{\sqrt{2}})\frac{1}{2}, (1+\frac{1}{\sqrt{2}})\frac{1}{2}, (1+\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}\right),$$

which makes some sense geometrically: this normal vectors has positive $x-, y-$ and z-components, so it points away from the origin, as the normal vector at the point we're at should be based on the picture above. In general, the expression we got for $\mathbf{X}_s \times \mathbf{X}_t$ gives outward pointing normal vectors, so $-(\mathbf{X}_s \times \mathbf{X}_t)$ would give inward pointing normal vectors.

Recalling that the general equation of a plane is

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0,$$

where (a, b, c) is a normal vector and (x_0, y_0, z_0) a point on the plane, in our case we get

$$(1+\frac{1}{\sqrt{2}})\frac{1}{2}\left(x-(1+\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}\right) + (1+\frac{1}{\sqrt{2}})\frac{1}{2}\left(y-(1+\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}\right), (1+\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}\left(z-\frac{1}{\sqrt{2}}\right) = 0$$

as the equation for the tangent plane we want. (And what a messy looking beast it is, but it works!)

Surface area. Going back to the case of a general surface S with parametric equations $\mathbf{X}(s,t)$, recall that $\mathbf{X}_s$ and $\mathbf{X}_t$ are vectors tangent to S at a given point, which is why $\mathbf{X}_s \times \mathbf{X}_t$ is normal to the surface at that point. But now, recalling what the length of a cross product tells us, we have:

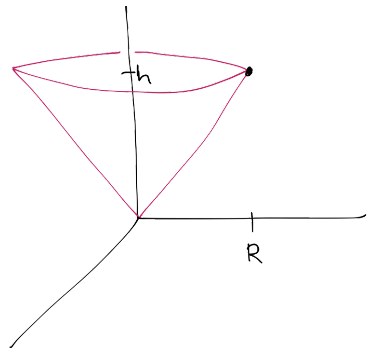
$$\|\mathbf{X}_s \times \mathbf{X}_t\| = \text{ area of the "infinitesimal" parallelogram with sides } \mathbf{X}_s \text{ and } \mathbf{X}_t.$$

Thus, $\|\mathbf{X}_s \times \mathbf{X}_t\|$ gives us the area of an "infinitesimal" piece of our surface, and so adding up all of these infinitesimal areas (using a double integral with respect to ds and dt) should give the total *surface area* of S:

$$\text{surface area of } S = \iint_D \|\mathbf{X}_s \times \mathbf{X}_t\| \, ds\, dt$$

where D is the region in the st-plane describing the bounds on s and t. (Of course, we could integrate in the order $dt\,ds$ instead if we wanted to.)

Example 1. We find the surface area of a cone of radius R and height h:



In cylindrical coordinates, the cone has equation $z = \frac{h}{R}r$ since at a radius of $r = R$ the height z should be h as in the picture above. Thus we get the parametric equations:

$$\mathbf{X}(r,\theta) = \left(r\cos\theta, r\sin\theta, \frac{h}{R}r\right),\ 0 \le r \le R,\ 0 \le \theta \le 2\pi.$$

Then

$$\mathbf{X}_r = (\cos\theta, \sin\theta, \frac{h}{R}) \text{ and } \mathbf{X}_\theta = (-r\sin\theta, r\cos\theta, 0),$$

so normal vectors are given by

$$\mathbf{X}_r \times \mathbf{X}_\theta = \left(-\frac{h}{R}r\cos\theta, -\frac{h}{R}r\sin\theta, r\right).$$

(Note that the z-component here is always positive, so this gives the inward/upward orientation of the cone. This is not important for the surface area computation, so is just an observation.)

We have

$$\|\mathbf{X}_r \times \mathbf{X}_\theta\| = \sqrt{\left(-\frac{h}{R}r\cos\theta\right)^2 + \left(-\frac{h}{R}r\sin\theta\right)^2 + r^2} = r\sqrt{\frac{h^2}{R^2} + 1}.$$

Thus the surface area of the cone is

$$\begin{aligned} \iint_D \|\mathbf{X}_r \times \mathbf{X}_\theta\|\, dr\, d\theta &= \int_0^{2\pi}\int_0^R r\sqrt{\frac{h^2}{R^2}+1}\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}R^2\sqrt{\frac{h^2}{R^2}+1}\, d\theta \\ &= \pi R^2\sqrt{\frac{h^2}{R^2}+1}. \end{aligned}$$

This can be rewritten as $\pi R\sqrt{h^2+R^2}$, which is more like the formula you would find on Wikipedia for the surface area of a cone.

Example 2. We find the surface area of a sphere of radius R. Using spherical coordinates, possible parametric equations for the sphere are:

$$\mathbf{X}(\phi,\theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi),\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$$

We computed last time that

$$\mathbf{X}_\phi \times X_\theta = (R^2 \sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\phi),$$

so

$$\|\mathbf{X}_\phi \times \mathbf{X}_\theta\| = \sqrt{R^4\sin^4\phi\cos^2\theta + R^4\sin^4\phi\sin^2\theta + R^4\sin^2\phi\cos^2\phi} = R^2\sin\phi,$$

where we use $\sin^2 + \cos^2 = 1$ multiple times. (Note the similarity between this and the Jacobian factor used when converting triple integrals into spherical coordinates—this is no accident! We'll soon see why.)

Thus the surface area of the sphere is:

$$\begin{aligned}\iint_D \|\mathbf{X}_r \times \mathbf{X}_\theta\|\, d\phi\, d\theta &= \int_0^{2\pi}\int_0^{\pi} R^2\sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} 2R^2\, d\theta \\ &= 4\pi R^2,\end{aligned}$$

which agrees with the usual formula for the surface area of a sphere of radius R.

Scalar surface integrals. So, $\|\mathbf{X}_s \times \mathbf{X}_t\|\, ds\, dt$ should give the area of an infinitesimal piece of our surface S. Using dS to denote such an infinitesimal area, we have

$$dS = \|\mathbf{X}_s \times \mathbf{X}_t\|\, ds\, dt,$$

which you should view as an analog of $ds = \|\mathbf{x}'(t)\|\, dt$ in the case of scalar line integrals. From this point of view, $\|\mathbf{X}_s \times \mathbf{X}_t\|$ is an "expansion factor" telling us how infinitesimal areas in the st-plane are translated into infinitesimal areas on S, which helps to explain why we got $\|\mathbf{X}_\phi \times \mathbf{X}_\theta\| = R^2\sin\phi$ in the case of a sphere. So, the integral giving surface area can be expressed as

$$\iint_S dS,$$

which says that we add up the infinitesimal areas dS as we move along the entire surface S.

More generally, the *scalar surface integral* of a function $f(x, y, z)$ over a surface S is denoted by

$$\iint_S f(x, y, z)\, dS,$$

which we interpret as "adding" together the values of f over all points of S. This gives surface area in the case where $f(x, y, z) = 1$, similar to how $\int_C ds$ gives arclength in the scalar line integral case or $\iint_D dA$ gives area in the double integral case. Concretely, we compute a scalar surface integral we use parametric equations $\mathbf{X}(s, t)$ for S and the formula:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{X}(s, t))\, \|\mathbf{X}_s \times \mathbf{X}_t\|\, ds\, dt.$$

In other words, we plug the parametric equations for x, y, z into our function (denoted by the $f(\mathbf{X}(s, t))$ term), substitute $dS = \|\mathbf{X}_s \times \mathbf{X}_t\|\, ds\, dt$, and integrate over the region D describing the bounds on the parameters s and t in the st-plane.

Example 3. Consider the scalar surface integral

$$\iint_S xy\,dS$$

where S is the unit disk in the xy-plane. For such a flat, horizontal surface, the area term dS is precisely what we've previously called $dA = dx\,dy$, so that this scalar surface integral is nothing but an ordinary double integral!

$$\iint_S xy\,dS = \iint_S xy\,dA.$$

The point is that a scalar surface integral is meant to integrate a function over a surface, and if our surface is just a region in the xy-plane, this is the same thing which a double integral is meant to do. A scalar surface integral takes this double integral idea and generalizes it to arbitrary surfaces which aren't contained on the xy-plane. (Note that $dS = dA$ for a surface fully contained in the xy-plane is analogous to how in line integrals $ds = dx$ for a curve which is horizontal line segment or $ds = dy$ for a curve which is a vertical line segment.)

So, after we've written our surface integral as a double integral in this special case, we could then convert to polar coordinates:

$$\iint_S xy\,dS = \iint_S xy\,dA = \int_0^{2\pi}\int_0^1 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta,$$

where the extra r factor comes from converting dA. But, let's go ahead and set this up using parametric equations as well, to see directly that we end up with the same expression. Parametric equations for S are given by:

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 0),\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

Then

$$\mathbf{X}_r = (\cos\theta, \sin\theta, 0) \text{ and } \mathbf{X}_\theta = (-r\sin\theta, r\cos\theta, 0),$$

so

$$\|\mathbf{X}_r \times \mathbf{X}_\theta\| = \|(0,0,r)\| = r.$$

Thus (with $f(x,y,z) = xy$)

$$\iint_S xy\,dS = \int_0^{2\pi}\int_0^1 f(\mathbf{X}(r,\theta))\,\|\mathbf{X}_r \times \mathbf{X}_\theta\|\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^2\cos\theta\sin\theta)r\,dr\,d\theta,$$

where now the extra r comes from $\|\mathbf{X}_r \times \mathbf{X}_\theta\|$, leading for credence to the idea that this length term should be viewed as a type of Jacobian factor. Computing this gives:

$$\int_0^{2\pi}\int_0^1 r^3\cos\theta\sin\theta\,dr\,d\theta = \frac{1}{4}\int_0^{2\pi}\cos\theta\sin\theta\,d\theta = \left.\frac{\sin^2\theta}{8}\right|_0^{2\pi} = 0.$$

This makes sense: the function xy is odd with respect to x and S is symmetric across the y-axis, so the integral of xy over S should be zero. (Or, xy is odd with respect to y and S is symmetric across the x-axis.)

Important. The scalar surface integral of a function $f(x,y,z)$ over a surface S with parametric equations

$$\mathbf{X}(s,t) \text{ for } (s,t) \text{ in } D$$

is given by

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{X}(s,t))\,\|\mathbf{X}_s \times \mathbf{X}_t\|\,ds\,dt.$$

Two special cases of note: when $f(x,y,z)=1$ the value of $\iint_S dS$ is the surface area of S, and when S is a surface contained in the xy-plane, $dS = dx\,dy$ so the scalar surface integral $\iint_S f dS$ is just the ordinary double integral $\iint_S f dA$.

Analogies between curves and surfaces. Everything we've done for surfaces is a generalization of something we did for curves. Here's a quick table to point out these analogies:

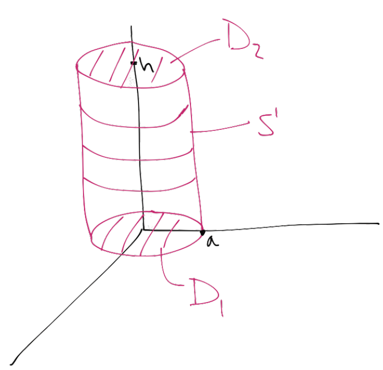
curves	surfaces
parametric equations $\mathbf{x}(t),\ a \le t \le b$	parametric equations $\mathbf{X}(s,t),\ (s,t)$ in D
tangent vector $\mathbf{x}'(t)$	normal vector $\mathbf{X}_s \times \mathbf{X}_t$
arclength $\int_a^b \|\mathbf{x}'(t)\|\,dt$	surface area $\iint_D \|\mathbf{X}_s \times \mathbf{X}_t\|\,ds\,dt$
scalar line integral $\int_C f\,ds$	scalar surface integral $\iint_S f\,dS$
vector line integral $\int_C \mathbf{F}\cdot d\mathbf{s}$	vector surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$

We'll talk about "vector surface integrals" next time.

Lecture 21: Vector Surface Integrals

Today we started talking about vector surface integrals, which arise when we integrate *vector fields* over surfaces. Just as with vector line integrals, vector surface integrals have nice geometric interpretations which hint at their possible applications.

Warm-Up 1. We compute the scalar surface integral of $f(x,y,z) = x^2+y^2+z$ over the surface S consisting of the cylinder of radius a and height h with top and bottom disks attached:



To do this we integrate f separately over the three individual surfaces making up S: the bottom disk D_1, the top disk D_2, and the cylinder S'.

Over the bottom disk, dS is the same as $dA = dx\,dy$ and z has a fixed value of 0 so

$$\iint_{D_1} (x^2+y^2+z)\,dS = \iint_{D_1} (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^a (r^2)r\,dr\,d\theta = \frac{\pi a^4}{2}.$$

Over the top disk, dS is still $dA = dx\,dy$ (since the top disk is still horizontally flat) but now $z=h$ at all points, so

$$\iint_{D_2} (x^2+y^2+h)\,dS = \iint_{D_2} (x^2+y^2+h)\,dA = \int_0^{2\pi}\int_0^a (r^2+h)r\,dr\,d\theta = \frac{\pi a^4}{2} + h\pi a^2.$$

For the cylinder S' we use the parametric equations

$$\mathbf{X}(\theta, z) = (a\cos\theta, a\sin\theta, z),\ 0 \le \theta \le 2\pi,\ 0 \le z \le h.$$

Then

$$\mathbf{X}_\theta \times \mathbf{X}_z = (-a\sin\theta, a\cos\theta, 0) \times (0,0,1) = (a\cos\theta, a\sin\theta, 0),$$

so

$$\iint_{S'} f(x,y,z)\, dS = \int_0^{2\pi}\int_0^h f(\mathbf{X}(\theta,z))\,\|\mathbf{X}_\theta \times \mathbf{X}_z\|\ dz\, d\theta = \int_0^{2\pi}\int_0^h (a^2+z)a\, dz\, d\theta = 2\pi a^3 h + \pi h^2 a.$$

Putting it all together,

$$\iint_S f\, dS = \iint_{D_1} f\, dS + \iint_{D_2} f\, dS + \iint_{S'} f\, dS = \pi a^4 + h\pi a^2 + 2\pi a^3 h + \pi h^2 a$$

is the value of the surface integral of f over S.

Warm-Up 2. Take S to be the same surface as before. Now we determine the value of

$$\iint_S (x+y)\, dS = \iint_S x\, dS + \iint_S y\, dS.$$

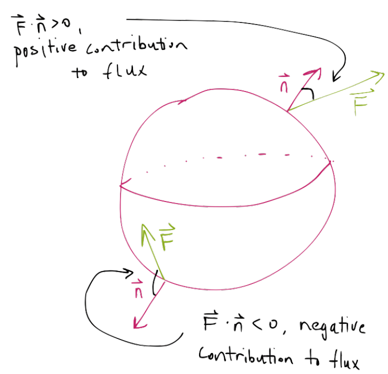
First, the integral of x over the top and bottom disks is zero since each of the are symmetric across the y-axis and the function x is odd with respect to x. Similarly, the integral of y over each of these is also zero.

The point is that the same is true even for the integrals over the cylinder. For the integral of x, for any point (x,y,z) on the cylinder the point $(-x,y,z)$ with opposite x-coordinate is also on the cylinder (the cylinder is symmetric across the yz-plane), and the contributions from these points to the integral of x over the entire cylinder cancel out. Similarly, for a point (x,y,z) on the cylinder the point $(x,-y,z)$ is also on the cylinder (using symmetry across the xz-plane), and the contributions from these points to the integral of y cancel out. Thus we get that the integral of x and y over each of the surfaces making up S are zero, so

$$\iint_S (x+y)\, dS = 0.$$

All integrals are the same. We took some time in class to point out that all integrals we've been looking at so far really involve the same concept, just with varying dimensions: the point is that all integrals boil down to the idea of adding up values of a function along some geometric object. I'll put up some additional notes about this separately.

Vector surface integrals. Suppose that $\mathbf{F}$ is a vector field and S a surface with a chosen orientation. The *vector surface integral* of $\mathbf{F}$ over S gives a way to measure the extent to which $\mathbf{F}$ flows "through" S. To be precise, at each point of S we compute the dot product of $\mathbf{F}$ with the unit normal vector $\mathbf{n}$ to S at that point: this dot product is positive when $\mathbf{F}$ flows through S in the same general direction as $\mathbf{n}$ and negative when $\mathbf{F}$ flows through S in the opposite general direction to that of $\mathbf{n}$:



Then we add up all of these dot product contributions as we move across the entire surface to give the *vector surface integral*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS.$$

In this notation, think of $d\mathbf{S}$ as representing normal vectors to S. The value of this integral is also called the *flux* of $\mathbf{F}$ across S.

We use *unit* normal vectors in this definition so that the value of the integral depends only on $\mathbf{F}$ and its direction relative to the normal vectors and not on the length of the normal vectors we choose. As we saw with vector line integrals, we'll soon see a way to compute this without having to rely on having normal vectors of length 1.

Example 1. We compute the vector surface integral

$$\iint_S (x^{y \cos xz}\mathbf{i} + xy\cos(\sin(e^{xyz}))\mathbf{j} + 3\mathbf{k})\, d\mathbf{S}$$

where S is the unit disk in the xy-plane with upward orientation. The vector field in question has complicated $\mathbf{i}$- and $\mathbf{j}$-components, but the point is that these terms don't actually matter: the disk S has unit normal vector $\mathbf{k}$, so after dotting with $\mathbf{F}$ with only get 3:

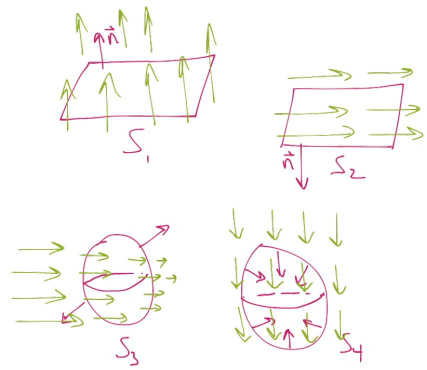
$$\iint_S (x^{y \cos xz}\mathbf{i} + xy\cos(\sin(e^{xyz}))\mathbf{j} + 3\mathbf{k}) \cdot \mathbf{k}\, dS = \iint_S 3\, dS.$$

Then, since S is a flat surface on the xy-plane, dS is $dA = dx\, dy$, so we get

$$\iint_S 3\, dS = \iint_S 3\, dA = 3(\text{area of } S) = 3\pi.$$

Thus the flux of $\mathbf{F}$ across S is 3π, and it makes sense that this value should be positive since $\mathbf{F}$ consists of vectors with positive z-component, which means that they always point in the same general direction as the orientation on S.

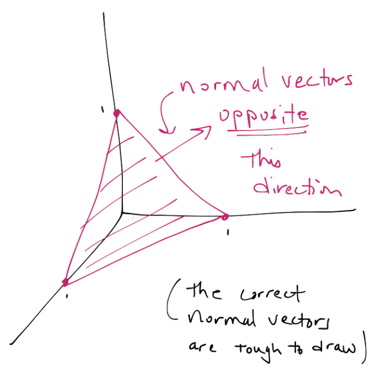
Example 2. Consider the following surfaces and vector fields (in green):



Over S_1, the vector field always flows through the surface in the same direction as the orientation, so $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} > 0$. Over S_2, the field does not flow through the surface at all, so $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0$. (More precisely, the dot product of the vector field and normal vector at any point is zero, and integrating zero along the surface gives zero.)

Now, over the left half of S_3 the field flows against the direction of the orientation whereas over the right half the field flows with the orientation; since the effect of the negative flow is stronger (due to the longer vectors on the left half), the net flux is still negative so $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} < 0$. Finally, over the top half of S_4 there is a positive flux whereas over the bottom half there is a negative flux, but these two effects exactly balance out since the field consists of vectors of the same length, so $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 0$.

Example 3. We compute the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the piece of the plane $x + y + z = 1$ in the first octant with downward orientation:



In this case, unit normal vectors to S are not as simple as $\mathbf{i}, \mathbf{j}, \mathbf{k}$ or the negatives of one of these, so we need to do something else.

Given parametric equations $\mathbf{X}(s,t)$ for S, we already know that $\mathbf{X}_s \times \mathbf{X}_t$ gives normal vectors to S. Thus

$$\frac{\mathbf{X}_s \times \mathbf{X}_t}{\|\mathbf{X}_s \times \mathbf{X}_t\|}$$

gives unit normal vectors. (Technically, we should assume that S is a "smooth" surface to guarantee that our normal vectors are never zero.) Recalling that $dS = \|\mathbf{X}_s \times \mathbf{X}_t\|\, ds\, dt$, we then have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$$

$$= \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \frac{\mathbf{X}_s \times \mathbf{X}_t}{\|\mathbf{X}_s \times \mathbf{X}_t\|} \, \|\mathbf{X}_s \times \mathbf{X}_t\| \, ds\, dt$$

$$= \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot (\mathbf{X}_s \times \mathbf{X}_t) \, ds\, dt$$

where D describes the region in the st-plane restricting the values of s and t. Note that in this formula now we no longer need a unit normal vector, but rather we just use whatever normal vector the parametric equations gives us, as long as we check for the correct orientation.

So, we need parametric equations for S. Using x and y as parameters, the equation of the plane gives $z = 1 - x - y$, so

$$\mathbf{X}(x,y) = (x, y, 1-x-y), \ (x,y) \in D,$$

where D is the shadow of S in the xy-plane, are parametric equations for S. (We use the shadow as D since this is the region describing the values of x and y we want to consider in order to give the piece of the plane we're interested in.) We have

$$\mathbf{X}_x \times \mathbf{X}_y = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1).$$

Now, this gives upward pointing normal vectors, so to get the orientation we want we simply multiply through by -1 and use

$$-\mathbf{X}_x \times \mathbf{X}_y = \mathbf{X}_y \times \mathbf{X}_x = (-1, -1, -1).$$

We thus get:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{X}(x,y)) \cdot (\mathbf{X}_y \times \mathbf{X}_x) \, dx\, dy \\
&= \iint_D (x, y, 1-x-y) \cdot (-1, -1, -1) \, dx\, dy \\
&= \iint_D (-x - y - 1 + x + y) \, dx\, dy \\
&= \int_0^1 \int_0^y -1 \, dx\, dy \\
&= -\frac{1}{2}
\end{aligned}$$

as the flux of $\mathbf{F}$ through S. Note again that it makes sense that this is negative since $\mathbf{F}$ always has positive $\mathbf{i}, \mathbf{j}, \mathbf{k}$-components, and so always point opposite the direction of the given orientation.

Important. The vector surface integral of $\mathbf{F}$ over S (also called the flux of $\mathbf{F}$ across S) is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot (\mathbf{X}_s \times \mathbf{X}_t) \, ds\, dt$$

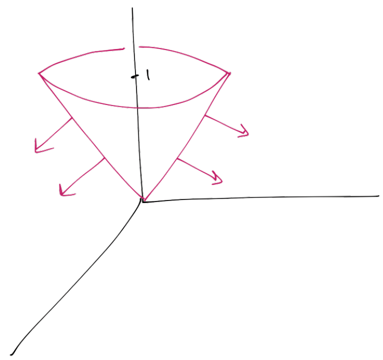
where "$\mathbf{X}(s,t)$, (s,t) in D" are parametric equations for S. The only thing to be careful of here is whether $\mathbf{X}_s \times \mathbf{X}_t$ gives the right orientation, but if not all we need to do is multiply $\mathbf{X}_s \times \mathbf{X}_t$ through by -1. The first definition in terms of unit normal vectors $\mathbf{n}$ will only be useful for flat horizontal or vertical surfaces where $\mathbf{n} = \pm\mathbf{i}, \pm\mathbf{j}$, or $\pm k$, otherwise the formula using parametric equations is the way to go.

Geometrically, a vector surface integral measures the net "flow" of $\mathbf{F}$ across S, where a positive value means that overall $\mathbf{F}$ flows through S in the same direction as the orientation more than against it, and a negative value means that $\mathbf{F}$ flows against the orientation more than with it.

Lecture 22: Stokes' Theorem

Today we started talking about Stokes' Theorem, which is a higher-dimensional analog of Green's Theorem. As with Green's Theorem, Stokes' Theorem relates two seemingly unrelated concepts—line integrals and surface integrals—and has diverse applications.

Warm-Up 1. Suppose that $\mathbf{E} = (xz, yz, y)$ is an *electric force field* and S is the piece of the cone $z = \sqrt{x^2+y^2}$ beneath $z = 1$ oriented downward, meaning away from the z-axis in this case:



We compute the *electric flux* of $\mathbf{E}$ across S, which measures the flow of electricity across S.

We use the following parametric equations for the cone:

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, r),\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1.$$

Then

$$\mathbf{X}_r \times \mathbf{X}_\theta = (\cos\theta, \sin\theta, 1) \times (-r\sin\theta, r\cos\theta, 0) = (-r\cos\theta, -r\sin\theta, r).$$

However, this gives upward pointing normal vectors due the positive z-component, so to match the orientation we want we use

$$-\mathbf{X}_r \times \mathbf{X}_\theta = \mathbf{X}_\theta \times \mathbf{X}_r = (r\cos\theta, r\sin\theta, -r).$$

The flux is thus:

$$\begin{aligned}\iint_S \mathbf{E}\cdot d\mathbf{S} &= \iint_D \mathbf{E}(\mathbf{X}(r,\theta)) \cdot (\mathbf{X}_\theta \times \mathbf{X}_r)\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (r^2\cos\theta, r^2\sin\theta, r\sin\theta)\cdot(r\cos\theta, r\sin\theta, -r)\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (r^3 - r^2\sin\theta)\,dr\,d\theta \\ &= \frac{\pi}{2}.\end{aligned}$$

Warm-Up 2. We compute

$$\iint_S (-4\mathbf{i} - x\mathbf{k})\cdot\, d\mathbf{S}$$

where S is the piece of the plane $x+z=5$ within the cylinder $x^2+y^2=9$ with upward orientation. First we use the parametric equations:

$$\mathbf{X}(x,y) = (x, y, 5-x),\ (x,y) \text{ in the disk } D \text{ of radius 3.}$$

This gives

$$\mathbf{X}_x \times \mathbf{X}_y = (1,0,-1)\times(0,1,0) = (1,0,1),$$

which gives normal vectors pointing in the correct direction. Thus:

$$\begin{aligned}\iint_S (-4\mathbf{i} - x\mathbf{k})\cdot\, d\mathbf{S} &= \iint_D (-4,0,-x)\cdot(1,0,1)\,dx\,dy \\ &= \iint_D (-4-x)\,dx\,dy \\ &= \int_0^{2\pi}\int_0^3 (-4 - r\cos\theta) r\,dr\,d\theta \\ &= -36\pi.\end{aligned}$$

Just to see another approach, let's instead use the parametric equations

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 5 - r\cos\theta),\ 0\le r\le 3,\ 0\le\theta\le 2\pi.$$

Then

$$\mathbf{X}_r \times \mathbf{X}_\theta = (\cos\theta, \sin\theta, -\cos\theta)\times(-r\sin\theta, r\cos\theta, r\sin\theta) = (r,0,r),$$

which gives the correct orientation. We get

$$\begin{aligned}\iint_S (-4\mathbf{i} - x\mathbf{k})\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^3 (-4,0,-r\cos\theta)\cdot(r,0,r)\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^3 (-4r - r^2\cos\theta)\,dr\,d\theta \\ &= -36\pi.\end{aligned}$$

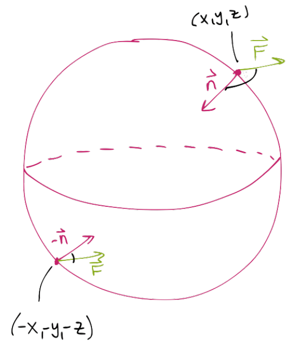
Note that in this case, the extra factor of "r" we got previously when converting polar coordinates now shows up in the normal vector, reflecting the fact that we started right off-the-bat using parametric equations based on polar coordinates.

Warm-Up 3. Finally, we compute the vector surface integral of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ over the unit sphere with inward orientation. We can try using parametric equations based on spherical coordinates, but we end up with a not-so-nice integral. Instead, we use symmetry to argue that the value of this surface integral should be zero.

Take a point (x,y,z) on the sphere, with inward pointing normal vector $\mathbf{n}$. The value of $\mathbf{F}$ at this point is (x^2,y^2,z^2). Now, at the point $(-x,-y,-z)$ on the opposite side of the sphere the vector field looks exactly the same since

$$\mathbf{F}(-x,-y,-z) = (-x)^2\mathbf{i} + (-y)^2\mathbf{j} + (-z)^2\mathbf{k} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

Moreover, the normal vector at $(-x,-y,-z)$ points in the direction exactly opposite that of $\mathbf{n}$, meaning that the normal vector at $(-x,-y,-z)$ is $-\mathbf{n}$:



Thus at the point $(-x,,-y,-z)$, the contribution to the vector surface integral is

$$\mathbf{F}\cdot(-\mathbf{n}) = -\mathbf{F}\cdot\mathbf{n},$$

so it cancels out with the contribution $\mathbf{F}\cdot\mathbf{n}$ from the point (x,y,z). Since this happens at every point along the sphere, the total flux of $\mathbf{F}$ across S is zero.

Note that we need to be looking at the entire sphere in order for this to work out, since given a point (x,y,z) on the sphere we need the point $(-x,-y,-z)$ to also be on the sphere.

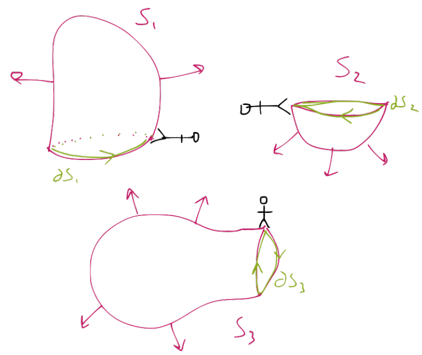
Stokes' Theorem. Suppose that $\mathbf{F}$ is a vector field and S an oriented surface, with $\mathbf{F}$ defined and having continuous partials throughout S. *Stokes' Theorem* arises when integrating $\operatorname{curl}\mathbf{F}$ over S and should be viewed as a higher-dimensional

$$\int_{\partial S}\mathbf{F}\cdot d\mathbf{s} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S},$$

where the left side is the *line integral* of $\mathbf{F}$ over the *boundary* ∂S of S and the right side is the *surface integral* of $\operatorname{curl}\mathbf{F}$ over S itself. That is, the flux of $\operatorname{curl}\mathbf{F}$ across a surface S is equal to the integral of $\mathbf{F}$ along its boundary.

Two remarks are in order. First, the boundary of a surface is the curve indicating where the surface "ends"; instead of trying to give a formal definition, it's probably easiest to just see what this looks like in some explicit examples. Second, as in Green's Theorem, the equality only holds when S have ∂S have "compatible" orientations. Given an orientation on S, the corresponding orientation on ∂S is the one with the following property: if you stand along the boundary ∂S with your head pointing in the direction of the normal vector determined by the orientation on S, the induced orientation on ∂S is the one such that walking in that direction will have S on your left side. Again, it's probably best to see what this looks like in explicit examples to get a feel for it.

Example 1. Consider the following surfaces with given orientations:



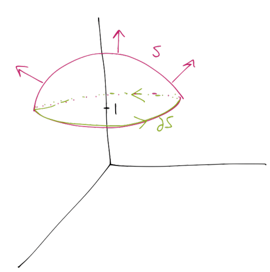
The boundary of S_1 is the bottom circle (in green) since this is where S_1 "ends". Now, standing along the boundary with your head point in the direction of the normal vector (as the stick figure shows), you would have to move in the indicated "counterclockwise" direction in order to have S_1 on your left side, so this is the orientation on ∂S_1 compatible with the given orientation on S_1.

The boundary of S_2 is the circle on top, and you would have to move in the indicated "clockwise" direction along that boundary to have S_2 on your left. Finally, the boundary of S_3 is the vertical circle on the right. Standing along this circle with your head pointed upward (in the same direction as the orientation), you'd have to move in the direction which goes towards the back first, so this is the induced orientation on the boundary. (This looks clockwise when viewing from the right side of the page.)

Example 2. Let $\mathbf{F} = -y\mathbf{i} + (x + (z-1)x^{x \sin x})\mathbf{j} + (x^2 + y^2)\mathbf{k}$. We compute the surface integral

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where S is the piece of the sphere $x^2 + y^2 + z^2 = 2$ lying above $z = 1$, with outward orientation:



Now, in order to compute this directly we would have to first of all compute $\operatorname{curl} \mathbf{F}$, but this is hard to do: when we get to the $\mathbf{k}$-component we would have to differentiate

$$x + (z-1)x^{x \sin x}$$

with respect to x, which is not easy. So, we really have no shot at computing this surface integral directly, and must appeal to Stokes' Theorem.

Stokes' Theorem says that the given surface integral is equal to

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s},$$

as long as we orient ∂S the right way. Standing on the boundary of S with our head pointed outward, we would have to walk counterclockwise (as viewed from the positive z-direction) in order to have S on our left, so this is the orientation we need in Stokes' Theorem. Parametric equations for this boundary circle are:

$$\mathbf{x}(t) = (\cos t, \sin t, 1),\ 0 \leq t \leq 2\pi,$$

so:

$$\begin{aligned}\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt \\ &= \int_0^{2\pi} (-\sin t, \cos t, 1) \cdot (-\sin t, \cos t, 0)\, dt \\ &= \int_0^{2\pi} dt \\ &= 2\pi.\end{aligned}$$

Thus the flux of $\operatorname{curl} \mathbf{F}$ across S is 2π as well. (Note that along the boundary z is always 1, so the coefficient of the complicated $x^{x \sin x}$ term is 0 in the line integral, which is what makes this line integral computable by hand.)

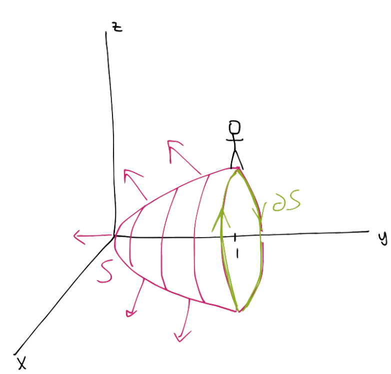
Lecture 23: More on Stokes' Theorem

Today we continued talking about Stokes' Theorem, and used it to show that surface integrals of curls have a certain "surface-independence" property, analogous to path-independence for line integrals of conservative vector fields. This is useful when integrating a not-so-simple curl over a not-so-simple surface, since we can replace that surface with a simpler one which has the same boundary.

Warm-Up 1. Let $\mathbf{F}$ be the field

$$(y-1)\sin e^{xz^y}\mathbf{i} + xyze^{xyz}\mathbf{j} + (xz+y)\mathbf{k}$$

and S the piece of the paraboloid $y = x^2 + z^2$ with $y \leq 1$, oriented with outward pointing normal vectors:



We compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Clearly, computing $\operatorname{curl} \mathbf{F}$ directly will not be easy, since at some point we'll have to differentiate $\sin e^{xz^y}$ with respect to y and z. But we don't need to compute $\operatorname{curl} \mathbf{F}$ since Stokes' Theorem says that

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

as long as the boundary ∂S of S has the appropriate orientation. In this case, ∂S is the circle of radius 1 in the plane $y = 1$ (the green circle in the picture), and the orientation corresponding to the outward one on S is the one which would move "into" the page when standing at the top point of the circle; indeed, standing at this point with our head pointing in the "outward" direction, moving into the page will have S on our left side.

We parametrize ∂S using

$$\mathbf{x}(t) = (\cos t, 1, \sin t), \ 0 \leq t \leq 2\pi.$$

As a check, at $t = 0$ we're at $(1, 1, 0)$ while at $t = \pi/2$ we're at $(0, 1, 1)$, meaning we're moving in the right direction to match the orientation we need. We have

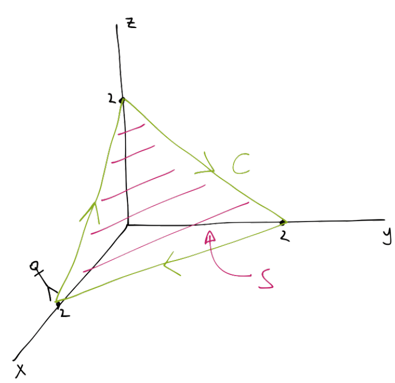
$$\begin{aligned}
\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt \\
&= \int_0^{2\pi} (0, \text{who cares}, \cos t \sin t + 1) \cdot (-\sin t, 0, \cos t)\, dt \\
&= \int_0^{2\pi} (\cos^2 t \sin t + \cos t)\, dt \\
&= 0.
\end{aligned}$$

Thus the flux of $\operatorname{curl} \mathbf{F}$ across S is zero as well.

Warm-Up 2. Let $\mathbf{F}$ be the field

$$(x \sin e^x - xz)\mathbf{i} - 2xy\mathbf{j} + (z^2 + y)\mathbf{k}$$

and C the triangle with vertices $(2, 0, 0), (0, 2, 0)$, and $(0, 0, 2)$ with orientation as drawn:



We compute $\int_C \mathbf{F} \cdot d\mathbf{s}$. To do this directly would require that we parametric the three different segments making up C separately and that we compute three different line integrals. This is a lot of work, and the $x \sin e^x$ term in the $\mathbf{i}$-component should also scare you away from attempting this.

Instead, Stokes' Theorem says that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

as long as S is a surface having C as its boundary, with an appropriate orientation. The simplest surface having C as its boundary is probably the triangular region enclosed by C, which lies on the plane

$$x + y + z = 2.$$

If we stood along C with our head pointed away from the z-axis and walked in the direction given by the orientation on C, the triangular region S would be on our *right* side, meaning that normal vectors pointing in this direction do not give the right orientation for Stokes' Theorem, so we orient S with normals pointing down towards the z-axis.

Parametric equations for S are given by

$$\mathbf{X}(x,y) = (x, y, 2 - x - y), \ (x,y) \text{ in the shadow } D \text{ of } S \text{ on the } xy\text{-plane.}$$

(This shadow is the region in terms of x and y which restricts the values of (x,y) we want to consider.) We these we get

$$\mathbf{X}_x \times \mathbf{X}_y = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1),$$

so get the correct orientation we use $\mathbf{X}_y \times \mathbf{X}_x = (-1, -1, -1)$. Since

$$\operatorname{curl} \mathbf{F} = (\mathbf{i} - x\mathbf{j} - 2y\mathbf{k}) \cdot d\mathbf{S},$$

we have:

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \operatorname{curl} \mathbf{F}(\mathbf{X}(x,y)) \cdot (\mathbf{X}_y \times \mathbf{X}_x)\, dx\, dy \\ &= \int_0^2 \int_0^{2-y} (1, -x, -2y) \cdot (-1, -1, -1)\, dx\, dy \\ &= \int_0^2 \int_0^{2-y} (-1 + x + 2y)\, dx\, dy \\ &= 2. \end{aligned}$$

Thus $\int_C \mathbf{F} \cdot d\mathbf{s} = 2$ as well.

Green's is Stokes'. Suppose that $\mathbf{F}$ is a vector field of the form $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ and S is a surface fully contained on the xy-plane with upward orientation. Then ∂S is a curve on the xy-plane with the counterclockwise orientation (as viewed from the positive z-direction) and Stokes' Theorem says that

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

But now, since S is fully contained on the xy-plane it has normal vector $\mathbf{k}$, so the surface integral becomes

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dS = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA$$

where the last equality comes from the fact that $dS = dA$ on the xy-plane. But for the type of vector field we're looking at,

$$\operatorname{curl}\mathbf{F} = (Q_x - P_y)\mathbf{k}, \text{ so } \operatorname{curl}\mathbf{F}\cdot\mathbf{k} = Q_x - P_y.$$

Thus we get

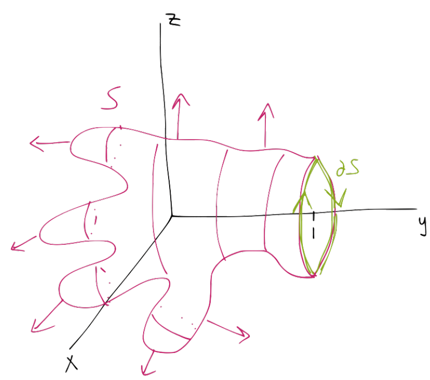
$$\int_{\partial S} \mathbf{F}\cdot d\mathbf{s} = \iint_S (Q_x - P_y)\,dA,$$

which is precisely the statement of Green's Theorem! In other words, Green's Theorem is nothing but a special case of Stokes' Theorem: namely the case of Stokes' Theorem applied to a surface fully contained on the xy-plane and a vector field with no dependence on z and no $\mathbf{k}$-component. So, in this sense, Stokes' Theorem really is a 3-dimensional generalization of Green's Theorem.

Example. Finally, consider a vector field $\mathbf{F}$ whose curl is

$$\operatorname{curl}\mathbf{F} = y^{y^y}\sin e^{z^2}\mathbf{i} + ((y-1)e^{x^{x^x}} + 2)\mathbf{j} + ze^{x^{x^x}}\mathbf{k},$$

(such a field actually exists, even though it can't be written down explicitly) and the surface S drawn below:



We compute $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$. Of course, we have no hope of doing this directly since we can't hope to find parametric equations for this random looking surface, and even if we could $\operatorname{curl}\mathbf{F}$ is complicated enough that we would not be able to compute the resulting double integral.

Instead we try using Stokes' Theorem. The boundary of S is the same circle C as in the first Warm-Up, and with the same orientation as in that Warm-Up, so the surface integral we want is equal to

$$\int_C \mathbf{F}\cdot d\mathbf{s}.$$

However, this is still not something we can actually compute since we don't even know what $\mathbf{F}$ is! So, it seems as if we're stuck.

Stokes' Theorem to the rescue once again! Given C, Stokes' Theorem says that the above line integral is equal to the flux of $\operatorname{curl}\mathbf{F}$ across *any* surface having C as its boundary, and the point is that there are many other simpler surfaces having C as their boundary apart from the original S. In particular, the unit disk S_1 in the plane $y = 1$ also has C as its boundary, and if we orient S_1 with leftward pointing normal vectors then this orientation is compatible with the one we already had on C. Thus we have:

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{\partial S = \partial S_1} \mathbf{F}\cdot d\mathbf{s} = \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S},$$

illustrating a general fact that when integrating the curl of a vector field over surfaces having the same boundary (and with compatible orientations), the values will be the same. Thus, to compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ we can "replace" S by a simpler surface having the same boundary.

So, we're left computing the flux of $\operatorname{curl} \mathbf{F}$ across S_1. Even though $\operatorname{curl} \mathbf{F}$ is pretty complicated, it turns out that this computation is very easy. The unit normal vector to S_1 is given by $\mathbf{n} = -\mathbf{j}$ since S_1 has the leftward orientation, meaning that only the $\mathbf{j}$-component of $\operatorname{curl} \mathbf{F}$ matters:

$$\operatorname{curl} \mathbf{F} \cdot (-\mathbf{j}) = -(y-1)e^{x^{x^x}} - 2.$$

Moreover, $y = 1$ on S_1 so the above expression simplifies further to -2. Thus:

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot (-\mathbf{j})\, dS = \iint_{S_1} -2\, dx\, dz = -2(\text{area of } S_1) = -2\pi.$$

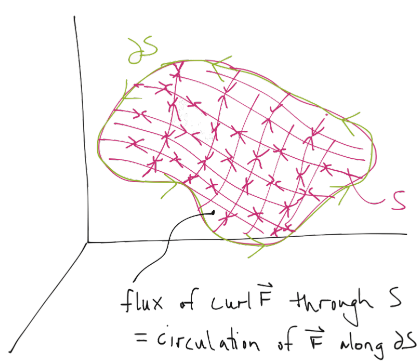
Thus $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where S is the original surface, is -2π as well.

Important. If S_1 and S_2 are two surfaces with the same boundary and with orientations which induce the same orientation on this common boundary, then for a field $\mathbf{F}$ we have:

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

since by Stokes' Theorem each of these is equal to the same line integral. View this as analogous to the fact that for conservative fields, $\int_{C_1} \nabla f \cdot d\mathbf{s} = \int_{C_2} \nabla f \cdot d\mathbf{s}$ whenever C_1 and C_2 have the same "boundary", i.e. the same initial and final points.

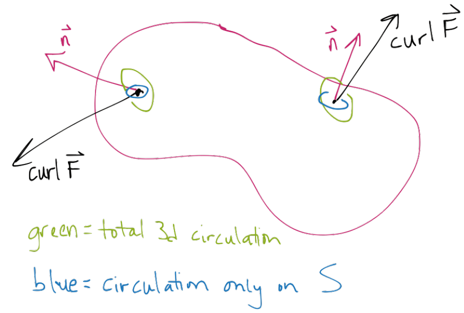
Intuition behind Stokes' theorem. Previously we looked at the intuition behind Green's Theorem, where the idea was of that taking a region in the xy-plane and filling it up with more and more, smaller and smaller curves. (Check the "Notes on Green's Theorem" for a reminder of this.) The same picture works for Stokes' Theorem, only we fill up a general *curved* surface with more and more curves:



As with Green's Theorem, the point is that adding up the line integrals of $\mathbf{F}$ around all these individual curves gives the line integral of $\mathbf{F}$ along the boundary ∂S on the one hand (the line integrals over the pieces of the curves "inside" will all cancel out), but on the other hand, as the curves get smaller and smaller we end up looking at the circulation of $\mathbf{F}$ around individual points and adding up all of these circulations over the entire surface:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S (\text{circulation of } \mathbf{F} \text{ at points})\, dS.$$

Recall that at a point $\operatorname{curl} \mathbf{F}$ points in the direction of the axis around which the 3-dimensional circulation is occurring, so $\operatorname{curl} \mathbf{F} \cdot \mathbf{n}$ at a point on S measures how much of the total circulation actually occurs on S itself:



Thus,

$$\iint_S (\text{circulation of } \mathbf{F} \text{ at points})\, dS = \iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{n})\, dS = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

and we have Stokes' Theorem. So, intuitively Stokes' Theorem says that how $\mathbf{F}$ behaves along the boundary of S determines and is determined by how $\operatorname{curl} \mathbf{F}$ flows through S.

To give one concrete use of this which is central to our everyday lives, suppose that $\mathbf{F}$ is an electric field. Then $\operatorname{curl} \mathbf{F}$ actually describes a *magnetic* field, so Stokes' Theorem says that the electric circulation along the boundary of a surface is equal to the magnetic flux through the surface. This is something you would see explicitly in an intro physics course covering electromagnetism: take a piece of wire bent into the shape of a circle and run an electric current through it—this in turn produces a magnetic flux over the region enclosed by the wire, so moving a magnetic compass over it will cause the needle to go crazy since it is responding to the newly created magnetic field. This idea is at the core of all modern uses of electricity and magnetism.

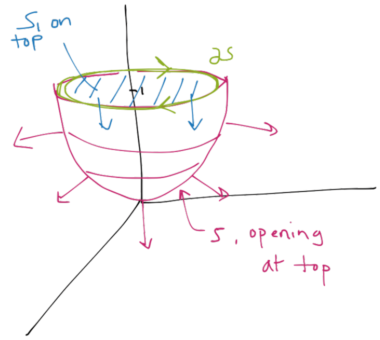
Lecture 24: Gauss's Theorem

Today we started talking about Gauss's Theorem, our final topic. Gauss's Theorem gives yet another way to compute surface integrals, this time by relating them to good ol' triple integrals. Combining this with the idea of "closing off" a surface gives an incredibly useful way of computing surface integrals.

Warm-Up. Suppose that $\mathbf{F}$ is a field whose curl is

$$\operatorname{curl} \mathbf{F} = ye^{y^{z^z}}\mathbf{i} + (xz \sin \cos z - y^2)\mathbf{j} + 2yz\mathbf{k}.$$

(We'll see later why such a field $\mathbf{F}$ must exist.) We compute the flux of $\operatorname{curl} \mathbf{F}$ across the piece of the paraboloid $z = x^2 + y^2$ with $z \leq 1$, oriented using outward pointing normal vectors:



Clearly we should not attempt to do this directly since curl $\mathbf{F}$ is a mess. Instead we use the fact that we can replace S by any surface having the same boundary and oriented appropriately. The boundary of S is the unit circle in the plane $z = 1$, and the unit disk S_1 in the plane $z = 1$ has the same boundary. The orientation on S gives the "clockwise" orientation on $\partial S = \partial S_1$ as viewed from above, so to give the same orientation on the boundary S_2 must have downward pointing normal vectors.

Thus we have

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1, \mathbf{n}=-\mathbf{k}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Since S_1 has an easy normal vector, we have

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = \operatorname{curl} \mathbf{F} \cdot (-\mathbf{k}) = -2yz,$$

which equals $-2y$ on S_1 since $z = 1$. Thus

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} -2y \, dx \, dy.$$

At this point we can convert to polar coordinates or note that $-2y$ is odd with respect to y and S_1 is symmetric with respect to y, so this final double integral is zero. Thus

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$$

is our desired value.

Gauss's Theorem. Gauss's Theorem (also called the *Divergence Theorem*) gives a way to compute surface integrals using triple integrals instead. Say that $\mathbf{F}$ is a vector field which has continuous partial derivatives throughout some solid region E in $\mathbb{R}^3$, where the boundary ∂E of E has the outward orientation. Then

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

Note that div $\mathbf{F}$ is just a function, so it makes to integrate it over the solid E using a triple integral, and that the boundary ∂E of E is a *closed* surface, i.e. a surface which itself has no boundary, so no "openings".

Of course, if the closed surface in question has inward orientation all we have to do is change the sign of the resulting triple integral. The main thing we need is to be integrating over a closed surface on the surface integral side, even though we'll soon see a way to get around this.

The intuition behind Gauss's Theorem is similar to that for Green's or Stokes' Theorem: $\operatorname{div}\mathbf{F}$ measures how $\mathbf{F}$ flows "through" individual points, and the triple integral $\iiint_E \operatorname{div}\mathbf{F}\, dV$ adds all of these individual flows together, so the net result should be the flow of $\mathbf{F}$ through the boundary ∂E as measured by the surface integral $\iint_{\partial E} \mathbf{F}\cdot d\mathbf{S}$. For more precise reasoning you would break E up into smaller and smaller pieces, adding up the surface integrals of $\mathbf{F}$ over all these pieces; this sum should equal the flux of $\mathbf{F}$ through the outer surface ∂E alone since the "inside" contributions will cancel out, and as your inside surfaces get smaller and smaller you end up measuring the "flux" through individual points, which is what $\operatorname{div}\mathbf{F}$ gives.

Example 1. We compute

$$\iint_S (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})\cdot d\mathbf{S}$$

where S is the unit sphere with outward orientation. This can be done directly by parametrizing S using spherical coordinates, but we end up with an integral computation which will require various trig identities. Instead, Gauss's Theorem gives:

$$\iint_S (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})\cdot d\mathbf{S} = \iiint_E (2x + 2y + 2z)\, dV$$

where E is the solid unit ball. This final triple integral can be computed by converting to spherical coordinates as well, or by noting that each term in the integrand is odd with respect to one variable and E is symmetric across all axes, so the value of the triple integral should be 0. Thus

$$\iint_S (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})\cdot d\mathbf{S} = 0$$

as well. (This makes sense since at points (x, y, z) and $(-x, -y, -z)$ on the sphere, the vector field looks the same but the normal vectors point in opposite directions, so the contribution from these points to the total vector surface integral cancel out.)

Example 2. We compute

$$\iint_S (y^{123}e^{\sin\cos(yz)}, y - x^{z^x}, z^2 - z)\, dS$$

whee S is the closed cylinder (with top and bottom disks attached) of radius 1 and height 3 with inward orientation. This surface is closed, but has the opposite orientation we need for Gauss's Theorem. Thus

$$\iint_S (y^{123}e^{\sin\cos(yz)}, y - x^{z^x}, z^2 - z)\, dS = -\iiint_E 2z\, dV$$

where E is the region enclosed by S.

To compute this triple integral we convert to cylindrical coordinates, in which E has bounds:

$$0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 0 \le z \le 3.$$

Thus

$$\begin{aligned}\iiint_E 2z\, dV &= \int_0^{2\pi}\int_0^1\int_0^3 2zr\, dz\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 9r\, dr\, d\theta\end{aligned}$$

$$= \int_0^{2\pi} \frac{9}{2}\, d\theta$$
$$= 9\pi,$$

so
$$\iint_S (y^{123} e^{\sin\cos(yz)}, y - x^{z^x}, z^2 - z)\, dS = -\iiint_E 2z\, dV = -9\pi$$
is the desired value.

Example 3. We compute
$$\iint_S (\text{some mess involving only } y \text{ and } z,\ \text{some mess involving only } x \text{ and } z, 3z) \cdot d\mathbf{S}$$
where S is most of the unit cube $[0,1] \times [0,1] \times [0,1]$ only excluding the bottom square, oriented using outward pointing normals. Now, we can't compute this directly integrating over the five faces of S separately since I didn't write down an explicit field. Also, S is not closed due to the missing bottom square, so Gauss's Theorem might not seem to apply. However, we can "close off" our surface by attaching the bottom square, apply Gauss's Theorem, and then subtract off the surface integral over the piece we attached. (Think back to a similar thing we did for line integrals using Green's Theorem.)

Letting S_1 denote the bottom square with downward orientation, we have
$$\iint_S (P(y,z), Q(x,z), 3z) \cdot d\mathbf{S} = \iint_{S+S_1} (P(y,z), Q(x,z), 3z) \cdot d\mathbf{S} - \iint_{S_1} (P(y,z), Q(x,z), 3z) \cdot d\mathbf{S},$$
where now we're using P and Q to denote the unknown components of $\mathbf{F}$ and $S + S_1$ denotes the closed surface obtained by attaching S_1 to S; we had to orient S_1 downward in order to have a consistent outward orientation on $S + S_1$.

For the surface integral over $S + S_1$ we can apply Gauss's Theorem. Note that when computing the divergence we never differentiate $P(y, z)$ with respect to y and z, and we never differentiate $Q(x, z)$ with respect to x and z. Thus
$$\operatorname{div}(P(y,z), Q(x,z), 3z) = 3,$$
and we get
$$\iint_{S+S_1} (P(y,z), Q(x,z), 3z) \cdot d\mathbf{S} = \iiint_{[0,1]\times[0,1]\times[0,1]} 3\, dV = 3(\text{volume}) = 3.$$

For the surface integral over S_1, note that the normal vector is simply $-\mathbf{k}$, so
$$(\text{vector field}) \cdot (\text{normal vector}) = -3z,$$
which is zero on S_1 since S_1 is on the xy-plane. Thus
$$\iint_{S_1} (P(y,z), Q(x,z), 3z) \cdot d\mathbf{S} = \iint_{S_1} 0\, dA = 0.$$
Hence all together we have
$$\iint_S (P(y,z), Q(x,z), 3z) \cdot d\mathbf{S} = 3 - 0 = 3.$$

Important. When a surface S isn't closed, Gauss's Theorem can still be applied after closing the surface by attaching a piece S_1:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \pm \iiint_E \operatorname{div} \mathbf{F} \, dV - \iiint_{S_1} \mathbf{F} \cdot d\mathbf{S}$$

where the $\pm$ depends on the orientation we have on the closed surface $S + S_1$. Also, this only works when $\mathbf{F}$ has continuous partials throughout the region E enclosed by $S + S_1$, since these are the assumptions we have in Gauss's Theorem.

Example 4. Finally, consider a field $\mathbf{F}$ continuous on all of $\mathbb{R}^3$ and the surface integral

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where S is *any* closed surface, with whatever orientation. Gauss's Theorem gives

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \pm \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F}) \, dV = \pm \iiint_E 0 \, dV = 0$$

since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$ for any $\mathbf{F}$. Hence, the flux of $\operatorname{curl} \mathbf{F}$ through any closed surface is zero.

View this as analogous to the fact that the line integral of a conservative field over any closed curve is always zero. This together with the "surface independence" property we previously saw for curls suggests that curls play a similar role for surface integrals as conservative fields do for line integrals: i.e. "curls are to surface integrals as conservative fields are to line integrals" is a good analogy to include on an SAT exam.

Checking if a given field is the curl of another. Pushing the analogy between curls and conservative fields further, the facts above suggest a method for determining whether or not a given field is the curl of another. For instance, we ask: is there a vector field $\mathbf{F}$ such that

$$\operatorname{curl} \mathbf{F} = y^{199} \sin \cos z \mathbf{i} + x^{e^{z^4}} \mathbf{j} + z^3 x \mathbf{k}?$$

The answer is no, since this given field has divergence $3z^2x$, whereas if it was the curl of another field $\mathbf{F}$ it would necessarily have divergence zero. (This is analogous to the fact that a field with nonzero curl is not conservative.)

But, we already saw that for a field $\mathbf{F}$ with $\operatorname{curl} \mathbf{F} = 0$, if $\mathbf{F}$ is defined and has continuous partials on all of $\mathbb{R}^3$ say, then $\mathbf{F}$ *is* in fact conservative. In this new setting, the analogous fact is: if $\mathbf{G}$ is defined with continuous partials on all of $\mathbb{R}^3$ and has divergence 0, then there *is* a field $\mathbf{F}$ such that $\operatorname{curl} \mathbf{F} = \mathbf{G}$. For instance, this is how we know that the field

$$ye^{y^{z^z}} \mathbf{i} + (xz \sin \cos z - y^2)\mathbf{j} + 2yz\mathbf{k}$$

from the first Warm-Up can be written as the curl of another field: it has divergence 0 and has continuous partials everywhere. However, the rub is that while potentials of conservative fields can be found without too much work, finding $\mathbf{F}$ such that $\operatorname{curl} \mathbf{F} = \mathbf{G}$ is an incredibly difficult task in general. The point is that often times knowing that one field is the curl of another is enough to let us make some conclusions about it, without knowing what that other field is.

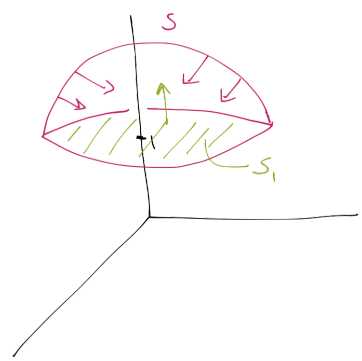
Lecture 25: More on Gauss's Theorem

Today, unfortunately, was the last class of the entire year. Kudos to you all for making it through Math 290! We looked at a few more examples, and then tried to put everything we've done these last few weeks in the right context.

Warm-Up 1. We compute

$$\iint_S [(2xy - z^z)\mathbf{i} + (y - y^2)\mathbf{j} - z\mathbf{k}] \cdot d\mathbf{S}$$

where S is the piece of the sphere $x^2 + y^2 + z^2 = 2$ above $z = 1$ with inward orientation:



The surface S is not closed, so in order to apply Gauss's Theorem we must close it off by attaching the disk S_1 of radius 1 in the plane $z = 1$. (This has radius 1 since the sphere equation gives $x^2 + y^2 = 1$ when $z = 1$.) To have a consistent orientation throughout the entire closed surface $S + S_1$ we gives S_1 the upward orientation. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S+S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = (2xy - z^z)\mathbf{i} + (y - y^2)\mathbf{j} - z\mathbf{k}$.

For the first term we use Gauss's Theorem with a negative to account for the inward orientation:

$$\iint_{S+S_1} \mathbf{F} \cdot d\mathbf{S} = -\iiint_E \operatorname{div} \mathbf{F}\, dV = -\iiint_E (2y + 1 - 2y - 1)\, dV = 0$$

where E is the solid enclosed by $S + S_1$. The surface S_1 has unit normal vector $\mathbf{n} = \mathbf{k}$, so

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} (\mathbf{F} \cdot \mathbf{n})\, dS = \iint_{S_1} -z\, dA = -\text{area of } S_1 = -\pi$$

since $z = 1$ on S_1. Thus all together we have:

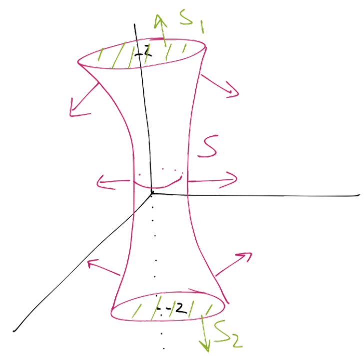
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S+S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = 0 - (-\pi) = \pi.$$

Note that it makes this value is positive: throughout S we have $z > 0$, so the $\mathbf{k}$-component of the field always points downward, meaning that the angle between $\mathbf{F}$ and the inward normal vector on S is always smaller than $90°$.

Warm-Up 2. We compute

$$\iint_S (ye^{\cos\sin z}, x^{100}e^{z^{xz}}, x - z^2) \cdot d\mathbf{S}$$

where S is the piece of the hyperboloid $x^2 + y^2 - z^2 = 1$ between $z = -2$ and $z = 2$ with outward orientation:



Again, S is not closed, only in this case we have to attached *two* pieces in order to close off S: the unit circle S_1 in the plane $z = 2$ and the unit circle S_2 in the plane $z = -2$. We orient S_1 using upward normal vector and S_2 using downward so that the closed surface $S + S_1 + S_2$ has outward orientation. Then, setting $\mathbf{F} = (ye^{\cos\sin z}, x^{100}e^{z^{xz}}, x - z^2)$ we have:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S+S_1+S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

Using Gauss's Theorem we have

$$\iint_{S+S_1+S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E -2z\, dV = 0$$

since the region E enclosed by $S + S_1 + S_2$ is symmetric across the xy-plane. Next, on S_1 we have $\mathbf{F} \cdot \mathbf{n} = x - z^2 = x - 4$, so

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} (x - 4)\, dA = -4(\text{area of } S_1) = -4\pi$$

since the integral of the x piece is zero due to symmetry. Similarly, on S_2 we have $\mathbf{F} \cdot \mathbf{n} = -x + z^2 = -x + 4$, so

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} (-x + 4)\, dA = 4\pi$$

since the integral of $-x$ is zero due again to symmetry. Thus all together we have:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S+S_1+S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0 - (-4\pi) - (4\pi) = 0$$

as the desired value.

Important. Some surfaces might require attaching more than one piece in order to "close it off" before we apply Gauss's Theorem.

Example. Finally, we compute

$$\iint_S (\mathbf{F} + \operatorname{curl}\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S}$$

where S is the piece of the paraboloid $x = y^2 + z^2$ for $x \leq 1$ with outward orientation and

$$\mathbf{F} = [(x-1)e^{\sin\cos e^y} - z^2 + y^2]\mathbf{i} + y\mathbf{j} + ze^{\sin\cos e^y}\mathbf{k}.$$

The point is that this will involve both Gauss's Theorem and Stokes' Theorem, so a nice computation bringing together our final two topics. It's definitely too long to be considered for the final, but provides a good review of various concepts nonetheless. (**Note**: In my 10am class I wrote the integrand as $\mathbf{F} + \operatorname{curl}\mathbf{F}$ instead of using a double curl, which is what led to me getting stuck. I meant it to be a double curl as written here.) The original integral breaks up as:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_S \operatorname{curl}\operatorname{curl}\mathbf{F} \cdot d\mathbf{S},$$

and we compute each term separately.

Since S is not closed, we attach the unit disk D at $x = 1$ with orientation in the positive $\mathbf{i}$-direction to get:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S+D} \mathbf{F} \cdot d\mathbf{S} - \iint_D \mathbf{F} \cdot d\mathbf{S}.$$

The closed surface $S + D$ has outward orientation, so with E the solid enclosed by $S + D$ we have

$$\iint_{S+D} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\, dV = \iiint_E dV = \text{volume of } E.$$

To compute this volume, we make the change of variables:

$$x = x,\ y = r\cos\theta,\ z = r\sin\theta$$

inspired by cylindrical coordinates only with the "polar" change happening in the y and z directions. (Or instead, you can note that E has the same volume as the region enclosed by the paraboloid $z = x^2 + y^2$ and $z = 1$, and this volume we can find using ordinary cylindrical coordinates.) The Jacobian factor still works out to be r, and throughout E the value of x starts along the paraboloid $x = y^2 + z^2 = r^2$ and increases to $x = 1$, so

$$\begin{aligned} \iiint_E dV &= \int_0^{2\pi}\int_0^1\int_{r^2}^1 r\, dx\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 (r - r^3)\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{4}\, d\theta \\ &= \frac{\pi}{2}. \end{aligned}$$

Thus $\iint_{S+D} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2}$ as well. Now, D has normal vector $\mathbf{n} = \mathbf{i}$, so

$$\iint_D \mathbf{F} \cdot d\mathbf{S} = \iint_D (\mathbf{F} \cdot \mathbf{n})\, dS = \iint_D ((x-1)e^{\sin\cos e^y}) - z^2 + y^2)\, dA = \iint_D (y^2 - z^2)\, dA$$

since $x = 1$ on D. This integral can be computed using polar coordinates, or by arguing that the integrals of z^2 and y^2 over D will be exactly the same due to symmetry, so the integral of $y^2 - z^2$ will be zero:

$$\iint_D (y^2 - z^2)\, dA = 0.$$

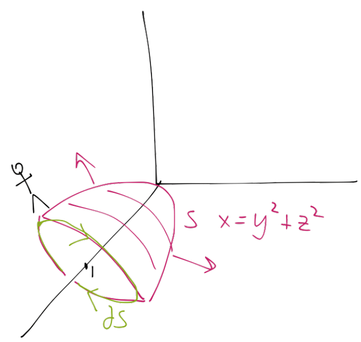
Putting it all together we have:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S+D} \mathbf{F} \cdot d\mathbf{S} - \iint_D \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

For the remaining surface integral, by Stokes' Theorem we get

$$\iint_S \operatorname{curl} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \operatorname{curl} \mathbf{F} \cdot \, ds$$

as long as we orient ∂S in the correct way. The boundary of S is the unit circle in the plane $x = 1$, and with outward orientation on S this circle should have clockwise orientation when viewed from the positive x-direction:



Thus possible parametric equations for ∂S are

$$\mathbf{x}(t) = (1, \cos t, -\sin t),\ 0 \le t \le 2\pi$$

where the $-\sin t$ handles the clockwise orientation. (Or you can orient the curve counterclockwise and change the sign of your final value.) We have

$$\operatorname{curl} \mathbf{F} = (\text{something}, -2z, -(x-1)\text{something} + 2y).$$

(Note that the terms labeled "something" don't actually matter in the computation which follows.) Thus we get:

$$\begin{aligned}
\int_{\partial S} \operatorname{curl} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} (\operatorname{curl} \mathbf{F})(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt \\
&= \int_0^{2\pi} (\text{something}, 2\sin t, 2\cos t) \cdot (0, -\sin t, -\cos t)\, dt \\
&= \int_0^{2\pi} (-2\sin^2 t - 2\cos^2 t)\, dt \\
&= \int_0^{2\pi} -2\, dt
\end{aligned}$$

$$= -4\pi.$$

Thus all together we get

$$\iint_S (\mathbf{F} + \operatorname{curl}\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_S \operatorname{curl}\operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} - 4\pi = -\frac{7\pi}{2}.$$

So, a super fun (not sarcastic!) computation, but again too long for the final.

Summary. Finally, we spoke about how to put all the theorems we've looked at recently (Fundamental Theorem of Line Integrals, Stokes' Theorem, Gauss's Theorem) in the right context to emphasize their similarities. The main point is that each has the form:

$$\int_{\text{boundary (object)}} \text{something} = \int_{\text{object}} \text{derivative (something)}.$$

Check the notes titled "The Wonders of Integration" online to see how this works. Thanks for reading!