MATH 325: Complex Analysis
Northwestern University, Lecture Notes

Written by Santiago Cañez

These are notes which provide a basic summary of each lecture for MATH 325, "Complex Analysis", taught by the author at Northwestern University. The book used as a reference is the 9th edition of *Complex Variables and Applications* by Brown and Churchwell. Watch out for typos! Comments and suggestions are welcome.

Contents

Lecture 1: Complex Numbers

Complex analysis is the study of functions of complex variables, and in particular doing calculus with such functions. The basic definitions we will see, such as that of the derivative and the integral of a complex function, are the same or very similar to ones you have seen before for functions of real variables, but the resulting theory has some stark differences to what happens in the real case. This happens because the notion of being "differentiable" places severe restrictions on the properties a complex can have, and these restrictions turn out to lead to tools and techniques that have no direct analog in the real case. But, as we will see, these purely complex tools have important things to say about purely real phenomena as well, so complex analysis is a subject with crucial practical consequences.

Let us give a sense of some of the behaviors we will see in the complex setting as contrasted with what we have seen before in the real setting:

- We will see that the (complex) integral of a complex differentiable function over a closed curve is always zero, a result known as *Cauchy's theorem* and which is at the heart of most every other result we will see in this course. Complex integrals are most analogous to the line integrals you would have seen in a multivariable calculus course like MATH 230-2, and there you would have seen examples where sometimes integrating over a closed curve gave zero and sometimes not. In the complex setting, however, the answer will *always* be zero as long as the function at hand has a complex derivative.
- In the real setting, the value of a differentiable function at a point has no relation in general to the values of the function at points further away, and indeed you can have multiple differentiable functions whose values at some $x = a$ are all the same but which have drastically different values away from a. But things are more restrictive in the complex case where we will see that the value of a complex function at some point z can be fully determined from knowing only the values of the function on a curve which encloses z, even if the points on this curve are very far away from z itself so that the behavior of the function far away from z has a direct bearing on its behavior at z itself. This will come from what's called *Cauchy's integral formula*, one of the cornerstone results we will derive.
- In the real case, there are examples of functions which have a first derivative but not a second derivative, or which have second derivative but not a third derivative, and so on. In the complex case however, we will see that once a function has a first complex derivative it will automatically have a second derivative, a third derivative, etc as well. This will also lead to the fact that complex differentiable functions are always expressible as what are called *power series* (which we will review when needed), which is not true in the real case.
- Finally, in the real case there are plenty of differentiable functions which are *bounded*—meaning there is a restriction on how large or small the values of the function can be—with $\sin x$ and $\cos x$ (both bounded by 1) being the main examples. However, in the complex case we will see that the only bounded differentiable functions are the constant ones, a result which is known as *Liouville's theorem.*

At this point, it is not expected that you fully grasp what the results above mean precisely, but they should give a sense of a restrictive but at the same time *rich* theory which arises in the study of functions of complex variables. As a result of this theory, we will be able to compute things like

$$\int_0^\infty \frac{\sin x}{x}\,dx, \text{ or } \int_{-\infty}^\infty \frac{\cos x}{1+x^4}\,dx,$$

which are *real* integrals and yet cannot be easily computed via only real methods. We will also gain a better understanding of standard functions we all know and love—exponentials, sine, and cosine—where much of their true essence is hidden from sight when considering only real phenomena.

Complex number. Before we get into any *analysis* (i.e., calculus), we need a firm understanding of complex numbers themselves as they and their properties are at the core of everything we will do. A complex number is an expression of the form

$$z = a + ib$$

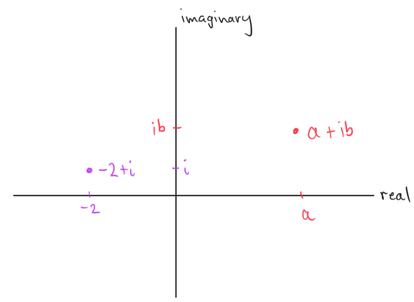
where a, b are real numbers and i has the property that $i^2 = -1$. We can perform standard algebraic operations on complex numbers, such as addition and multiplication, simply as one might expect when manipulating such expressions; for example, we have

$$(1 + 2i) + (\pi - 4i) = (1 + \pi) - 2i$$

by grouping like terms, and

$$(1 + 2i)(\pi - 4i) = \pi - 4i + 2\pi i - 8i^2 = \pi - 4i + 2\pi i + 8 = (\pi + 8) + i(2\pi - 4)$$

by expanding out the product on the left as normal and using $i^2 = -1$ to simplify. Geometrically, we visualize complex numbers as points in the *complex plane*, where we keep track of the "real" part on the horizontal axis and the "imaginary" part on the vertical axis:



We should clarify that there is no "number" in usual sense which satisfies $i^2 = -1$, but rather we are introducing a new type of object that satisfies this property by definition. One might ask why we are allowed to do this? The simple answer is that it is no different than anything else we do in mathematics, where abstract "things" are introduced by definition all the time in order to have some useful property we care about. After all, what is the number "1" but an abstract notion we introduce to capture some idea of "quantity", and introducing i in this way is no different. Anything is fair as long as it leads to useful results and applications. But, if we want to be pedantic, we can in fact give a more proper definition of "complex number" using the idea of points in a plane. From this perspective, we simply define a complex number to be such a point (a, b) in $\mathbb{R}^2$ with the stipulation that multiplication of complex numbers is then defined by

$$(a, b)(c, d) = (ac - bd, ad + bc),$$

which is motivated by the outcome $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$ we would expect anyway. Here then, i is defined to be the point $(0, 1)$, where with this definition of multiplication we then have

$$(0, 1)(0, 1) = (-1, 0),$$

where $(-1, 0)$ is then the interpretation of the real number -1 in this context. But, working with points this way and recalling this multiplication can be cumbersome, so we simply use the previous $a + ib$ notation with the $i^2 = -1$ stipulation as a much more convenient way to keep track of it all.

Example. We determine the complex numbers $z = a + ib$ (with a, b real) satisfying $z^2 = i$. (In other words, we determine the "square roots" of i.) We compute

$$z^2 = (a + ib)(a + ib) = a^2 + abi + abi + i^2b^2 = (a^2 - b^2) + i2ab,$$

using $i^2 = -1$ in the last step and then collecting like terms. In order to this resulting expression to equal $i = 0 + 1i$, we need the real part $a^2 - b^2$ to be zero and the imaginary part $2ab$ to be 1, so we get the requirements that

$$a^2 - b^2 = 0 \quad \text{and} \quad 2ab = 1.$$

The first then gives $a = \pm b$, but the choice $a = -b$ then turns the second condition into

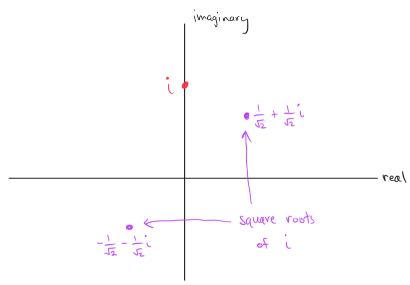
$$-2a^2 = 1,$$

which has no solutions since a is supposed to be real. Thus we need only consider the $a = b$ case, in which case the second condition is

$$2a^2 = 1, \text{ so } b = a = \pm\tfrac{1}{\sqrt{2}}.$$

Thus we get two complex numbers satisfying $z^2 = i$, namely $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

We can visualize these two square roots of i in the complex plane as follows:



It is natural to wonder what the geometric meaning of $z^2 = i$ is, and whether we would have expected to find roots as in the picture without going through the brute force algebraic computation we carried out. We will come back to this later when discussing roots of complex numbers more generally, and we will see that the picture above does indeed make sense.

Inverses. Nonzero complex numbers have inverses (i.e., reciprocals) just as do nonzero real numbers. For example, let us compute

$$\frac{1}{1 + 2i},$$

by which we mean we seek to express this complex number in the standard form $a + ib$. One approach is to solve for the desired a, b by determining the values which satisfy

$$(1 + 2i)(a + ib) = 1.$$

After expanding the left side this becomes $(a - 2b) + i(b + 2a) = 1 + 0i$, so we need

$$a - 2b = 1 \quad \text{and} \quad b + 2a = 0.$$

This can now be solved using standard (linear) algebraic methods.

But instead we can compute the desired reciprocal more easily by multiplying numerator and denominator in $\frac{1}{1+2i}$ by something which will get rid of the imaginary term in the denominator. Note that $(1+2i)(1-2i) = (1+4) + i(-2+2) = 5$, so we have

$$\frac{1}{1+2i} = \frac{1}{1+2i}\left(\frac{1-2i}{1-2i}\right) = \frac{1-2i}{5} = \frac{1}{5} - \frac{2}{5}i$$

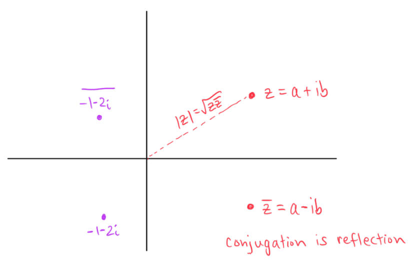
as the desired inverse. For another example, we compute

$$\frac{1-3i}{5+2i}.$$

Now we multiply numerator and denominator by $5-2i$ since $(5+2i)(5-2i) = 29$, so

$$\frac{1-3i}{5+2i} = \frac{1-3i}{5+2i}\left(\frac{5-2i}{5-2i}\right) = \frac{(1-3i)(5-2i)}{(5+2i)(5-2i)} = \frac{-1-17i}{29} = -\frac{1}{29} - \frac{17}{29}i.$$

Conjugates and moduli. The complex number obtained from $z = a+ib$ obtained by changing the sign of the imaginary part (as in turning $1+2i$ into $1-2i$ or $5+2i$ into $5-2i$ in the examples above) is useful enough that we give it a special name and notation: $\overline{z} = a - ib$ is called the complex *conjugate* of $z = a+bi$, again assuming that a and b here are real. Geometrically, taking the conjugate of a complex number corresponds to reflecting it across the horizontal real axis:



The product of a complex number and its conjugate is always real and nonnegative since

$$z\overline{z} = (a+ib)(a-ib) = a^2 + b^2.$$

In particular, the square root of this product is what gives the usual notion of distance from (a,b) to the origin, and this is what we call the *modulus* of z and denote by $|z|$:

$$|z| = \sqrt{z\overline{z}}.$$

The modulus will give us a way to measure how "large" a complex number is, and will be useful in coming up with b ounds of various expressions later on. With these notations, we can then cleanly write the inverse of a nonzero z as

$$\frac{1}{z} = \frac{1}{z}\left(\frac{\overline{z}}{\overline{z}}\right) = \frac{\overline{z}}{|z|^2}.$$

Lecture 2: Polar Forms

Warm-Up 1. We solve the equation

$$4iz^2 - 4z - i = 4i$$

for z. One approach is to use the standard quadratic formula, which is still valid in the complex setting, but instead we use factoring to highlight some key algebraic properties. First, we factor i out of the left side to get

$$i(4z^2 + 4iz - 1) = 4i.$$

Note here that factoring i out of the $-4z$ term in the initial expression leaves us with $4iz$ using $-1 = i^2$. Then we can divide through by i, or equivalently multiply through by $\frac{1}{i} = -i$ to get

$$4z^2 + 4iz - 1 = 4.$$

The left side is now $(2z + i)^2$ (again using $i^2 = -1$, so we we have

$$(2z + i)^2 = 4.$$

The square roots of 4 are ± 2, so we get

$$2z + i = \pm 2, \text{ and thus } z = \tfrac{1}{2}(\pm 2 - i) = \pm 1 - \tfrac{1}{2}i$$

as the solutions.

Warm-Up 2. We verify that the operation of taking complex conjugates is multiplicative in the sense that

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}.$$

(As a consequence, since the modulus of z can be written as $|z| = \sqrt{z\overline{z}}$, we also get that the operation of taking moduli is multiplicative:

$$|zw| = \sqrt{zw\overline{zw}} = \sqrt{z\overline{z}w\overline{w}} = \sqrt{z\overline{z}}\sqrt{w\overline{w}} = |z||w|.$$

This will be a useful tool later when we need to develop bounds on various types of expressions.) This is meant to be a purely algebraic verification, where we compute both sides of the desired equality to see that they are same. For $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, with a_1, a_2, b_1, b_2 all real, we have

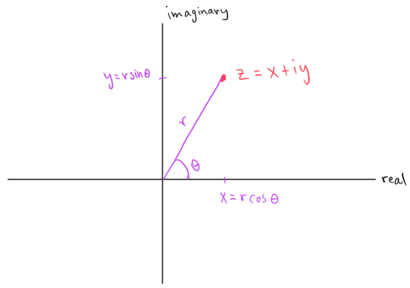
$$\begin{aligned} z_1 z_2 &= (a_1 + ib_1)(a_2 + ib_2) \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + i^2 b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1), \text{ so} \\ \overline{z_1 z_2} &= (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1). \end{aligned}$$

On the other hand,

$$\begin{aligned} \overline{z_1}\,\overline{z_2} &= (a_1 - ib_1)(a_2 - ib_2) \\ &= a_1 a_2 + i^2 b_1 b_2 - i a_1 b_2 - i b_1 a_2 \\ &= (a_1 a_2 - b_1 b_1) - i(a_1 b_2 + a_2 b_1), \end{aligned}$$

which agrees with what we got for $\overline{z_1 z_2}$ above.

Polar coordinates. Being able to visualize complex numbers as points in the complex plane leads to an important way of expressing them, namely via polar coordinates. Recall that polar coordinates are defined as in the picture

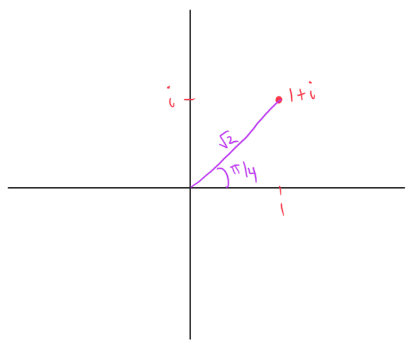


The radius value $r = |z|$ is what we called the modulus last time. The polar angle θ is what we will now call the *argument* of z and denote by $\arg z = \theta$. Arguments are not unique, since adding 2π to one argument value gives another, so $\arg z$ really denotes a collection of angles and not a specific one. But at times we will want to single out a specific argument, and we take the argument value satisfying $-\pi < \theta \leq \pi$ to be what we call the *principal argument* of z and denote it by $\operatorname{Arg} z$.

With the standard conversions $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

Here we have thus separated z into its "size" as measured by the modulus $r = |z|$ and its "direction" as measured by $\theta = \arg z$. For example, the complex number $1 + i$



has modulus $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and principal argument $\operatorname{Arg}(1 + i) = \frac{\pi}{4}$, so

$$1 + i = \sqrt{2}(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}).$$

Any angle of the form $\frac{\pi}{4} + 2\pi k$ with k and integer also serves as a valid argument.

Imaginary exponentials. The expression $\cos\theta + i\sin\theta$ can get cumbersome to work with. Luckily, there is a much simpler way of keeping track of this argument piece via the identity

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Now, what does this mean exactly, or in other words how does taking e to the power of a purely imaginary number (meaning one with real part zero) give something like $\cos\theta + i\sin\theta$. One answer, which is the one our book uses, is that we simply take this expression as the *definition* of what $e^{i\theta}$ means. We will later that there really is no choice: if we want a definition of $e^{i\theta}$ in a way that is consistent with everything we know about e^x for real x, this is what it must be.

But we prefer, along similar lines, to start with a different definition of $e^{i\theta}$ and instead *derive* the $\cos\theta, \sin\theta$ expression above as a consequence. This alternative definition is also one that, as we will see, is forced on us if we want to be consistent with e^x. We recall that for real x, e^x can be expressed as an infinite summation

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

(We will recall some details of such *series* expressions later.) With this in mind, we then define $e^{i\theta}$ to be what we get if we take the same series expansion only with $i\theta$ in place of x:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}.$$

To simplify this, we use the fact that

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ \ldots$$

and so on with the pattern repeating. In particular, for even powers we get ± 1 and for odd powers we get $\pm i$, or more precisely

$$i^{2n} = (-1)^n \quad \text{and} \quad i^{2n+1} = (-1)^n i.$$

If we thus separate our series definition of $e^{i\theta}$ in the terms occuring with even exponents vs those occuring with odd exponents, we have

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n i \theta^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}.$$

The two resulting series on the right are precisely the usual series expansions of $\cos\theta$ and $\sin\theta$ respectively, so we do get

$$e^{i\theta} = \cos\theta + i \sin\theta$$

as a result. This expression is known as *Euler's identity.*

Polar form. With $e^{i\theta} = \cos\theta + i\sin\theta$, the polar coordinate expression we had before for a complex number z becomes

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta},$$

which is what we call the *polar form* of z. For example,

$$1 + i = \sqrt{2} e^{i\pi/4}$$

using what we computed before. The complex number i has modulus 1 and principal argument $\frac{\pi}{2}$, so

$$i = e^{i\pi/2}$$

is the polar form of i; of course, $i = e^{i5\pi/2}$ is another valid polar form of i if we allow ourselves to use a non-principal argument.

Polar products. The complex number $-2 + 2i$ has modulus $\sqrt{2^2 + 2^2} = \sqrt{8}$ and argument $3\pi/4$, while the complex number $1 + i\sqrt{3}$ has modulus $\sqrt{1+3} = 2$ and argument $\pi/3$, so

$$-2 + 2i = \sqrt{8} e^{i(3\pi/4)} \quad \text{and} \quad 1 + i\sqrt{3} = 2e^{i\pi/3}.$$

The product of these is

$$(-2+2i)(1+i\sqrt{3}) = (-2-2\sqrt{3}) + i(2-2\sqrt{3}),$$

which should also be obtained by multiplying the polar forms above. In fact, exponentials like $e^{i\theta}$ behave as you would expect of any exponential, in particular meaning that when multiplying two such things the exponents add together:

$$\sqrt{8}e^{i(3\pi/4)} \cdot 2e^{i\pi/3} = 2\sqrt{8}e^{i(3\pi/4)}e^{i\pi/3} = 2\sqrt{8}e^{i(\frac{3\pi}{4}+\frac{\pi}{3})},$$

so

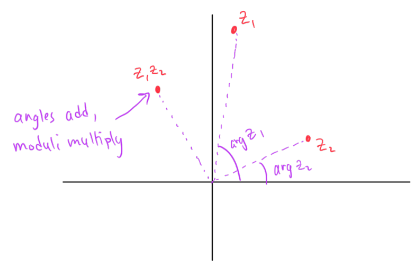
$$(-2-2\sqrt{3}) + i(2-2\sqrt{3}) = 2\sqrt{8}e^{i(\frac{3\pi}{4}+\frac{\pi}{3})}$$

is a valid polar form for the complex number on the left.

The fact that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ is true is not at all obvious. For something like $e^2e^3 = e^5$ the reasoning is clear, since taking 2 e's and 3 e's gives 5 e's over all, but for $e^{i\theta}$ it is not as if we literally multiplying e by itself "$i\theta$" many times, since this makes no sense. Rather, this identity is reflective a basic trigonometric fact we will clarify next time. Assuming this for now, we have

$$r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)},$$

with the point being that when multiplying complex numbers, the moduli multiply together but the arguments *add* together. That is, geometrically complex multiplication corresponds to angle addition, up to some scaling factor. Given a picture of two points in the complex plane, we can thus reasonably determine where their product should be:



Lecture 3: More on Complex Numbers

Warm-Up 1. We justify the identity

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

As we said last time, this is not obvious since complex exponentials like this do not amount to multiplying e by itself some number of times. One approach is via the original definition we gave

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!},$$

where via some series manipulations one can verify that the product of the two series characterizing $e^{i\theta_1}e^{i\theta_2}$ does give the same series as the one defining $e^{i(\theta_1+\theta_2)}$. But this is overkill as there is a simpler trigonometric interpretation. We have

$$e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1).$$

The standard angle addition formulas for cosine and sine say

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \quad \text{and} \quad \sin(\theta_1 + \theta_2) = \cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1,$$

so in fact the expression for $e^{\theta_1} e^{\theta_2}$ above is equal to

$$\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) = e^{i(\theta_1+\theta_2)}.$$

Thus $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ is true, with the upshot being that this nothing but a way to encode the angle addition formulas for sine and cosine. (Side remark: working out both sides of this identity is precisely how I am able to remember these angle addition formulas when needed.)

Warm-Up 2. We find the polar form of

$$\frac{1 + \sqrt{3}i}{2\sqrt{3} - 2i}.$$

On the one hand, we could simply compute this quotient directly and then determine the polar form of the result. But let us first approach this using polar forms from the get-go. We have

$$1 + \sqrt{3}i = 2e^{\pi/3} \quad \text{and} \quad 2\sqrt{3} - 2i = 4e^{-i\pi/6},$$

by computing moduli and arguments. (I determined the arguments here by visualizing these points and comparing to standard angles on the unit circle, but if nothing else arguments can be found by the usual $\theta = \arctan(\frac{y}{x})$ polar equation. We also chose to use the principal argument $-\pi/6$ for the second term here rather than, say, $11\pi/6$.) The polar form of a reciprocal is quick to determine:

$$e^{i\theta} e^{-i\theta} = e^{i(\theta-\theta)} = e^0 = 1, \text{ so } \frac{1}{e^{i\theta}} = e^{-i\theta}.$$

In other words, inversion simply changes the sign of the argument. (This can also be interpreted via complex conjugation since the conjugate of $e^{i\theta} = \cos\theta + i \sin\theta$ is

$$\cos\theta - i \sin(\theta) = \cos(-\theta) + i \sin(-\theta) = e^{-i\theta}$$

where we use that cosine is even and sine is odd. The point is that the inverse of a complex number of modulus 1—i.e., on the unit circle—is just its conjugate.) Thus

$$2\sqrt{3} - 2i = 4e^{-i\pi/6} \implies \frac{1}{2\sqrt{3} - 3i} = \frac{1}{4} e^{i\pi/6},$$

so

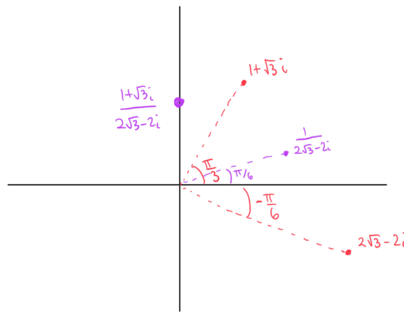
$$\frac{1 + \sqrt{3}i}{2\sqrt{3} - 2i} = 2e^{i\pi/3} \cdot \frac{1}{4} e^{i\pi/6} = \frac{1}{2} e^{i(\frac{\pi}{3} + \frac{\pi}{6})} = \frac{1}{2} e^{i\pi/2}$$

is the desired polar form.

Now, in fact $\frac{1}{2} e^{i\pi/2} = \frac{1}{2} i$, as a direct computation with the original quotient verifies:

$$\frac{1 + \sqrt{3}i}{2\sqrt{3} - 2i} = \left(\frac{1 + \sqrt{3}i}{2\sqrt{3} - 2i}\right)\left(\frac{2\sqrt{3} + 2i}{2\sqrt{3} + 2i}\right) = \frac{(2\sqrt{3} - 2\sqrt{3}) + i(2 + 6)}{16} = \frac{8}{16} i = \frac{1}{2} i.$$

So, the polar form is simple enough to find via this direct computation as well, but the point of doing it the first way was to illustrate the use of polar forms in general as an alternative way of performing computations. From the polar perspective, the fact that the given quotient should be purely imaginary with positive imaginary part makes sense geometrically: we have



so keeping in mind that inversion reflects across the real axis and rescales (conjugation is reflection) and that multiplication corresponds to angle addition, it makes sense that multiplying $1+\sqrt{3}i$ with $\frac{1}{2\sqrt{3}-2i}$ should result in a point with argument $\pi/2$ on the positive imaginary axis.

de Moivre's formula. Taking θ_1 and θ_2 to be the same angle in the identity derived in the first Warm-Up gives

$$(e^{i\theta})^2 = e^{i\theta}e^{i\theta} = e^{i2\theta}.$$

(The real part of the left side $(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$ is $\cos^2\theta - \sin^2\theta$ and the imaginary part is $2\sin\theta\cos\theta$, so comparing with the real and imaginary parts on the right gives

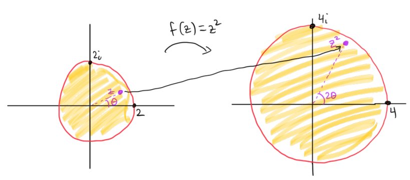
$$\cos^2\theta - \sin^2\theta = \cos(2\theta) \quad \text{and} \quad 2\sin\theta\cos\theta = \sin(2\theta),$$

which are the standard double-angle formulas for sine and cosine.) Applying this repeatedly gives

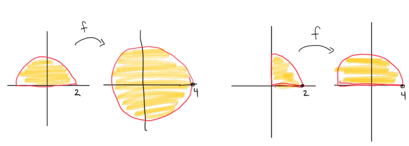
$$(e^{i\theta})^n = \underbrace{e^{i\theta}e^{i\theta}\cdots e^{i\theta}}_{n \text{ times}} = e^{in\theta}.$$

This is known as *de Moivre's formula*, and says that taking n-th powers corresponds to n-fold addition of angles.

In particular then, the operations that send z to z^n for positive integers n have nice geometric interpretations. For example, the transformation that sends $z = re^{i\theta}$ to $z^2 = r^2e^{i2\theta}$ has the effect of squaring the modulus and doubling the angle. If we apply this to all points within the disk of radius 2 centered at 0 (the region enclosed by a circle of radius 2), we obtain as a result a disk of radius 4 centered at the origin: if $|z| \leq 2$, then $|z^2| \leq 4$, and doubling an angle just rotates a point (albeit by different amounts for each point), so disks are sent to disks:



We say here that the disk of radius 4 centered at the origin is the *image* of the disk of radius 2 centered at the origin under the map $f(z) = z^2$. (We will talk more about viewing complex functions as transformations next time.) In fact, the full disk of radius 4 is also the image of the upper-half disk of radius 2 alone, since doubling the angles between 0 and π already gives all angles between 0 and 2π; as another example, the image of the quarter disk of radius 2 would then be the upper-half disk of radius 4:

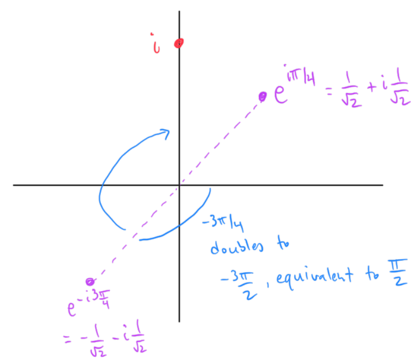


We have similar results for the map sending z to z^3, or z^4, etc.

Complex roots. Back on the first day we computed the square roots of i—i.e., the complex numbers satisfying $z^2 = i$—and found that they were

$$\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}} \quad \text{and} \quad -\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}.$$

We found these by a brute-force algebraic computation, but now we can see more simply why it makes sense geometrically that these are the correct roots. Since i has modulus 1 and (principal) argument $\pi/2$, its square roots should also have moduli 1 (since squaring a complex number squares the modulus) and arguments that double to give $\pi/2$ or something equivalent to $\pi/2$. So, having an argument of $\pi/4$ makes sense, and this is what gives $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = e^{i\pi/4}$, and the other choice for the principal argument is $-3\pi/4$ since this doubles to $-3\pi/2$, which is equivalent to $\pi/2$, and this is what gives the square root $-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = e^{-i3\pi/4}$ in the third quadrant:



In general, following the same idea we can find arbitrary roots of any complex number: in order to have $w^n = z = re^{i\theta}$, w should have polar form $w = r^{1/n}e^{i(\theta/n + 2\pi k/n)}$, with $r^{1/n}$ being the usual nonnegative real n-th root of the nonnegative real number r, since this gives

$$w^n = [r^{1/n}e^{i(\theta/n+2\pi k/n)}]^n = (r^{1/n})^n e^{in(\theta/n+2\pi k/n)} = re^{i(\theta+2\pi k)} = re^{i\theta} = z.$$

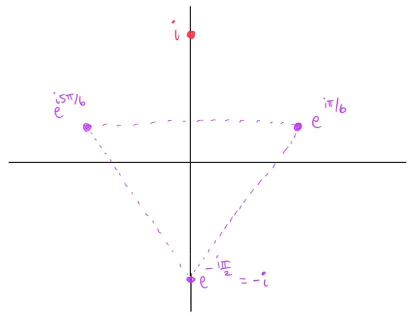
(The $2\pi k$ term in the third-to-last expression just results in non-principal arguments of z.) For example, let us find the cube roots of i. Since $i = e^{i\pi/2}$, the cube roots should have modulus i, and we get a first cube root using the argument $\pi/6$, which triples to $\pi/2$; this gives

$$e^{i\pi/6} = \cos(\tfrac{\pi}{6}) + i\sin(\tfrac{\pi}{6}) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$

as one cube root. This is what we call the *principal* cube root of i, since it was obtained (via dividing by 3) from the principal argument $\pi/2$ of i. To get other cube roots of i, we use non-principal arguments:

$$i = e^{i(\pi/2+2\pi)} = e^{i5\pi/2} \rightsquigarrow \sqrt[3]{i} = e^{i5\pi/6} \text{ and } i = e^{i(\pi/2+4\pi)} = e^{i9\pi/2} \rightsquigarrow \sqrt[3]{i} = e^{i9\pi/6} = e^{-i\pi/2} = -i.$$

Other choices for the initial arguments of i will result in one of these three, so altogether then we have three distinct cube roots of i:



The angle between one and the next is always $4\pi/6$, which is why visually the three roots are the vertices of an equilateral triangle. Note that the cube root $e^{i\pi/6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ at least is possible to find solely from the picture by dividing the principal argument of i by 3, and the other two can be found by visualizing the triangle.

Example. We solve the equation

$$(z+i)^5 = 1 + i\sqrt{3}$$

for z. The key point is that here $z+i$ should be a fifth root of $1+i\sqrt{3}$, so first we find these fifth roots. We have

$$1 + i\sqrt{3} = 2e^{i\pi/3} = 2e^{i(\pi/3+2\pi k)},$$

so the fifth roots of $1+i\sqrt{3}$ are

$$z + i = 2^{1/5}e^{i(\pi/15+2\pi k/5)} \text{ for } k = 0, 1, 2, 3, 4.$$

(Other integers k just give one of these five roots.) The choice with $k = 0$, namely $2^{1/5}e^{i\pi/15}$, is the principal fifth root of $1+i\sqrt{3}$.

Thus, the solutions to our original equation are

$$z = 2^{1/5}e^{i\frac{\pi}{15}} - i,\ 2^{1/5}e^{i(\frac{\pi}{15}+\frac{2\pi}{5})} - i,\ 2^{1/5}e^{i(\frac{\pi}{15}+\frac{4\pi}{5})} - i,\ 2^{1/5}e^{i(\frac{\pi}{15}+\frac{6\pi}{5})} - i,\ 2^{1/5}e^{i(\frac{\pi}{15}+\frac{8\pi}{5})} - i.$$

Lecture 4: Complex Functions

Warm-Up 1. We justify the fact that if w_1 and w_2 are both n-th roots of a nonzero complex number z, then

$$w_1 = \zeta w_2$$

for some ζ satisfying $\zeta^n = 1$. Such ζ are called *n-th roots of unity*, so the claim is that we can always get from one root of $z \neq 0$ to another by multiplying by a root of unity. This is actually apparent in the explicit form we found last time for these roots: if $z = re^{i\theta}$, then the n-th roots are

$$r^{1/n}e^{i(\theta/n+2\pi k/n)} = r^{1/n}e^{i\theta/n}\underbrace{e^{i2\pi k/n}}_{\zeta},$$

where we get an arbitrary n-th root from the principal one $r^{1/n}e^{i\theta/n}$ by multiplying by the n-th root of unity $\zeta = e^{i2\pi k/n}$. But the goal here is to reach this conclusion without knowing ahead of time what the roots explicitly look like.

The argument is simple. We have $w_1^n = z = w_2^n$, so

$$\left(\frac{w_1}{w_2}\right)^n = \frac{w_1^n}{w_2^n} = \frac{z}{z} = 1.$$

(Note that $w_2 \neq 0$ since $z \neq 0$.) Thus $\frac{w_1}{w_2}$ is an n-th root of unity, and

$$w_1 = \underbrace{\left(\frac{w_1}{w_2}\right)}_{\zeta} w_2$$

is our desired expression.

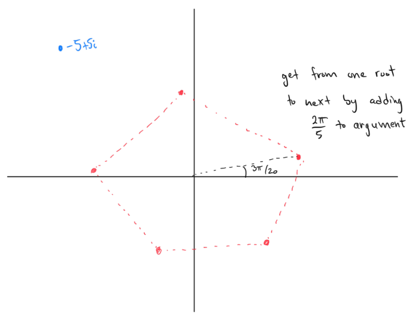
Warm-Up 2. We find the complex numbers z satisfying

$$(iz - 1)^5 = -5 + 5i.$$

This equation requires that $iz - 1$ be a fifth root of $-5 + 5i$, so we first find these roots. We have $-5 + 5i = \sqrt{50}e^{i3\pi/4}$, so

$$iz - 1 = \sqrt{50}^{1/5} e^{i(3\pi/4 + 2\pi k)/5} = 50^{1/10} e^{i(3\pi/20 + 2\pi k/5)} \text{ for } k = 0, 1, 2, 3, 4.$$

Note that these five roots form the vertices a regular pentagon:



Solving for z thus gives

$$z = -i(50^{1/10} e^{i(3\pi/20 + 2\pi k/5)} + 1) \text{ for } k = 0, 1, 2, 3, 4$$

as the desired solutions of $(iz - 1)^5 = -5 + 5i$.

Complex functions. We are now ready to get to the "analysis" part of complex analysis, which is all about doing calculus with complex functions. A complex function is a function that takes as input a complex number z and outputs a complex number $f(z)$. For example, $f(z) = (2 + 2i)z$ is the function that inputs z and outputs the product $(2 + 2i)z$, and $f(z) = \frac{1}{z}$ is the function that inverts a (nonzero) complex number.

One basic technique we will use to study such functions is to consider their real and imaginary parts via

$$f(x + iy) = u(x, y) + iv(x, y).$$

Here, $u(x, y)$ and $v(x, y)$ are both *real-valued* functions of two variables, namely the variables giving the real and imaginary parts of $z = x + iy$; $u(x, y)$ gives the real part of $f(z)$ and $v(x, y)$ gives the imaginary part. For example,

$$f(z) = (2 + 2i)z = (2 + 2i)(x + iy) = (2x - 2y) + i(2y + 2x)$$

has real part $u(x, y) = 2x - 2y$ and imaginary part $v(x, y) = 2y + 2x$, and

$$f(z) = \frac{1}{z} = \frac{1}{x + iy}\left(\frac{x - iy}{x - iy}\right) = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}$$

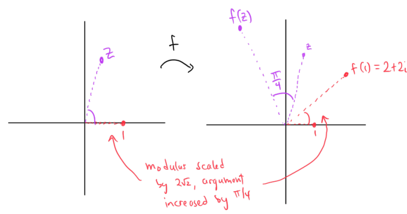
has real part $u(x, y) = x/(x^2 + y^2)$ and imaginary part $v(x, y) = -y/(x^2 + y^2)$.

Functions as transformations. In the usual single-variable real case, a function $f(x)$ can be analyzed via its graph, but this notion has no easy analog in the complex case since such a graph would live in four dimensions because we need two dimensions to keep track of the input $z = x + iy$ and two to keep track of the output $f(z) = u(z) + iv(z)$. Instead, we can gauge the geometric behavior of a complex function by viewing it as a *transformation* of the complex plane (or a region there within) to the complex plane.

For example, we consider the transformation defined by $f(z) = (2 + 2i)z$. In polar form with $2 + 2i = 2\sqrt{2}e^{i\pi/4}$, this transformation is

$$f(re^{i\theta}) = 2\sqrt{2}e^{i\pi/4}re^{i\theta} = (2\sqrt{2}r)e^{i(\frac{\pi}{4}+\theta)}.$$

Thus, f has the effect of scaling moduli by a factor of $2\sqrt{2}$ and increasing arguments by $\frac{\pi}{4}$, so geometrically this transformation is the composition of a scaling with a rotation:



If we had a constant term added on like $f(z) = (2 + 2i)z + 3i$, we would get the composition of a scaling, rotation, and translation since adding $3i$ to a complex number has effect of translating it. More general *linear* functions $f(z) = az + b$ with a and b complex and $a \neq 0$ can also be interpreted geometrically as combinations of scalings, rotations, and translations.

Example. Consider the function $f(z) = z^2$. Last time we saw the effect this has on a disk when viewed as a transformations, and now we determine the images of some other objects under f. Let us write f in terms of its real and imaginary parts as

$$f(z) = (x + iy)^2 = (x^2 - y^2) + i2xy.$$

We consider first the image of the line $x = 1$ under f. For points on the this line, the real part $u = x^2 - y^2$ of the output becomes $u = 1 - y^2$ and the imaginary part $v = 2xy$ becomes $v = 2y$, so (since $y = \frac{v}{2}$ by the second equation) we can see that the real and imaginary parts of such points are related by

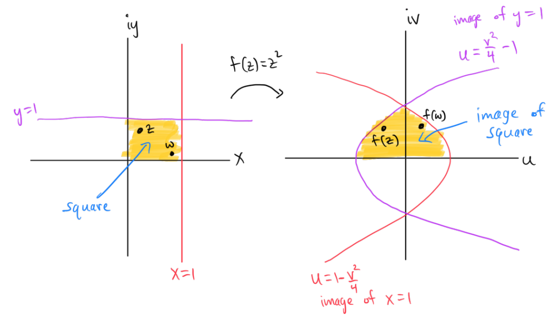
$$u = 1 - y^2 = 1 - \tfrac{1}{4}v^2.$$

The image of the line $x = 1$ is thus the parabola $u = 1 - \frac{1}{4}v^2$. (We'll draw this parabola below.)

For points on the line $y = 1$, the real and imaginary parts of $f = u + iv$ become $u = x^2 - 1$ and $v = 2x$, so $u + iv$ satisfies

$$u = x^2 - 1 = \tfrac{1}{4}x^2 - 1 \text{ since } x = \tfrac{v}{2}.$$

Thus the image of the line $y = 1$ is the parabola $u = \frac{1}{4}x^2 - 1$. Finally, since $f(z) = z^2$ doubles angles, points in the first quadrant get transformed into points in the first and second quadrants, so the image of the square in the first quadrant bounded by the lines $x = 1$ and $y = 1$ and the axes is the region in the first and second quadrants bounded by the parabolas $u = 1 - \frac{1}{4}v^2$ and $u = \frac{1}{4}v^2 - 1$. These images thus all looks like



Lecture 5: Complex Derivatives

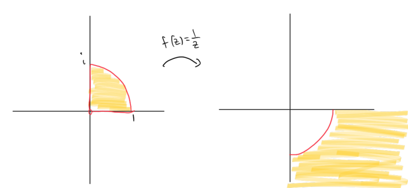
Warm-Up. We determine the image of the portion of the closed unit disk (but not including the origin) in the first quadrant under the inversion map $f(z) = \frac{1}{z}$, and the image of the portion of the line $y = x$ in the first quadrant that lies beyond the unit circle under this same map. The inversion map is given by

$$f(z) = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

The conjugation in the numerator has the effect of reflecting across the real axis and the denominator, since it is real and positive, has the effect of scaling. To be clear, if $z \neq 0$ satisfies $|z| \leq 1$, then $f(z)$ satisfies

$$|f(z)| = \frac{|\overline{z}|}{|z|^2} = \frac{|z|}{|z|^2} = \frac{1}{|z|} \geq 1,$$

where we use the fact that z and $\overline{z}$ have the same modules. This means that points within the unit circle are inverted to points lying outside the unit circle, so altogether we get that the image of the closed unit disk (origin excluded) in the first quadrant under inversion is



For the image of the desired portion of the line $y = x$ in the first quadrant, let us use the explicit real and imaginary parts of f:

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}.$$

If $z = x + ix$ is on the line $y = x$, we have that $f(z)$ looks like

$$f(x+ix) = \frac{x}{2x^2} + i\frac{-x}{2x^2} = \frac{1}{2x} + i\frac{-1}{2x}.$$

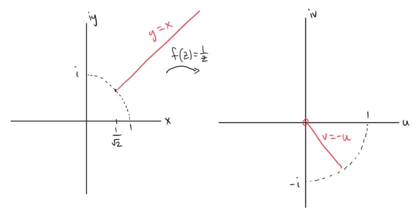
Thus, $f(z) = u + iv$ is on the line $v = -u$ since $u = \frac{1}{2x}$ and $v = -\frac{1}{2x}$ satisfy this equation. (This also makes sense from thinking of the conjugation in the numerator of $f(z) = \frac{\overline{z}}{|z|^2}$ as reflection across the real axis.) Moreover, if $z = x + ix$ lies beyond the unit circle, we have

$$\frac{1}{\sqrt{2}} \leq x$$

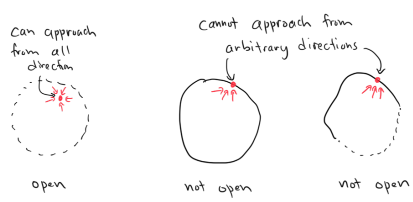
by considering the x-coordinate of the point where $y = x$ intersects the unit circle in the first quadrant, so after manipulating this inequality we get

$$\frac{1}{x} \leq \sqrt{2} \rightsquigarrow \frac{1}{2x} \leq \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}},$$

which says that the real part $u = \frac{1}{2x}$ of $f(x + ix)$ is never larger than that of the point where $u = -v$ intersects the unit circle in the fourth quadrant. Thus, the desired image is

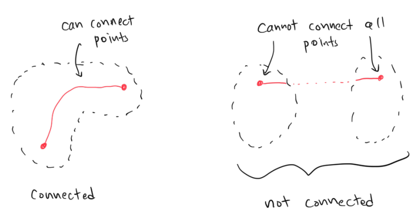


Domains. At times we will need to be clear about the types of regions in the complex plane we are considering, and in particular the types of regions that will serve as valid domains of differentiable functions. First, we will typically want such regions to be *open*, which means that they include no portion of their boundary:



For open regions, the point is that any z within it is "surrounded" from all possible directions by points still within that region, which will be a desired property when taking limits. For non-open regions, we cannot guarantee that we can approach points in that region from arbitrary directions while still remaining within that region.

Second, we will typically want our regions to be *connected*, which visually means that they consist of a single "piece":



For open connected regions, what essentially matters is that any two points within it can be connected by a continuous path, but this is not true for non-connected regions since a continuous path cannot "jump" passed a gap between the pieces making up the region. Going forward then, we will use the term *domain* to refer to an open connected region of the complex plane.

Limits. For a function f defined on a domain, we can thus make sense of limits like

$$\lim_{z \to z_0} f(z).$$

The intuition is the same as it is for every other limit you have seen in your life: the limit above exists and has value w if the complex number $f(z)$ gets closer and closer to w as z gets closer and closer to z_0. As with the 2-dimensional limits you saw in a multivariable calculus course, in order for this limit to exist the value should not depend on *how* we approach z_0, meaning that the limit along any possible direction towards z_0 would necessarily have to result in the same limiting value. (As mentioned before, being able to consider arbitrary directions near z_0 in this way is why we take our domain to be open.)

For example, let us compute the limit

$$\lim_{z \to 2+i} (z^2 + iz - 1).$$

The point here is that the the function $f(z) = z^2 + iz - 1$ is *continuous*, which means that values of limits are simply the values of the function at point being approached. So

$$\begin{aligned}\lim_{z \to 2+i} (z^2 + iz - 1) &= (2+i)^2 + i(2+i) - 1 \\ &= 3 + 4i + 2i - 1 - 1 \\ &= 1 + 6i.\end{aligned}$$

(Continuity of z^2+iz-1 can be determined, if nothing else, from considering the real and imaginary parts: $z^2 + iz - 1 = (x+iy)^2 + i(x+iy) - 1 = (x^2 - y^2 - y - 1) + i(2xy + x)$, so since the real and imaginary parts are continuous in the sense of a multivariable calculus course, $f(z) = z^2 + iz - 1$ is continuous as well.)

Derivatives. We are now ready to give the main definition for the entire course and begin to study its properties. For a function f defined on some domain D, we say that f is *complex differentiable* at z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If so, we call the value of this limit the *complex derivative* of f at z_0 and denote it by $f'(z_0)$.

So, the definition of what it means for a complex function to be differentiable uses the exact same limit expression as what you would have seen for a usual single-variable derivative in a first calculus course. In a sense, this is giving us the "(complex) rate of change" of f at z_0. We will note, however, that we cannot really interpret this as a "slope" in the complex setting, and we will come back to the proper geometric interpretation a bit later. Even though this definition is the same as the real one, we will see—which is the entire point of this course!—that in the complex case this definition places severe restrictions on the behavior of functions which satisfy it.

For one final piece of terminology, when f is (complex) differentiable at all points of a domain D, we say that f is *holomorphic* on D. We could also simply say that f is differentiable on D, but "holomorphic" is a more modern term motivated by the fact that we will often want to distinguish between being differentiable at only a specific point versus on a whole collection of points. Note that the book only briefly mentions the word "holomorphic", and instead prefers to say that f is *complex analytic* on D when it is differentiable at all points of D. This is common choice of terminology as well, but we prefer to use holomorphic because, technically, "analytic" refers to a different property we will get to in due time. The point (which is a major result of complex analysis) is that "holomorphic" as we have defined it and "complex analytic" as we will eventually define it end up being equivalent, so the book loses no generality in using the term analytic from the get-go. But, we prefer here to reserve analytic for this latter property we will describe and use holomorphic at the start since, as we have said, the fact that holomorphic functions are analytic is a BIG THING. (For a contrast, in the real case, it is very very very far from true that real differentiable functions are always real analytic!)

Examples. For n a positive integer, the function $f(z) = z^n$ is differentiable at all points—hence holomorphic on $\mathbb{C}$, the entire complex plane—with derivative, as you might guess, given by $f'(z) = nz^{n-1}$. Let us go through this argument which, since the limit defining the complex derivative is just analogous to the one defining the real derivative, is the same as the one you might have seen before for $f(x) = x^n$ in the real case.

For fixed z_0 in $\mathbb{C}$, we must compute the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0}.$$

The point is that we can factor the numerator as

$$z^n - z_0^n = (z - z_0)(z^{n-1} + z^{n-2}z_0 + \cdots + zz_0^{n-2} + z_0^{n-1}),$$

where in the final expression we decrease the power of z and increase the power of z_0 as we go. Multiplying out the right-hand side will give an initial $zz^{n-1} = z^n$, a bunch of intermediate terms that all cancel out, and a final $-z_0z_0^{n-1} = -z_0^n$, which is why we get the left side. (The intermediate terms all cancel out since each will appear twice, only with opposite signs. For example, if we take z from the first set of parentheses times $z^{n-2}z_0$ from the second we get $z^{n-1}z_0$ overall, but this cancels with what we get from taking $-z_0$ in teh first parentheses times z^{n-1} in the second.) Thus

$$\begin{aligned}\lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} &= \lim_{z \to z_0} \frac{(z - z_0)(z^{n-1} + z^{n-2}z_0 + \cdots + zz_0^{n-2} + z_0^{n-1})}{z - z_0} \\ &= \lim_{z \to z_0} (z^{n-1} + z^{n-2}z_0 + \cdots + zz_0^{n-2} + z_0^{n-1}).\end{aligned}$$

What remains is continuous, so the limit is obtained by setting $z = z_0$, and there are n terms overall, so $f'(z_0)$ exists and equals

$$f'(z_0) = z_0^{n-1} + z_0^{n-2}z_0 + \cdots + z_0z_0^{n-2} + z_0^{n-1} = nz_0^{n-1}$$

as claimed.

For another example, we claim that $f(z) = \frac{1}{z}$ is holomorphic on the set of nonzero complex numbers, which call the *punctured* complex plane (punctured because it is the plane with the origin removed) and denote by $\mathbb{C}^*$. For a nonzero z_0 in $\mathbb{C}^*$, we must compute

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\frac{1}{z} - \frac{1}{z_0}}{z - z_0}.$$

Some algebraic manipulation gives

$$\frac{\frac{1}{z} - \frac{1}{z_0}}{z - z_0} = \frac{\frac{z_0 - z}{zz_0}}{z - z_0} = -\frac{1}{zz_0}.$$

This resulting expression is continuous at $z_0 \neq 0$, so the limit defining $f'(z_0)$ exists and equals

$$f'(z_0) = \lim_{z \to z_0} \frac{\frac{1}{z} - \frac{1}{z_0}}{z - z_0} = \lim_{z \to z_0} -\frac{1}{zz_0} = -\frac{1}{z_0^2},$$

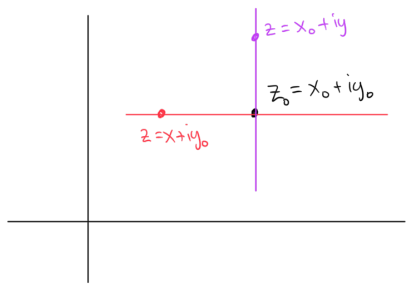
just as you might have expected for the derivative of $\frac{1}{z}$.

Non-example. The conjugation function $f(z) = \overline{z}$ is a fairly simple one that is easy to interpret geometrically, but we claim that in fact there are no points in $\mathbb{C}$ at which this is differentiable in the way we have defined. It might seem surprising at first that such a simple function is nowhere differentiable, but this example starts to hint at some of the restrictions that differentiability forces. After we discuss what differentiability means geometrically, we will be able to see more intuitively what the problem is.

In order for $f(z) = \overline{z}$ to be differentiable at z_0, we would need the limit

$$\lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0}$$

to exist. For this to exist, the value would have to be the same regardless of how we approach z_0, so we will consider the limiting values first when approaching z_0 along the horizontal direction and then along the vertical direction:



Approaching along the horizontal direction means that we consider points of the form $z = x + iy_0$ with the same imaginary part as $z_0 = x_0 + iy_0$ and x approaching x_0. For such points, we have

$$\lim_{z \to z_0} \frac{\overline{x + iy_0} - \overline{x_0 + iy_0}}{(x + iy_0) - (x_0 + iy_0)} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$$

since all terms involving y_0 cancel out. However, when approaching $z_0 = x_0 + iy_0$ vertically, meaning along points of the form $z = x_0 + iy$, we have

$$\lim_{z \to z_0} \frac{\overline{x_0 + iy} - \overline{x_0 + iy_0}}{(x_0 + iy) - (x_0 + iy_0)} = \lim_{y \to y_0} \frac{-iy - (-iy_0)}{iy - iy_0} = \lim_{y \to y_0} -\frac{i(y - y_0)}{i(y - y_0)} = -1.$$

Since the horizontal and vertical limits are different, we conclude that

$$\lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0}$$

does not exist, so f is not differentiable at z_0, and hence nowhere differentiable on $\mathbb{C}$.

Lecture 6: Cauchy-Riemann Equations

Warm-Up 1. We determine the points at which the function

$$f(z) = z\overline{z}$$

is complex differentiable. (Note that this just the modulus-squared function $f(z) = |z|^2$.) Differentiability at z_0 requires the existence of

$$\lim_{z \to z_0} \frac{z\overline{z} - z_0\overline{z_0}}{z - z_0}.$$

Now, first note that this limit does exist at $z_0 = 0$:

$$\lim_{z\to 0} \frac{z\overline{z} - 0}{z - 0} = \lim_{z\to 0} \overline{z} = 0$$

since conjugation is continuous. Thus f is differentiable at 0 at least, with $f'(0) = 0$.

But we claim that f is not differentiable at any $z_0 \neq 0$. To justify this, we consider the same horizontal vs vertical approach as in the conjugation example from last time. Approaching $z_0 = x_0 + iy_0$ horizontally among points of the form $z = x + iy_0$ gives

$$\begin{aligned}
\lim_{z\to z_0} \frac{(x+iy_0)(x-iy_0) - (x_0+iy_0)(x_0-iy_0)}{(x+iy_0)-(x_0+iy_0)} &= \lim_{x\to x_0} \frac{x^2+y_0^2-(x_0^2+y_0^2)}{x-x_0} \\
&= \lim_{x\to x_0} \frac{x^2-x_0^2}{x-x_0} \\
&= \lim_{x\to x_0} (x+x_0) = 2x_0.
\end{aligned}$$

Instead, approaching vertically among points of the form $z = x_0 + iy$ gives

$$\begin{aligned}
\lim_{z\to z_0} \frac{(x_0+iy)(x_0-iy) - (x_0+iy_0)(x_0+iy_0)}{(x_0+iy)-(x_0+iy_0)} &= \lim_{y\to y_0} \frac{x_0^2+y^2-(x_0^2+y_0^2)}{i(y-y_0)} \\
&= \lim_{y\to y_0} \frac{y^2-y_0^2}{i(y-y_0)} \\
&= \lim_{y\to y_0} -i(y+y_0) = -i2y_0.
\end{aligned}$$

(Note we use $\frac{1}{i} = -i$ in the final step.) The only way these horizontal and vertical limits agree is if

$$x_0 = -iy_0,$$

but if x_0, y_0 are both real this only happens when $x_0 = y_0 = 0$. Thus for $z_0 = x_0 + iy_0 \neq 0$, these limits are not the same, so f is not differentiable at any nonzero point.

Warm-Up 2. We determine the points at which the function

$$g(z) = z^2\overline{z}$$

is differentiable. This can be done using similar, but a bit more involved, computations as in the first Warm-Up, but instead we exploit basic properties of derivatives to avoid the extra work; namely, we use the fact that the usual *product* and *quotient rules* hold in the complex setting as well. (The *chain rule* is also valid!) Now, the product and quotient rules are just statements about how to compute derivatives of products and quotients, they are also statements about the existence of such derivatives as well. In our case, if we think about g as

$$g(z) = z^2\overline{z} = z(z\overline{z}) = zf(z)$$

where $f(z) = z\overline{z}$ is the function from the first Warm-Up, then since z is differentiable at 0 and f is as well (by the first Warm-Up), we get immediately that the product $g(z) = zf(z)$ is differentiable at 0 by the product rule.

If g were differentiable at a nonzero z_0, then

$$\frac{g(z)}{z}$$

would be as well by the quotient rule since the numerator and nonzero denominator would both be differentiable at z_0. But this quotient is

$$\frac{g(z)}{z} = \frac{z^2\overline{z}}{z} = z\overline{z} = f(z),$$

which we know from the first Warm-Up is not differentiable at z_0. Hence g could not have been differentiable at $z_0 \neq 0$ either, so $g(z) = z^2\overline{z} = z|z|^2$ is differentiable at 0 and nowhere else.

Horizontal vs vertical in general. The idea of considering horizontal and vertical limits when checking differentiability is a crucial one, so let us work out the details in the most general setting. We write a general function f in terms of its real and imaginary parts as

$$f(z) = u(x, y) + iv(x, y).$$

Approaching $z_0 = x_0 + iy_0$ horizontally (i.e. setting $y = y_0$) in the limit defining differentiability of f at z_0 gives

$$\begin{aligned}\lim_{\substack{z\to z_0\\ \textit{horizontally}}} \frac{f(z)-f(z_0)}{z-z_0} &= \lim_{x\to x_0} \frac{[u(x,y_0)+iv(x,y_0)]-[u(x_0,y_0)+iv(x_0,y_0)]}{(x+iy_0)-(x_0+iy_0)} \\ &= \lim_{x\to x_0} \frac{[u(x,y_0)-u(x_0,y_0)]+i[v(x,y_0)-v(x_0,y_0)]}{x-x_0} \\ &= \lim_{x\to x_0} \left(\frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}\right),\end{aligned}$$

where in the second step we grouped real and imaginary parts together, and in the final step we split up the fraction. Now, the first fraction

$$\frac{u(x,y_0)-u(x_0,y_0)}{x-x_0}$$

in what remains measures the change in the values of the real function $u(x, y)$ but only with respect to varying x-coordinates since the y-coordinates are fixed at y_0 throughout; as $x \to x_0$, the resulting limit is precisely what you would have called the *partial derivative* of u with respect to x at (x_0, y_0) in a multivariable calculus course and denoted by $u_x(x_0, y_0)$. Similarly, the second fraction

$$\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}$$

in the limit above measures the change in $v(x, y)$ with respect to x, so the limit of this expression gives the partial derivative $v_x(x_0, y_0)$ of v with respect to x at (x_0, y_0). Thus, in the horizontal direction we get that

$$\begin{aligned}\lim_{\substack{z\to z_0\\ \textit{horizontally}}} \frac{f(z)-f(z_0)}{z-z_0} &= \lim_{x\to x_0} \left(\frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}\right) \\ &= u_x(x_0,y_0) + iv_x(x_0,y_0).\end{aligned}$$

If nothing else, if f is indeed differentiable at z_0, then this particular limit should give the value of $f'(z_0)$, so we get an explicit expression for the complex derivative in terms of real partial derivatives with respect to x:

$$f'(z_0) \text{ exists} \implies f'(z_0) = u_x(z_0) + iv_x(z_0).$$

Now we approach vertically. We have

$$\begin{aligned}\lim_{\substack{z\to z_0\\ \textit{vertically}}} \frac{f(z)-f(z_0)}{z-z_0} &= \lim_{y\to y_0} \frac{[u(x_0,y)+iv(x_0,y)]-[u(x_0,y_0)+iv(x_0,y_0)]}{(x_0+iy)-(x_0+iy_0)} \\ &= \lim_{y\to y_0} \frac{[u(x_0,y)-u(x_0,y_0)]+i[v(x_0,y)-v(x_0,y_0)]}{i(y-y_0)} \\ &= \lim_{y\to y_0} \left(\frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)} + \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0} \right).\end{aligned}$$

(Note the location of i throughout.) The remaining fractions measure the change in u, v with respect to y now, so the limits of these as $y \to y_0$ are partial derivatives with respect to y. Thus

$$\lim_{\substack{z\to z_0\\ \textit{vertically}}} \frac{f(z)-f(z_0)}{z-z_0} = \tfrac{1}{i}u_y(x_0,y_0)+v_y(x_0,y_0) = v_y(x_0,y_0)-iu_y(x_0,y_0),$$

where we use $\frac{1}{i} = -i$ at the end. If f is differentiable at z_0, then these horizontal and vertical limits we have computed must be the same, so we get as a consequence that

$$u_x(z_0)+iv_x(z_0) = v_y(z_0)-iu_y(z_0)$$

and that this common value is the derivative $f'(z_0)$.

Cauchy-Riemann equations. By comparing real and imaginary parts in the identity derived above, we get that the real and imaginary parts of $f = u + iv$ must satisfy the pair of equations

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

at any point at which f is differentiable. These are what are known as the *Cauchy-Riemann equations*, and, as we will see, place restrictions on the behaviors of holomorphic functions. If these equations are not satisfied at some z_0, then $f = u + iv$ will absolutely not be differentiable at z_0. (We will come back to the question as to whether satisfying the Cauchy-Riemann equations is enough to guarantee that $f = u + iv$ *is* differentiable next time!)

For example, the conjugation function $f(z) = \overline{z} = x - iy$ has real part $u(x,y) = x$ and imaginary part $v(x,y) = -y$. For these we have

$$u_x = 1 \text{ and } v_y = -1, \text{ so } u_x \neq v_y \text{ at all points.}$$

This means that $f(z) = \overline{z}$ cannot be differentiable anywhere, as we already knew. (Note that the second Cauchy-Riemann equation *is* satisfied in this case because u_y and v_x are both zero, but failure of one Cauchy-Riemann equation alone is enough to guarantee non-differentiability.)

Examples. For the function $f(z) = z\overline{z} = |z|^2 = x^2 + y^2$ from the first Warm-Up, we have $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. Thus the Cauchy-Riemann equations become

$$\begin{aligned} 2x &= u_x = v_y = 0 \\ 2y &= u_y = -v_x = 0. \end{aligned}$$

If $z \neq 0$, then at least one of these equations is not satisfied, so f is not differentiable at $z \neq 0$, just as we saw in the Warm-Up. The fact that the Cauchy-Riemann equations *are* satisfied at $z = 0 + 0i$

does guarantee differentiability at 0 (again, we will see why next time), which also agrees with what we saw in the Warm-Up.

The function from the second Warm-Up is

$$g(z) = z^2\overline{z} = (x+iy)^2(x-iy) = ([x^2-y^2]+i2xy)(x-iy) = (x^3+xy^2)+i(x^2y+y^3).$$

For $u(x,y) = x^3+xy^2$ and $v(x,y) = x^2y+y^3$, the Cauchy-Riemann equations become

$$\begin{gathered} 3x^2+y^2 = x^2+3y^2 \quad (\text{this is } u_x = v_y) \\ 2xy = -2xy \quad (\text{this is } u_y = -v_x). \end{gathered}$$

The second equation implies that $x=0$ or $y=0$, but then the first implies that the other variable must be zero as well: if $x=0$, the first equation gives $y^2 = 3y^2$, so $y=0$, while if $y=0$ the first equation is $3x^2 = x^2$, so $x=0$. Thus, the Cauchy-Riemann equations are satisfied only at $z=0$, which is why we can only expect $g(z) = z^2\overline{z}$ to be differentiable at 0, also agreeing with what we saw in the Warm-Up.

Derivative zero implies constant. Finally, we use the Cauchy-Riemann equations to justify a basic fact you might expect to be true, namely that if f is holomorphic on a domain D and $f' = 0$ at all points in D, then f must be constant in D. Using the expression $f' = u_x + iv_x$ for the derivative, the condition that the derivative is zero everywhere gives

$$u_x = 0 \quad \text{and} \quad v_x = 0.$$

But then the Cauchy-Riemann equations give

$$u_y = -v_x = 0 \quad \text{and} \quad v_y = u_x = 0$$

as well. Thus $u(x,y)$ has both partial derivatives equaling zero everywhere, so u is constant, and $v(x,y)$ has both partial derivatives equaling zero everywhere, so v is constant, and therefore $f = u+iv$ is constant as well. The upshot is that the Cauchy-Riemann equations here give a way to turn information about some of the partial derivatives into information about all of them.

Note that it is important here that D be a domain, specifically that it be connected. To be precise, the conclusions that

$$u_x = 0 = u_y \implies u \text{ is constant}$$

and similarly for v do not hold if the region in question is not connected, since in that case we could have f equal one constant over one piece of the non-connected region but a different constant over the other piece, so that f would be not be (the same) constant throughout the entire region.

Lecture 7: More on Derivatives

Warm-Up 1. We verify that $f(z) = z^3$ satisfies the Cauchy-Riemann equations at all points and that $f' = u_x + iv_x$ does give the correct expression for $f'(z) = 3z^2$. We have

$$f(z) = (x+iy)(x+iy)(x+iy) = ([x^2-y^2]+i2xy)(x+iy) = (x^3-3xy^2)+i(3x^2y-y^3).$$

Thus

$$u_x = 3x^2-3y^2 \quad \text{and} \quad v_y = 3x^2-3y^2,$$

so the first Cauchy-Riemann equation $u_x = v_y$ holds, and

$$u_y = -6xy \quad \text{and} \quad v_x = 6xy,$$

so the second Cauchy-Riemann equation $u_y = -v_x$ holds. Moreover,

$$u_x + iv_x = (3x^2 - 3y^2) + i6xy = 3[(x^2 - y^2) + i2xy].$$

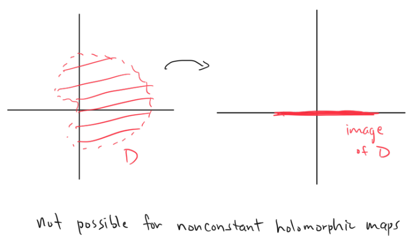
The expression in brackets is precisely $z^2 = (x+iy)(x+iy)$, so we indeed have $f'(z) = 3z^2 = u_x+iv_x$.

Warm-Up 2. We show that if f is holomorphic and real-valued on a domain D, then f is constant. To be real-valued means that $f(z)$ is always a real number, so the imaginary part of $f = u + iv$ is the constant zero function. Hence $v_x = 0$ and $v_y = 0$. But the Cauchy-Riemann equations then give

$$u_x = v_y = 0 \quad \text{and} \quad u_y = -v_x = 0,$$

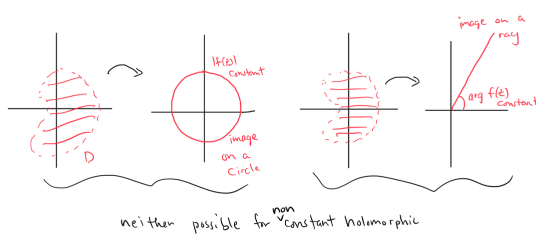
so u must be constant as well, and hence $f = u + iv = u$ is constant.

Geometrically, this says that the image of a domain D under a nonconstant holomorphic function can never lie completely on the real axis:



In a similar way, if a holomorphic function f is purely imaginary valued (so instead $u = 0$ in $f = u+iv$), then it too must be constant, so the image of D under a nonconstant holomorphic map can never lie fully on the imaginary access. View these types of results as restricting the geometric behaviors of holomorphic maps, or least ruling out certain types of behaviors.

On the homework you will show that having constant modulus also forces a holomorphic map to be constant, as does having constant argument. So, we cannot have something like



In general, nonconstant holomorphic maps can never "collapse" 2-dimensional regions onto 1-dimensional curves.

Linear approximations. In the real case, derivatives give slopes of tangent lines to graphs, but this notion has no easy analog in the complex setting, in particular because the idea of a "graph" is itself elusive. So, if we want to come up with a geometric interpretation of the derivative we need to look elsewhere. The answer comes from viewing the tangent line expression not as a geometric line but rather as the function that gives the best linear approximation to $f(x)$ near a:

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

To say that this the "best" linear approximation to f near a is the claim that the "error" (also called the "remainder") in this approximation $\epsilon(x) = f(x) - [f(a) + f'(a)(x-a)]$ goes to 0 as $x \to a$ more rapidly than does the distance between x and a in the sense that

$$\lim_{x \to a} \frac{\epsilon(x)}{x-a} = 0.$$

This all makes perfect sense in the complex case as well. If $f(z)$ is differentiable at z_0, we get a linear approximation

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0) \text{ for } z \text{ near } z_0,$$

which is "best" in the sense that the error $\epsilon(z) = f(z) - [f(z_0) + f'(z_0)(z - z_0)]$ satisfies

$$\lim_{z \to z_0} \frac{\epsilon(z)}{z - z_0}.$$

Note that $f(z_0) + f'(z_0)(z - z_0)$ is "linear" because the only variable dependence is in the z^1 term; everything else, including the coefficient $f'(z_0)$ of this z term, is just a constant complex number. This says that "infinitesimally" near z_0, f should behave like the linear function $f(z_0)+f'(z_0)(z-z_0)$.

But such a linear function has a nice geometric interpretation as a transformation, at least in the case when $f'(z_0) \neq 0$: the map that sends z to $f(z_0) + f'(z_0)(z - z_0)$ is the composition of a translation with scaling by a factor of $|f'(z_0)|$ and rotation by the angle $\arg f'(z_0)$, which comes from the fact that multiplying z by $f'(z_0) = |f'(z_0)|e^{i(\arg f'(z_0))}$ indeed has this geometric effect. Thus, to say that f is differentiable at z_0 is to say that f behaves roughly like a scaling/rotation near z_0, at least when $f'(z_0) \neq 0$. The derivative $f'(z_0) \neq 0$ is then not a slope, but instead what characterizes the scaling factor and rotation amount.

An aside on Jacobian matrices. (This is not something we mentioned in class, and I only include here to provide context for those who have seen Jacobian matrices before. You will not be responsible for understanding this notion in this course.) By ignoring anything have to do with complex notation, a complex functions $f(z) = u(z) + iv(z)$ can be thought of simply as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ via

$$f(x, y) = (u(x, y), v(x, y)).$$

In multivariable calculus (covered in some courses but not others), the behavior of f can be approximated by its *Jacobian matrix*, which is the matrix Df encoding the partial derivatives of the components of f:

$$Df(x, y) = \begin{bmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{bmatrix}.$$

The Cauchy-Riemann equations thus allow us to write this Jacobian matrix as

$$Df = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}.$$

You might have a similar type of matrix in a linear algebra course, namely

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

which are the matrices that describe rotations of the plane. The Jacobian matrix Df above has determinant $u_x^2 + v_x^2$, and by factoring out the scalar $\sqrt{u_x^2 + v_x^2}$ we can write this matrix as

$$Df = \sqrt{u_x^2 + v_x^2} \begin{bmatrix} u_x/\sqrt{u_x^2 + v_x^2} & -v_x/\sqrt{u_x^2 + v_x^2} \\ v_x/\sqrt{u_x^2 + v_x^2} & u_x/\sqrt{u_x^2 + v_x^2} \end{bmatrix}.$$

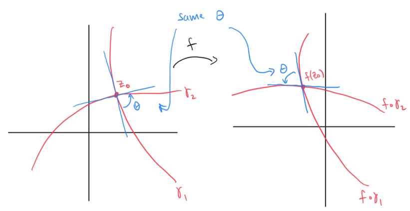
The entires in this matrix turn out to be precisely the cosine and sine values of the argument of $f' = u_x + iv_x$, and $\sqrt{u_x^2 + v_x^2}$ is the modulus of f', so this Jacobian matrix is

$$Df = |f'| \begin{bmatrix} \cos(\arg f') & -\sin(\arg f') \\ \sin(\arg f') & \cos(\arg f') \end{bmatrix},$$

which is *precisely* in the form of scaling by a factor of $|f'|$ and rotating by $\arg f'$. The upshot is that the Cauchy-Riemann equations really amount to saying that Df, or f', should indeed be a scaling/rotation type of transformation.

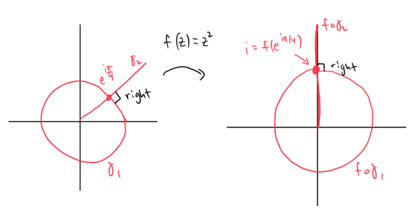
Conformality. Back to course material. We can in fact be more precise in the idea that differentiable functions behave infinitesimally like scaling rotations. A crucial property that scalings and rotations have is that they preserve angles between curves, in that the angle between two curves intersecting at a point is the same as the angle between their images at their point of intersection under a rotation or a scaling. If the existence of $f'(z_0) \neq 0$ is meant to say that f behaves like a scaling/rotation near z_0, we would expect f to have this angle-preservation property as well.

To be clear, take two curves γ_1 and γ_2 in the complex plane intersecting at z_0. The angle between the two curves at z_0 is defined to be the angle between their tangent lines at z_0. (We will say more about complex curves and their tangents—including existence of—next time. They will also play an important role in integration.) Applying f to all points of γ_i gives the image curve $f \circ \gamma_i$, and the claim is that the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z_0)$ is the same as that between γ_1 and γ_2 at z_0:

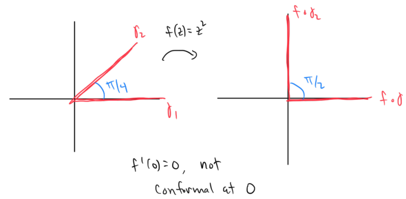


A map which preserves angles in this way is said to be *conformal*, so the result is that holomorphic functions with nonzero derivatives are conformal. The converse is also true: conformal maps are necessarily holomorphic with nonzero derivatives. The notion of complex differentiability is thus indeed a very geometric one, that amounts to encoding information about how angles behave.

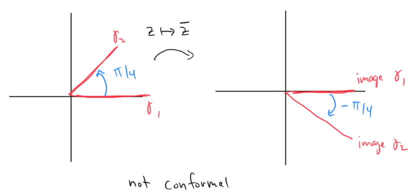
Examples. Consider $f(z) = z^2$. This has nonzero derivative $f'(z) = 2z$ at, say, $e^{i\pi/4}$, so we expect f to be conformal at this point. Indeed, the unit circle and the ray $y = x$ in the first quadrant intersect at a right angle at $e^{i\pi/4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$. The image of the unit circle under f is still the unit circle, and the image of the ray is the positive imaginary axes since the argument $\frac{\pi}{4}$ along the ray doubles to $\frac{\pi}{2}$. The angle between these two images at $f(e^{i\pi/4}) = e^{i\pi/2} = i$ is also $\frac{\pi}{2}$, so this angle was indeed preserved:



We cannot expect conformality (i.e., angle-preservation) at points where $f'(z) = 2z$ is zero, so at $z = 0$. Indeed, the positive x-axis and ray $y = x$ from above intersect at an angle $\frac{\pi}{/}4$ at 0, but their images under $f(z) = z^2$, which are the positive x-axis and positive imaginary axis respectively, intersect at an angle $\frac{\pi}{2}$, so this angle was not preserved.



The conjugation function $z \mapsto \overline{z}$ is nowhere differentiable, and so should be nowhere conformal. Geometrically this map is a reflection, and reflections do not preserve angles but rather flip (i.e., change the sign of) angles. The angle $\frac{\pi}{4}$ between the positive x-axis and ray from before becomes $-\frac{\pi}{4}$ after reflecting across the real axis, so conjugation is not conformal:



(It is important that the literal angle, sign and all, be preserved in the definition of conformal, not just the absolute value of the angle.)

Cauchy-Riemann and differentiability. We will do more with developing our geometric intuition for complex differentiability as we go, but for now we finish with clarifying the extent to which the Cauchy-Riemann equations guarantee differentiability. The precise claim we make is that if $f = u + iv$ satisfies the Cauchy-Riemann equations at z_0, *and* the partial derivatives of u and v are continuous at z_0, then f is indeed complex differentiable at z_0. So, under the mild assumption of continuity of u_x, u_y, v_x, v_y, Cauchy-Riemann is enough to determine existence of the derivative.

In fact, we will see later that being holomorphic (note, differentiable on an entire domain, not just at a single point) always guarantees continuity of the partial derivatives above, so we are not actually losing anything by assuming this continuity from the get-go. What continuity of these partial derivatives gives is the following. The function $u(x, y)$ is real-valued of two variables, and from this we get a "tangent plane", or better yet *linear*, approximation via

$$u(z) \approx u(z_0) + u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) \text{ for } z \text{ near } z_0$$

where $z = x + iy$ and $z_0 = x_0 + iy_0$. Continuity of u_x and u_y guarantees that this is a "good" approximation in the sense that the error

$$\epsilon_1(z) = u(z) - [u(z_0) + u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0)] \quad \text{satisfies} \quad \lim_{z \to z_0} \frac{\epsilon_1(z)}{z - z_0} = 0.$$

(If you have not seen this notion before, no big deal as it is not one we will work with heavily and you can just take it for granted. If you have seen this notion before—depending on which multivariable calculus course you took or whether you have taken a course in higher-dimensional real analysis—you might recognize the statement above as what it actually means for the function 2-variable function $u(x, y)$ to be real differentiable at (x_0, y_0), and the fact we are using here is that continuity of partial derivatives implies real differentiability. The usual limit definition of real differentiability in this sense actually uses $|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ in the denominator instead of just $z - z_0$, but the limit of a complex expression is zero if and only if the limit of its modulus is zero, which is why we can get away with using $z - z_0$ rather than its modulus in the denominator.) Similarly, continuity of v_x and v_y guarantees that the error

$$\epsilon_2(z) = v(z) - [v(z_0) + v_x(z_0)(x - x_0) + v_y(z_0)(y - y_0)]$$

in the linear approximation to $v(x, y)$ near (x_0, y_0) satisfies

$$\lim_{z \to z_0} \frac{\epsilon_2(z)}{z - z_0} = 0.$$

With this setup, we can now justify the fact that Cauchy-Riemann plus continuity of partials implies existence of the complex derivative. For $f = u + iv$ and $z_0 = x_0 + iy_0$, we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{[u(z) + iv(z)] - [u(z_0) + iv(z_0)}{z - z_0} = \frac{[u(z) - u(z_0)] + i[v(z) - v(z_0)]}{z - z_0}.$$

The error terms introduced above allow us to write

$$\begin{aligned} u(z) - u(z_0) &= u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) + \epsilon_1(z), \text{ and} \\ v(z) - v(z_0) &= v_x(z_0)(x - x_0) + v_y(z_0)(y - y_0) + \epsilon_2(z). \end{aligned}$$

Plugging these into our original quotient $\frac{f(z)-f(z_0)}{z-z_0}$ expression above turns it into

$$\frac{[u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) + \epsilon_1(z)] + i[v_x(z_0)(x - x_0) + v_y(z_0)(y - y_0) + \epsilon_2(z)]}{z - z_0}.$$

Now, here's the magic: the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ further turn this into

$$\frac{[u_x(z_0)(x - x_0) - v_x(z_0)(y - y_0) + \epsilon_1(z)] + i[v_x(z_0)(x - x_0) + u_x(z_0)(y - y_0) + \epsilon_2(z)]}{z - z_0}.$$

Grouping together the u_x terms and the v_x terms results in

$$\frac{u_x(z_0)[(x - x_0) + i(y - y_0)] + iv_x(z_0)[(x - x_0) + i(y - y_0)] + \epsilon_1(z) + i\epsilon_2(z)}{z - z_0}.$$

(There is some care to be taken in checking that the i's are all correct here, but they are!) The expression $(x - x_0) + i(y - y_0)$ is nothing but $z - z_0$, so we get

$$\frac{u_x(z_0)(z - z_0) + iv_x(z_0)(z - z_0) + \epsilon_1(z) + i\epsilon_2(z)}{z - z_0} = u_x(z_0) + iv_x(z_0) + \frac{\epsilon_1(z)}{z - z_0} + i\frac{\epsilon_2(z)}{z - z_0}.$$

Thus, the limit defining $f'(z_0)$ is

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \left(u_x(z_0) + iv_x(z_0) + \frac{\epsilon_1(z)}{z - z_0} + i\frac{\epsilon_2(z)}{z - z_0} \right).$$

The first two terms on the right are constants with respect to z, so they remain as is after taking the limit, and the remaining terms go to zero by the "good approximation" properties on the errors, so we get

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = u_x(z_0) + iv_x(z_0).$$

Thus, not only does $f'(z_0)$ exist but it equals $u_x(z_0) + iv_x(z_0)$, just as we would expect from our previous discussions. Huzzah!

Note that, in this case, existence of the 2-dimensional limit $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ *does* end up being completely determined solely from its existence along two directions (horizontal and vertical) alone, which is in stark contrast to what happens for usual 2-dimensional real limits, where checking two directions alone is never enough. This is also a reflection of the idea being developed that complex differentiability places big restrictions on complex behaviors, much more than what we might expect in the real setting.

Lecture 8: Exponentials and Trig

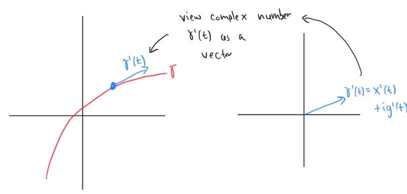
Warm-Up. Last time we hinted at the fact that functions with nonzero derivative at a point are conformal (i.e., angle-preserving) at that point, and now we justify this formally. This requires some background on complex curves, which we introduce here. A *curve* γ in $\mathbb{C}$ is given by parametric equations

$$\gamma(t) = x(t) + iy(t)$$

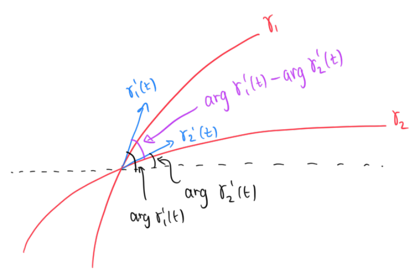
where $x(t), y(t)$ are real-valued functions. Note that this is just analogous to parametric equations $(x(t), y(t))$ you would have seen for curves in $\mathbb{R}^2$ in a multivariable calculus course, only that now we are describing points in the plane using complex notation. We say that a curve is *smooth* at $z_0 = \gamma(t_0)$ if the derivative

$$\gamma'(t) = x'(t) + iy'(t)$$

exists and is nonzero at $t = t_0$. In multivariable calculus such derivatives $(x'(t), y'(t))$ describe *tangent vectors*, which is thus the interpretation we give to the complex number $\gamma'(t)$ as well:



To be smooth just means that the tangent vector is nonzero, which is a desired property since otherwise we could not reasonably make sense of the notion of "angle". In the smooth case, the angle a curve makes with the horizontal direction at point is then the argument of the tangent vector $\gamma'(t)$, and we thus define the *angle of intersection* between two smooth curves at a point of intersection as the difference in the arguments of their tangent vectors at this point:



So, suppose $f'(z_0) \neq 0$ and that γ_1, γ_2 are two smooth curves intersecting at $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$. (If the point of intersection z_0 occurred at different values of the parameter t along the two curves, we can always reparametrize one to get a common value of t.) The image of γ_1 under f has parametrized form $f(\gamma_1(t))$, so the chain rule gives

$$\frac{d}{dt}\Big|_{t=t_0} f(\gamma_1(t)) = f'(\gamma_1(t_0))\gamma_1'(t_0) = f'(z_0)\gamma_1'(t_0)$$

as the tangent vector to the image at $f(z_0)$. This tangent vector is nonzero because of the $f'(z_0) \neq 0$ assumption, so the image curve is smooth as well. (This is why we need to assume the derivative is nonzero, since otherwise "angle between curves" does not make sense.) Similarly, we get

$$f'(z_0)\gamma_2'(t_0)$$

as the tangent vector to the image of γ_2 at $f(z_0)$. The angle between these two image curves at $f(z_0)$ is thus the difference

$$\arg[f'(z_0)\gamma_1(t_0)] - \arg[f'(z_0)\gamma_2(t_0)].$$

Arguments of products are sums of arguments, so the difference above is

$$\begin{aligned}\arg[f'(z_0)\gamma_1'(t_0)] - \arg[f'(z_0)\gamma_2'(t_0)] &= [\arg f'(z_0) + \gamma_1'(t_0)] - [\arg f'(z_0) + \arg \gamma_2'(t_0)] \\ &= \arg \gamma_1'(t_0) - \arg \gamma_2'(t_0)\end{aligned}$$

since the $\arg f'(z_0)$ terms cancel out. This resulting value is just the angle between γ_1 and γ_2 at $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$, so we get that the angle between these curves at z_0 is preserved as claimed, so f is conformal at z_0 when $f'(z_0) \neq 0$.

Exponentials. We now seek to expand our repertoire of holomorhic functions, starting with making sense of e^z for z an arbitrary complex number. (We already know what e^z means when z is real or purely imaginary.) One approach is to define e^z via a series, as we did for $e^{i\theta}$ previously, and derive its properties from there. Instead, we will be more explicit. As motivation, if we expect $e^z = e^{x+iy}$ to have the usual properties we would expect of an exponential, we should be able to split up the exponent as

$$e^z = e^{x+iy} = e^x e^{iy}.$$

We already know that $e^{iy} = \cos y + i \sin y$, so we should have

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Thus, we will take this final expression as a our definition, so that e^{x+iy} is defined to be

$$e^{x+iy} = e^x \cos y + ie^x \sin y.$$

We will see later, as mentioned for e^{iy} before as well, that we really have no choice here: if we want a definition of e^z which agrees with e^x when $z = x$ is real and which should be differentiable, the definition we have given is the only possible one that could work. For a quick example, we have

$$e^{2+3i} = e^2 \cos(3) + ie^2 \sin(3).$$

For this to be a good definition, it should first of all give the answer we expect when $z = x$ is real, but for $z = x + i0$ we get

$$e^z = e^x \cos(0) + ie^x \sin(0) = e^x,$$

so this checks out. Second, our definition should give a holomorphic function. For this we check the Cauchy-Riemann equations. The real and imaginary parts of e^z are

$$u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y$$

respectively, and we have

$$\begin{aligned} u_x &= e^x \cos y & u_y &= -e^x \sin y \\ v_x &= e^x \sin y & v_y &= e^x \cos y. \end{aligned}$$

All of these partial derivatives are continuous everywhere, and they do satisfy $u_x = v_y$ and $u_y = -v_x$, so $f(z) = e^z$ is indeed holomorphic on all of $\mathbb{C}$ as desired. Here's some terminology: a function which is holomorphic on $\mathbb{C}$ is said to be *entire*, so e^z is entire; polynomials in powers of z are also entire, but $\frac{1}{z}$ is not since it is not differentiable (or even defined) at 0.

Finally, using the real and imaginary parts, the derivative of $f(z) = e^z$ is given by

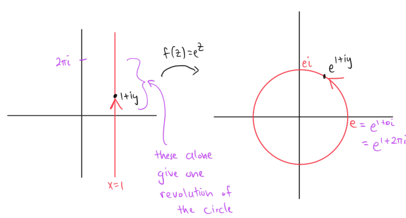
$$f'(z) = u_x(z) + iv_x(z) = e^x \cos y + ie^x \sin y,$$

which is just the definition of e^z again. Hence the derivative of e^z is e^z, just as you might hope.

Geometry of exponentials. The polar form of $e^z = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y)$ is $e^x e^{iy}$ (which we already used in motivating the definition of e^z), so

$$|e^z| = e^x = e^{\operatorname{Re} z} \quad \text{and} \quad \arg(e^z) = y = \operatorname{Im} z.$$

From this we get much geometric information about the behavior of e^z. For example, the line $x = 1$ consists of all complex numbers $1+iy$ with real part 1, so the image of this under the transformation $f(z) = e^z$ contains complex numbers of modulus $|e^{1+iy}| = e^1 = e$. This image is thus contained in the circle of radius e centered at the origin, and indeed we get the full circle as the image since the argument $\arg(e^{1+iy}) = y$ covers all possible values as y changes:



The images of other vertical lines $x = a$ are circles of other radii.

Note too that e^z is $2\pi i$-periodic in the sense that

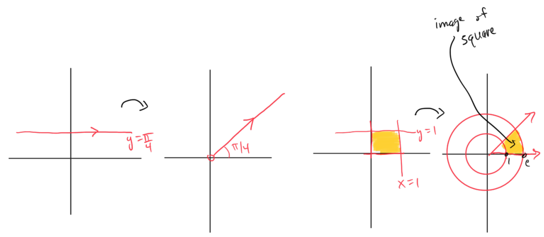
$$e^{z+2\pi i} = e^z \text{ for all } z,$$

so the vertical segment of $x = 1$ for $0 \leq y < 2\pi$ already gives the circle of radius e as the image, and moving beyond these values of y just traces out the circle more than once. (So, e^z is unbounded in real directions, but bounded and periodic in imaginary directions, which is essentially the opposite of what we will soon see for sine and cosine. To be clear, to be bounded means that there is a restriction on how large $|f(z)|$ can be, and e^z is bounded in imaginary directions since $e^x e^{iy} = e^x \cos y + ie^x \sin y$ with x fixed has bounded real and imaginary parts as y varies.)

The image of the horizontal line $y = \frac{\pi}{4}$ consists of points with argument

$$\arg(e^z) = \operatorname{Im}(x + \tfrac{\pi}{4}i) = \tfrac{\pi}{4},$$

so the image lies on the ray which is the portion of the line $y = x$ in the open first quadrant. (Note that the origin is not in the image since e^z is never zero, as can be seen from the fact that $|e^z| = e^x$ is never zero.) As x varies in $x + i\frac{\pi}{4}$, the modulus $|e^z| = e^x$ varies as well among all positive real numbers, so we get the entire ray as the image of $y = 1$:



The image of $x = 0$ is the circle of radius $e^0 = 1$, and the image of $y = 0$ is the horizontal ray at an angle of $\arg(e^{x+i0}) = 0$, so the image of the unit square in the first quadrant is the sector between two circles and two rays, as in the second picture above. Note also that $x = 1$ and $y = 1$ intersect at a right angle in the square picture, and so do the image ray and circle on the right, which makes sense because $f(z) = e^z$ will be conformal at all points as $f'(z) = e^z$ is never zero.

Trigonometric functions. How should we define $\cos z$ and $\sin z$ for a complex variable z? One approach is via a series definition, but instead we derive an alternative expression that, surprise surprise, will be forced on us. For y real, we already know that

$$\begin{aligned} e^{iy} &= \cos y + i \sin y \\ e^{-iy} &= \cos y - i \sin y, \end{aligned}$$

where the second line is the conjugate of the first. From this we can extract expressions for $\cos y$ and $\sin y$ either by adding or by subtracting:

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \quad \text{and} \quad \sin y = \frac{e^{iy} - e^{-iy}}{2i}.$$

(In other words, this comes from the fact that the real part of z can be written as $\frac{1}{2}(z + \overline{z})$, and the imaginary part is $\frac{1}{2i}(z - \overline{z})$.) We thus simply replace y by an arbitrary complex z, and take

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

as the definitions of our standard trig functions. For example,

$$\begin{aligned} \cos(2+i) &= \frac{e^{i(2+i)} + e^{-i(2+i)}}{2} \\ &= \frac{e^{-1+2i} + e^{1-2i}}{2} \\ &= \frac{1}{2}[e^{-1}(\cos 2 + i \sin 2) + e^{1}(\cos(-2) + i \sin(-2))] \\ &= \frac{1}{2}(e^{-1} + e) \cos 2 + \frac{i}{2}(e^{-1} - e) \sin 2, \end{aligned}$$

where we use $\cos(-2) = \cos 2$ and $\sin(-2) = -\sin 2$ in the last step to keep from having to write so many negatives.

The functions $\cos z, \sin z$ thus defined are entire (holomorphic on $\mathbb{C}$) since they are built by composing, adding, and scaling functions which are already known to be entire. (You can also check Cauchy-Riemann, it just gets a bit tedious to write out the real and imaginary parts in general.) Moreover, using the chain rule on $e^{\pm iz}$ we get

$$\begin{aligned}(\cos z)' &= \frac{ie^{iz} - ie^{-iz}}{2} \\ &= \frac{i(e^{iz} - e^{-iz})}{2} \\ &= -\frac{e^{iz} - e^{-iz}}{2i} \\ &= -\sin z\end{aligned}$$

just as you might expect, where we use $i = -\frac{1}{i}$ when rewriting. Similarly, you can verify that the derivative of $\sin z$ is $\cos z$, and that trig identities like

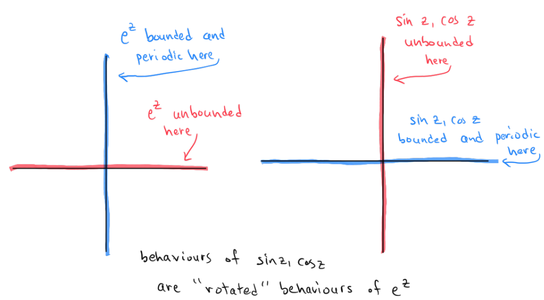
$$\sin^2 z + \cos^2 z = 1$$

still hold even with the complex definitions we have given.

Exponentials vs trig functions. The value of cosine at a purely imaginary number is

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2}.$$

You might recognize this resulting expression as the definition of what is called *hyperbolic cosine* at y, denoted by $\cosh y$. Hyperbolic functions are not ones we will study in any depth, but the important observation for us is the fact that $\cos(iy)$ is thus unbounded as y changes in either the positive or negative directions, so that $\cos z$ is unbounded in imaginary directions. The same is true of $\sin z$, so we get that cosine and sine are bounded and periodic in real directions but unbounded in imaginary directions, which, as mentioned before, is the opposite of what happened for e^z:



This is no coincidence, since the definitions we gave for all these functions explicitly have the others built into them:

$$e^z = e^x(\cos y + i \sin y), \ \cos z = \frac{e^{iz} + e^{-iz}}{2}, \ \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

To be more precise, note that iz is obtained by rotation z by $\frac{\pi}{2}$ since $i = e^{i\pi/2}$, so in constructing $\cos z$ or $\sin z$ above we essentially rotate z in both counterclockwise and clockwise (for $-iz$) directions and then apply exponentials. In a sense, $\cos z$ and $\sin z$ are "rotated" (albeit using two rotations in opposite directions) versions of e^z, and e^z is a "rotated" version of $\cos z, \sin z$, which is what helps to explain the bounded and periodic vs unbounded behaviors above. The moral is that this such relations between exponentials and trig functions only become apparent in the setting of complex analysis, as they are not noticeable in the strictly real setting alone.

Lecture 9: Logarithms and Branches

Warm-Up 1. We determine the complex numbers z for which e^z is real and negative. Note that in the purely real case, e^x is never negative, so this problem highlights one thing that the complex exponential allows us to do that the real exponential does not. We have

$$e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y.$$

In order for this to be real, the imaginary part must be zero, so this requires that $\sin y = 0$ since e^x is never zero. Thus we must have $y = n\pi$ for an integer n. But then, in order for

$$e^{x+in\pi} = e^x \cos(n\pi) = e^x(-1)^n$$

to be negative, we need n to be odd since e^x is always positive. Thus we get that e^z is real and negative only when $z = x + i(2n+1)\pi$ for n an integer, and x can be anything.

Warm-Up 2. We describe the z which satisfy $\cos z = 2$. (In other words, we are finding the values of "$\arccos 2$", but inverse trig functions are not something we will study in this course so we will avoid this notation.) In the purely real case, $\cos x$ can never equal 2, so this highlights a way in which complex cosine differs from real cosine. We have

$$\begin{aligned}
\cos z &= \tfrac{1}{2}(e^{iz} + e^{-iz}) \\
&= \tfrac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) \\
&= \tfrac{1}{2}(e^{-y+ix} + e^{y-ix}) \\
&= \tfrac{1}{2}[(e^{-y}\cos x + e^y \cos x) + i(e^{-y}\sin x - e^y \sin x)],
\end{aligned}$$

where at the end we use $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. In order for this to equal 2, we need the following to hold:

$$(e^{-y} + e^y)\cos x + i(e^{-y} - e^y)\sin x = 4,$$

which thus requires that

$$(e^{-y} + e^y)\cos x = 4 \quad \text{and} \quad (e^{-y} - e^y)\sin x = 0.$$

The second requirements says that

$$e^{-y} - e^y = 0 \quad \text{or} \quad \sin x = 0.$$

The first condition holds only when $y = 0$ since otherwise one of e^y and e^{-y} is greater than 1 and the other less than 1, but in this case the second equation from before becomes

$$(e^{-0} + e^0)\cos x = 4 \rightsquigarrow 2\cos x = 4,$$

which never holds since $\cos x$ is never 2. Thus we can rule out $y = 0$, so we must have

$$\sin x = 0, \quad \text{so} \quad x = n\pi.$$

The remaining equation $(e^{-y} + e^y)\cos x = 4$ then becomes

$$(e^{-y} + e^y)\cos(n\pi) = 4.$$

But $e^{-y}+e^{y}$ is always positive, so $\cos(n\pi)=(-1)^n$ must be positive, so n must be even. If so, we are left with

$$e^{-y}+e^{y}=4,$$

which has two solutions as can be seen from plotting the graphs. (The solutions are actually $y=\ln(2+\sqrt{3})$ and $y=\ln(2-\sqrt{3})$, but giving explicit values is not important here. If interested, you can get the explicit values after multiplying through by e^y to get

$$1+e^{2y}=4e^{y},$$

and then treating this as a quadratic in terms of e^y and using the quadratic formula.) We thus conclude that the z satisfying $\cos z=2$ are

$$z=2n\pi+iy$$

where y is one of the two numbers satisfying $e^{-y}+e^{y}=4$.

Logarithms. After exponentials and trig functions, the next "standard" function to try to make sense of is the logarithm $\log z$. To be clear, by $\log z$ we mean w which satisfies $e^w=z$. We actually already saw an instance of this in the first Warm-Up. Indeed, according to the work we had there we can see that the w which satisfy $e^w=-1$ are

$$w=i(2n+1)\pi \text{ for } n \text{ an integer.}$$

(The fact that the real part should be zero comes from the the requirement that $e^x(-1)^{2n+1}=-1$ in one of the equations we derived, which forces $e^x=1$.) Thus, we would say that

$$\log(-1)=i(2n+1)\pi.$$

(So, we can take logarithms of negative numbers, which cannot be done in the real setting alone!) A key thing to note here is that there are actually *multiple* values of $\log(-1)$: $i\pi$ works, but so does $i3\pi$, or $i5\pi$, and so on. This is an inherent fact about how logarithms behave for complex numbers, where "logarithm" will always be a multi-valued quantity.

In general, if $w=u+iv$, in order to have

$$e^w=z \rightsquigarrow e^u e^{iv}=|z|e^{i\arg z}$$

be true, we need $e^u=|z|$ and $v=\arg z$. The condition $e^u=|z|$ is the same as $u=\ln|z|$, so we get that $w=\log z$ looks like

$$\log z=\ln|z|+i\arg(z),$$

and this is what we will take as the definition of $\log z$ for z complex. (Note that we use ln here to denote the usual natural logarithm for *real* numbers, so as to reserve log for the complex logarithm. Also, e^z is never zero, so 0 does not have a logarithm and thus $|z|\neq 0$ in the definition above, which means that $\ln|z|$ makes sense.) Since $\arg(z)$ has multiple values, $\log z$ is multi-valued as well.

For example, the values of $\log(1+i)$, where $|1+i|=\sqrt{2}$ and $\arg(1+i)=\frac{\pi}{4}+2n\pi$, are

$$\log(1+i)=\ln\sqrt{2}+i(\tfrac{\pi}{4}+2n\pi)$$

for n an integer. We can verify that indeed

$$e^{\log(1+i)}=e^{\ln\sqrt{2}+i(\frac{\pi}{4}+2n\pi)}=e^{\ln\sqrt{2}}e^{i\frac{\pi}{4}}\underbrace{e^{i2n\pi}}_{1}=\sqrt{2}e^{i\frac{\pi}{4}}=1+i.$$

Careful with identities. The identity

$$e^{\log z} = z$$

holds for all $z \neq 0$ because we determined the values of $\log z$ precisely so that this equation would hold. However, note that

$$\log(e^z) = z$$

is *not* true, with one reason being that the left side is multi-valued but the right side is not. Rather, z is but only one of the values of the left side; since e^z is $2\pi i$-periodic, $z + i2n\pi$ exponentials to the same thing as does z, so in fact

$$\log(e^z) = z + i2n\pi.$$

Here's another identity one might expect to be true but is not true in the literal sense. Consider $\log(i^2)$ versus $2\log(i)$. If the logarithm were to behave as expected, we might think that these two expressions are the same, but in fact we have

$$\log(i^2) = \log(-1) = i(2n+1)\pi \quad \text{and} \quad 2\log(i) = 2(\underbrace{\ln|i|}_{0} + i\arg(i)) = 2i(\tfrac{\pi}{2} + 2n\pi) = i(4n+1)\pi.$$

Thus, $\log(i^2) \neq 2\log(i)$ as the values of the right side are only *some* of the values of the left but not all; for example, $i3\pi$ is a value of $\log(i^2)$ which is not a value of $2\log(i)$. The upshot is that the multi-valued nature of $\log z$ forces us to be careful about assuming that identities we might expect to be true based on the behavior of $\ln x$ are actually true in the complex setting.

Logarithms as functions. This all thus poses an issue when trying to think of $\log z$ as an actual *function* on $\mathbb{C}^*$ (the punctured plane, we must certainly exclude 0 since $\log 0$ is undefined), which requires that we get only single unique values for anything we plug in. To get single values for $\log z$ we must pick unique values of the argument $\arg z$ for each z, so let us specify, for now, that we will choose to use the principal values of the argument. The principal value of $\arg z$ is denoted $\operatorname{Arg} z$, and it is common to call the resulting value of $\log z$ the *principal value* and to denote it by $\operatorname{Log}(z)$:

$$\operatorname{Log}(z) = \ln|z| + \operatorname{Arg}(z) \text{ for } z \neq 0.$$

So, for example,

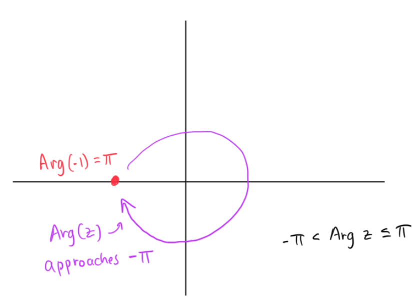
$$\operatorname{Log}(i) = \ln|i| + i\operatorname{Arg}(i) = i\tfrac{\pi}{2} \quad \text{and} \quad \operatorname{Log}(-1) = \ln|-1| + i\operatorname{Arg}(-1) = i\pi$$

where we recall that $-\pi < \operatorname{Arg}(z) \leq \pi$.

The question is whether this definition gives us a "nice" function on $\mathbb{C}^*$, where by "nice" we mean *continuous*. The answer is no since we claim

$$\lim_{z \to -1} \operatorname{Log}(z) \neq \operatorname{Log}(-1),$$

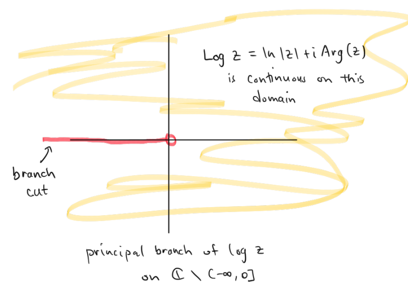
whereas equality would be required by the definition of "continuous". The issue is that -1 has principal argument π, but if we approach -1 from among points in the third quadrant, the principal argument actually approaches $-\pi$:



So, $\lim_{z\to -1} \operatorname{Log}(z)$ when approaching -1 from the third quadrant is $-i\pi$, which is different from $\operatorname{Log}(-1) = i\pi$. A similar thing happens at any point on the negative real axis.

Logarithm branches. If we want to get a continuous function giving values of $\log z$, we must thus exclude all points on the negative real axis from our domain altogether to avoid the issue of limits not matching up. (Why do we care about having a function be continuous? Because, just as in the usual real calculus case, the existence of a derivative always implies continuity, so if we want our function to end up being holomorphic, they had better be continuous to begin with! We'll discuss derivatives of $\log z$ next time.) Thus, we can say that $\operatorname{Log}(z)$ does define a continuous function on $\mathbb{C}\backslash(-\infty, 0]$, which is notation for $\mathbb{C}$ with points on the (real) interval $(-\infty, 0]$ excluded.

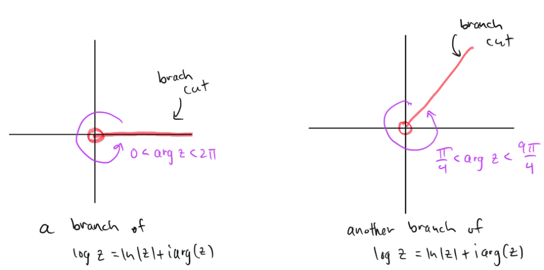
A continuous function giving values of $\log z$ on a domain is called a *branch* of $\log z$ on that domain, so the principal branch $\operatorname{Log}(z)$ of $\log z$ is the one defined on $\mathbb{C}\backslash(-\infty, 0]$. The set $(-\infty, 0]$ we had to exclude in order to define this (continuous) branch is called a *branch cut*:



There are many other branches of $\log z$ we could use, each with their own branch cuts. For example, we can restrict our choice of argument in

$$\log z = \ln |z| + i \arg(z)$$

to be those for which $0 < \arg(z) < 2\pi$, and get a branch of $\log z$ on $\mathbb{C}\backslash[0, \infty)$, which is $\mathbb{C}$ with 0 and the positive real axis excluded. (This excluded half axis is thus the "branch cut" of this particular branch. We must exclude these points since allowing $0 = \arg(z)$ as an argument will lead to a function which is not continuous at these points.) If we instead use $\frac{\pi}{4} < \arg z < \frac{9\pi}{4}$, we get a branch of $\log z$ defined on the domain with branch cut described by $\arg z = \frac{\pi}{4}$:



We will say more about branches of $\log z$, and branches of other functions, next time.

Lecture 10: More on Branches

Warm-Up 1. We find all values of $\log(\log i)$. First, we have

$$\log i = \ln |i| + i \arg(i) = i(\tfrac{\pi}{2} + 2\pi n)$$

for n an integer. Thus

$$\log(\log i) = \log(i(\tfrac{\pi}{2} + 2\pi n)) = \ln|i(\tfrac{\pi}{2} + 2\pi n)| + i\arg(i(\tfrac{\pi}{2} + 2\pi n)).$$

Now, note that $\frac{\pi}{2} + 2\pi n$ can be negative depending on whether n is negative, so the modulus of $i(\frac{\pi}{2} + 2\pi n)$ is the absolute value of $\frac{\pi}{2} + 2\pi n$ and the argument of $i(\frac{\pi}{2} + 2\pi n)$ is $\pm\frac{\pi}{2}$ plus multiples of $2\pi i$. (We need $-\frac{\pi}{2}$ here to account for the cases where $\frac{\pi}{2} + 2\pi n$ is negative.) Hence

$$\log(\log i) = \ln|\tfrac{\pi}{2} + 2\pi n| + i(\pm\tfrac{\pi}{2} + 2\pi m)$$

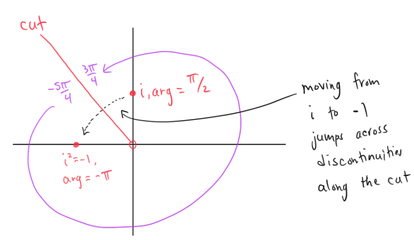
where n and m are integers. (Different integer m for the arguments of $i(\frac{\pi}{2} + 2\pi n)$ than for the initial n in $\arg(i)$.) These are thus all the values satisfying $e^{e^z} = i$.

Warm-Up 2. We show that $\log(i^2) = 2\log i$ does not hold for the branch of log defined using $-\frac{5\pi}{4} < \arg z < \frac{3\pi}{4}$, and then find a branch of log for which this identity *is* true. Last time we showed that $\log(i^2) = 2\log i$ does not hold when we interpret both sides as multi-valued expressions, but that's not to say that it never holds for a specific branch. The takeaway is that whether or not an identity involving log is actually true sometimes depends on the branch we use. This will show up later when computing certain integrals, where whether or not we can use a specific technique might depend on the branch we pick.

For the branch using $-\frac{5\pi}{4} < \arg z < \frac{3\pi}{4}$ (thus with branch cut along the ray $\arg z = \frac{3\pi}{4}$), we have $\arg(i^2) = \arg(-1) = -\pi$ and $\arg(i) = \frac{\pi}{2}$, so

$$\log(i^2) = \log(-1) = \ln|-1| + i\arg(-1) = -i\pi \quad \text{and} \quad 2\log(i) = 2(\ln|i| + i\arg(i)) = 2(i\tfrac{\pi}{2}) = i\pi,$$

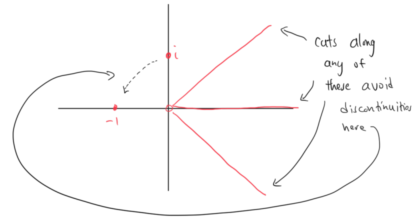
which do not agree as claimed. Now, essentially the reason why this happens is because the branch cut occurs "between" $i^2 = -1$ and i, so that argument values do not vary continuously when moving from i to -1:



The same would happen if we took a branch cut along some other ray in the second quadrant, but if we instead take the branch cut along a ray in a different quadrant, the given identity should in fact hold. For example, with the branch of log defined on $\mathbb{C}\backslash[0,\infty)$ with $0 < \arg z < 2\pi$, we have

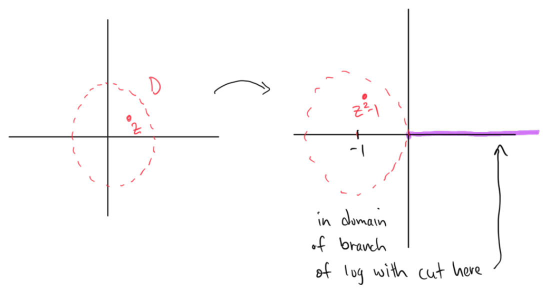
$$\log(i^2) = \log(-1) = i\pi \quad \text{and} \quad 2\log(i) = 2(i\tfrac{\pi}{2}) = i\pi,$$

so that $\log(i^2) = 2\log(i)$ is true for this branch:



We could consider other branches as well, say with cut along $\arg z = -7\pi/4$ and using $-\frac{7\pi}{4} < \arg z < \frac{\pi}{4}$, but for our purposes sticking with more standard argument ranges will be enough.

Another branch example. We now want to define a branch of the function $\log(z^2-1)$ on the open unit disk D defined by $|z| < 1$. The definition is the same as the one we gave for $\log z$: a branch of $\log(z^2-1)$ on D is a continuous function $f(z)$ whose value at any point is a value of $\log(z^2-1)$, meaning that $e^{f(z)} = z^2 - 1$ should be true on D. For a first approach to constructing this, let us consider where z^2-1 lives when z is in D. If z is in D, then so is z^2 since $|z| < 1$ implies that $|z^2| = |z|^2 < 1$. But then $z^2 - 1$ will also be in a disk, namely the disk of radius 1 centered at -1 on the real axis which subtracting 1 from z^2 has the effect of translating it to the left:



Since we want to evaluate log on such a z^2-1, we thus need to use a branch of log which is defined on this translated disk. Let us therefore use the branch of log from the second part of the second Warm-Up, meaning the branch $\ell(z)$ defined using $0 < \arg z < 2\pi$. Since $z^2 - 1$ is never in the branch cut $[0,\infty)$ for this branch, evaluating ℓ at z^2-1 makes sense, so $\ell(z^2-1)$ is our desired branch of $\log(z^2-1)$ on D; this is continuous since it is ℓ and z^2-1 are continuous, and its value at any point is a value of $\log(z^2-1)$.

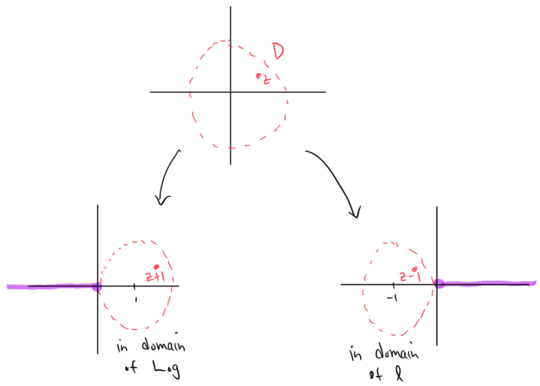
Let us now give a second approach in order to highlight the idea that sometimes using multiple branches of log is necessary. (It is not strictly necessary here since in the first approach we were able to use only one approach, but in general we might have to use different branches at the same time.) Since $z^2 - 1 = (z+1)(z-1)$, this second approach is motivated by

$$\log(z^2-1) = \log((z+1)(z-1)) = \log(z+1) + \log(z-1)$$

as an identity we might expect of log. Of course, since log is multi-valued there is a question as to whether something like

$$\log(w_1 w_2) = \log(w_1) + \log(w_2)$$

is actually true (in fact it is!), but we are only using this desired identity as motivation without worrying about whether it literally holds. The point is that if we want to define $\log(z^2-1)$, we can instead try to define $\log(z+1)$ and $\log(z-1)$. Defining $\log(z+1)$ requires a branch of log for which $\log(z+1)$ makes sense when z is in D, and defining $\log(z-1)$ requires a branch for which $z+1$ is in its domain when z is in D. If z is in D, $z+1$ is in the open disk of radius 1 centered at 1 and $z-1$ is in the open disk of radius 1 centered at -1:



The disk centered at 1 is in the domain of the principal branch Log of log (with cut along $(-\infty, 0]$), and the disk centered at -1 is in the domain of the branch ℓ we used before (with cut along $[0, \infty)$ and $0 < \arg z < 2\pi$), so

$$\mathrm{Log}(z+1) + \ell(z-1)$$

is defined on D. This expression is continuous since it is built from continuous things, and it satisfies the property required of a branch of $\log(z^2-1)$, namely

$$e^{\mathrm{Log}(z+1)+\ell(z-1)} = e^{\mathrm{Log}(z+1)}e^{\ell(z-1)} = (z+1)(z-1) = z^2 - 1$$

where in the first step we use $e^{w_1+w_2} = e^{w_1}e^{w_2}$ which still holds in the complex setting, and in the second step we use that Log and ℓ are branches of log, so that $e^{\mathrm{Log}\, w} = w$ and $e^{\ell(w)} = w$. Thus, $f(z) = \mathrm{Log}(z+1) + \ell(z-1)$ is a valid branch of log on D.

Derivatives of logarithms. Recall that the reason for considering branches of $\log z$ is to ensure that the functions with which we are working are continuous, which is a necessary property if we hope that our functions will be holomorphic. But it turns out that once we have dealt with the continuity issue, differentiability is no problem since all branches of $\log z$ will in fact be holomorphic on their domains, which we can check using the Cauchy-Riemann equations. A standard way of doing this is to use (as the book does) the expression for the Cauchy-Riemann equations in polar coordinates: for $f = u + iv$ with $u(r, \theta)$ and $v(r, \theta)$ written in terms of polar coordinates, the Cauchy-Riemann equations look like

$$ru_r = v_\theta \quad \text{and} \quad u_\theta = -rv_r.$$

These can be derived from the usual Cauchy-Riemann equations using the multivariable chain rule, but we will not go through the derivation here as we will not have need to use this polar form going forward. With these at hand, for a branch of $\log z = \ln r + i\theta$ where $u(r, \theta) = \ln r$ and $v(r, \theta) = \theta$, we have

$$ru_r = r(\tfrac{1}{r}) = 1 = v_\theta \quad \text{and} \quad u_\theta = 0 = -rv_r,$$

so the Cauchy-Riemann equations in polar coordinates are satisfied.

Instead, let us stick with rectangular coordinates and the usual Cauchy-Riemann equations. For a branch of $\log z$, we have

$$\log z = \ln|z| + i \arg z = \underbrace{\ln\sqrt{x^2+y^2}}_{u(x,y)} + i\underbrace{\arctan(\tfrac{y}{x})}_{v(x,y)}.$$

Now, there are a few subtleties here. One is that arctan is itself also a multi-valued expression as there are many angles that will satisfy $\tan\theta = \frac{y}{x}$ for a given $\frac{y}{x}$. Typically in a previous calculus course one restricts arctan to outputing angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (this is what would be called the "principal branch" of arctan, although the term "branch" is not usually mentioned in a first calculus course), but other choices are possible. It is true that any such "branch" of $\arctan t$ for t real will be differentiable with derivative $\frac{d}{dt}(\arctan t) = \frac{1}{1+t^2}$, so we will take this real fact for granted here. The second subtlety is that $\frac{y}{x}$ in $\arctan(\frac{y}{x})$ is not defined when $x = 0$ on the imaginary axis, but such points could certainly be in the domain of a branch of log. This can be dealt with by noting that, for such points, $-\arctan(\frac{x}{y}) + \frac{\pi}{2}$ gives the correct arctangent value instead, which stems from the fact that $\tan(\theta + \frac{\pi}{2}) = -\cot(\theta)$. The expression $-\arctan(\frac{x}{y}) + \frac{\pi}{2}$ has the same partial derivatives as $\arctan(\frac{y}{x})$, so this $\frac{y}{x}$ being undefined when $x = 0$ issue can be fixed by changing the arctangent expression we use without changing the output of the Cauchy-Riemann equations. We will thus stick with using $\frac{y}{x}$ in our computation.

We have

$$u_x = \frac{1}{\sqrt{x^2+y^2}}\left(\frac{2x}{2\sqrt{x^2+y^2}}\right) = \frac{x}{x^2+y^2} \qquad v_y = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{x}{x^2}\right) = \frac{x}{x^2+y^2}$$
$$u_y = \frac{1}{\sqrt{x^2+y^2}}\left(\frac{2y}{2\sqrt{x^2+y^2}}\right) = \frac{y}{x^2+y^2} \qquad v_x = \frac{1}{1+(\frac{y}{x})^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}.$$

These expressions are all continuous on the domain of any branch (since no branch domain contains zero), and $u_x = v_y, u_y = -v_x$ is true, so any branch is holomorphic as claimed. Moreover, the derivative of any branch of $\log z$ is

$$u_x + iv_x = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z},$$

so $\frac{d}{dz}(\log z) = \frac{1}{z}$, just as we might expect!

Branches of other functions. The issue regarding branches shows up when considering other multi-valued expressions as well. For example, to define $\sqrt{z} = z^{1/2}$ as a (holomorphic) function, we must specify which value of $\sqrt{z}$ we are taking, and hence must choose a branch. By a branch of $\sqrt{z} = z^{1/2}$ on a domain D we mean, just as before, a continuous function $f(z)$ on D whose value at any point is a value of $z^{1/2}$, meaning that $f(z)$ should satisfy

$$f(z)^2 = z \text{ on } D.$$

Rather than go through the theory all over again, we will exploit what we already know about branches of log, since in fact we can (hope to) write $z^{1/2}$ as

$$z^{1/2} = e^{\log(z^{1/2})} = e^{\frac{1}{2}\log z}.$$

(The first equality here is certainly true, but we have to be careful about whether the identity $\log(z^{1/2}) = \frac{1}{2}\log(z)$ used in the second equality is actually true. This will not matter for our discussion here, however, since we only use this to motivate how we will define $z^{1/2}$ as a function.) The idea is that if we can define the right side, then we can give (a) definition of the left side, so defining the function $z^{1/2}$ comes down to picking a branch of $\log z$.

For example, the *principal branch* of $z^{1/2}$ on $\mathbb{C}\backslash(-\infty, 0]$ is defined by

$$f(z) = e^{\frac{1}{2}\operatorname{Log} z}$$

where Log is the principal branch of log. This is continuous on $\mathbb{C}\backslash(-\infty,0]$ since it is the composition of continuous things, and we can check that

$$f(z)^2 = e^{\frac{1}{2}\operatorname{Log}(z)}e^{\frac{1}{2}\operatorname{Log}(z)} = e^{\frac{1}{2}\operatorname{Log}(z)+\frac{1}{2}\operatorname{Log}(z)} = e^{\operatorname{Log}(z)} = z,$$

so that $f(z) = e^{\frac{1}{2}\operatorname{Log} z}$ is indeed a value of $z^{1/2}$. Since $\operatorname{Log}(1) = 0$ and $\operatorname{Log}(i) = i\frac{\pi}{2}$, we get for example that the principal square roots of 1 and i are respectively

$$f(1) = e^{\frac{1}{2}\operatorname{Log} 1} = e^0 = 1 \quad \text{and} \quad f(i) = e^{\frac{1}{2}\operatorname{Log}(i)} = e^{\frac{1}{2}(i\frac{\pi}{2})} = e^{i\pi/4} = \tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}.$$

As for the derivative of this principal branch of $z^{1/2}$, we have by the chain rule that

$$f'(z) = \tfrac{d}{dz}(e^{\frac{1}{2}\operatorname{Log} z}) = e^{\frac{1}{2}\operatorname{Log} z}\left(\tfrac{1}{2z}\right).$$

If we write z in the denominator as $z = e^{\operatorname{Log} z}$ and use $e^w = \frac{1}{e^{-w}}$ (still valid in the complex case since $e^w e^{-w} = e^{w-w} = 0 = 1$), we have

$$f'(z) = \frac{1}{e^{-\frac{1}{2}\operatorname{Log} z}[2e^{\operatorname{Log} z}]} = \frac{1}{2e^{-\frac{1}{2}\operatorname{Log} z+\operatorname{Log} z}} = \frac{1}{2e^{\frac{1}{2}\operatorname{Log} z}} = \frac{1}{2f(z)},$$

which exactly says that the derivative of $z^{1/2}$ is $\frac{1}{2z^{1/2}}$, just as you would hope.

Branches of other power functions z^c, like z^i, can be defined in the same way, and we will look at an example next time.

Lecture 11: Contour Integrals

Warm-Up 1. We determine some values of $z^{1/2} = e^{\frac{1}{2}\log z}$ for some appropriate branches of $\log z$. For example, we first see what value this assigns to $i^{1/2}$ when using the principal branch of log:

$$i^{1/2} = e^{\frac{1}{2}\operatorname{Log} i} = e^{\frac{1}{2}(i\frac{\pi}{2})} = e^{i\pi/4},$$

which is what we previously called the principal square root of i. In order to obtain the non-principal square root if i, namely $e^{-i3\pi/4}$, as a value of (a branch of) the function $z^{1/2} = e^{\frac{1}{2}\log z}$, we must use a different branch of log. We want $\frac{1}{2}\log i$ to end up giving $-\frac{3\pi}{4}$ as an argument, which means that we need $\log i$ to have $-\frac{3\pi}{2}$ as an argument since it is this argument that gives $-\frac{3\pi}{4}$ after multiplication by $\frac{1}{2}$. Thus, any branch of $\log z$ that uses $-\frac{3\pi}{4}$ as the argument of i will work. Take for example the branch $\ell(z)$ of $\log z$ defined by picking $-2\pi < \arg z < 0$ as argument values with branch cut along the nonnegative real axis $[0,\infty)$. For this branch we have

$$\ell(i) = \ln|-i| + i\arg(i) = -i\tfrac{3\pi}{2}, \text{ so } i^{1/2} = e^{\frac{1}{2}\ell(i)} = e^{\frac{1}{2}(-i3\frac{\pi}{2})} = e^{-i3\pi/4}$$

as desired. Taking other branch cuts that do not exclude i on the imaginary axis also work as long as $\arg(i) = -\frac{3\pi}{2}$ for that branch.

For the principal branch of log we get

$$1^{1/2} = e^{\frac{1}{2}\operatorname{Log}(1)} = e^0 = 1$$

as the (principal) square root of 1. If we want instead a branch that will result in $1^{1/2} = -1$, we just need to be careful about argument values as above. Since $-1 = e^{-i\pi}$, we want $\arg(1)$ to be such that $\frac{1}{2}\arg(1) = -\pi$, which means that we need $\arg(1) = -2\pi$. Thus picking a branch of log that uses this argument of 1 should work. For example, take a branch cut $\ell(z)$ along $(-\infty, 0]$ with $-3\pi < \arg z < -\pi$. For this branch we get $\ell(1) = \ln|1| + i\arg(1) = -2\pi i$, so

$$1^{1/2} = e^{\frac{1}{2}\ell(1)} = e^{\frac{1}{2}(-2\pi i)} = e^{-i\pi} = -1.$$

In general, this particular branch gives $-\sqrt{x}$ as the square root of any real $x > 0$. The upshot is that by varying the branch of log we use, we can obtain holomorphic functions that give whatever square root values we need.

Here's some terminology we did not introduce in class yet but will in a few weeks: the function $z^{1/2} = e^{\frac{1}{2}\ell(z)}$ which uses the branch of log defined by $-3\pi < \arg z < -\pi$ is the *analytic continuation* of the real function $f(x) = -\sqrt{x}$ with domain $(0, \infty)$. Given a real differentiable function f defined on some portion of the real axis, an *analytic continuation* of f is a holomorphic function defined on a domain in $\mathbb{C}$ containing the domain of f which agrees with the value of f at any real input. The idea is that given f with real inputs, we want to know whether it is possible to extend the domain of f to include complex numbers while still being differentiable. The particular branch of $z^{1/2}$ described above is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$ and gives the value $-\sqrt{x}$ for any x in $(0, \infty)$ (here $\sqrt{x}$ denotes the nonnegative square root of x), so it is an analytic continuation of $-\sqrt{x}$. The principal branch $z^{1/2} = e^{\frac{1}{2}\operatorname{Log} z}$ is an analytic continuation of the usual nonnegative square root function $\sqrt{x}$. (We will see that analytic continuations, when they exist, are unique, which is what lies behind the fact that there is essentially only one way in which $e^z, \cos z$, and $\sin z$ could have been defined, at least if we wanted to get holomorphic functions that give the usual value we would expect at real inputs!)

Warm-Up 2. For a complex number c, we define the "c-th power of z" by

$$z^c = e^{c \log z}.$$

The motivation, as in the $c = \frac{1}{2}$ case, comes from

$$z^c = e^{\log(z^c)} \text{ which "should" equal } e^{c \log z}.$$

Again, it might not be literally true that $\log(z^c) = c \log z$, so we only take this as motivation for how z^c should be defined. (We need to have *some* definition of what z^c means when c is complex, because it does not make sense for example to multiply z by itself "i many times".) The resulting expression $z^c = e^{c \log z}$ is multivalued since $\log z$ is multivalued, so for example we can determine all the values of $(1+i)^i$ to be

$$\begin{aligned}
(1+i)^i &= e^{i \log(1+i)} \\
&= e^{i(\ln\sqrt{2} + i(\frac{\pi}{4} + 2\pi n))} \\
&= e^{i \ln\sqrt{2} - (\frac{\pi}{4} + 2\pi n)} \\
&= e^{-\frac{\pi}{4} - 2\pi n} e^{i \ln\sqrt{2}} \\
&= e^{-\frac{\pi}{4} - 2\pi n}(\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2}))
\end{aligned}$$

where n is an integer. The value obtained using the principal branch of log in particular is the one where $n = 0$ for example.

To give one reason why $z^i = e^{i \log z}$ is a good definition, we check that it gives what you might expect as the derivative of z^i when we work with a particular branch of $\log z$. Let us denote such a branch just by $\log z$ itself (so, for our purposes in this specific problem the notation $\log z$ is not multi-valued), where we use the fact that any such branch is holomorphic with derivative $\frac{1}{z}$. The function $z^i = e^{i \log z}$ defined using this branch is also holomorphic as it is the composition of holomorphic things, and the chain rule gives

$$\tfrac{d}{dz}(z^i) = \tfrac{d}{dz}(e^{i \log z}) = e^{i \log z}\left(\tfrac{i}{z}\right).$$

To put this into a more recognizable form, we use $z = e^{\log z}$ (true for any branch of log and properties of exponentials to write

$$e^{i \log z}\left(\frac{i}{z}\right) = e^{i \log z}\left(\frac{i}{e^{\log z}}\right) = ie^{i \log z}e^{-\log z} = ie^{i \log z - \log z} = ie^{(i-1)\log z}.$$

For the chosen branch of $\log z$, $e^{(i-1)\log z}$ is a branch of z^{i-1} by the same definition $z^c = e^{c \log z}$, so we do get that

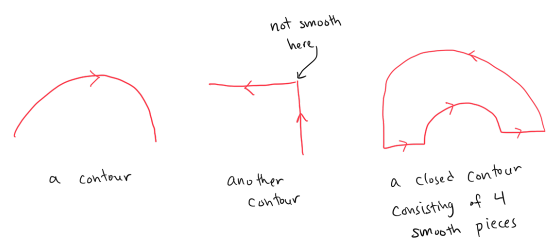
$$\frac{d}{dz} z^i = iz^{i-1}$$

as one might hope. The same works for the derivative of any complex power z^c (i.e., bring the exponent down and then subtract 1), so the definition $z^c = e^{c \log z}$ does give the types of properties we would expect.

Contour integrals. We are now ready to discuss complex integration, which will form the basis of everything else we will do in this course. The type of integral we will care about is what is known as a *contour integral* and is denoted by

$$\int_C f(z)\, dz.$$

Here $f(z)$ is a complex function with z the variable of integration, and C is a complex curve, or more specifically what will be called a *contour*. A contour is a piecewise smooth curve with a specific orientation, or in other words a collection of oriented smooth curves joined together:



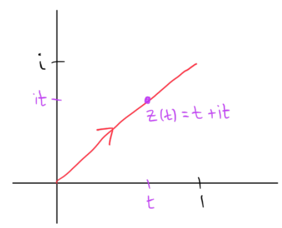
Recall that to be "smooth" means that we have a nonzero tangent vector at each point, and to be piecewise smooth means that we allow for "sharp" edges at points where one smooth piece finishes and the next begins. A contour is *closed* if it ends at the same point at which it began.

Given a parametrization $z(t) = x(t) + iy(t),\ a \le t \le b$ of C (with the appropriate orientation), we define the contour integral of f over C by

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

So, on the right side we evaluate f at points $z(t)$ along the curve, multiply the value by the tangent vector $z'(t)$ (we discussed how complex numbers like $z'(t) = x'(t) + iy'(t)$ can be treated as tangent vectors back when discussing conformality), and then integrate as the curve parameter t varies. The resulting integrals can be computed using the same integration techniques you have used all you lives since t in particular will be a real parameter. (The definition above can be shown to be independent of parametrization in the sense that using different parametric equations for C will result in the same value for $\int_C f(z)\,dz$.)

Example. Let us compute $\int_C z^2\,dz$ where C is the line segment from 0 to $1+i$:



We parametrize C as $z(t) = t + it$ for $0 \le t \le 1$, so that $z'(t) = 1 + i$ and hence

$$\begin{aligned}
\int_C z^2\,dz &= \int_0^1 \underbrace{z(t)^2}_{f(z)}\,\underbrace{z'(t)\,dt}_{dz} \\
&= \int_0^1 (t+it)^2(1+i)\,dt \\
&= \int_0^1 t^2(1+i)^2(1+i)\,dt \\
&= (1+i)^3 \int_0^1 t^2\,dt \\
&= (-2+2i)\tfrac{1}{3} \\
&= -\tfrac{2}{3} + i\tfrac{2}{3}
\end{aligned}$$

is our desired value. (Note that constants like $(1+i)^3$ factor out just as with any other type of integral.) If we instead orient C in the opposite direction, so moving from $1+i$ to 0, the sign of the tangent vector changes so we get

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dx = \tfrac{2}{3} - i\tfrac{2}{3},$$

where $-C$ denotes C with the opposite orientation.

More examples. Now we consider $\int_{C_1} z^2\,dz$ where C_1 is the upper-half of the unit circle defined by $|z| = 1$ oriented counterclockwise. We can parametrize C_1 by $z(t) = e^{it}$ for $0 \le t \le \pi$, which is just the usual cosine and sine parametric equations for a circle

$$z(t) = \cos t + i \sin t$$

only written in the more convenient polar form. We have $z'(t) = ie^{it}$, so we get

$$\begin{aligned}
\int_{C_1} z^2\,dz &= \int_0^\pi (e^{it})^2(ie^{it})\,dt \\
&= \int_0^\pi ie^{2it}e^{it}\,dt \\
&= i\int_0^\pi e^{3it}\,dt
\end{aligned}$$

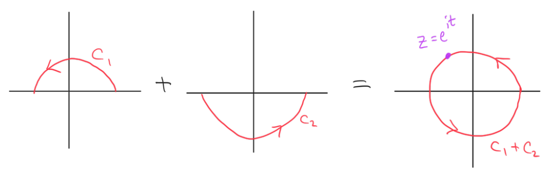
$$= i\left(\frac{e^{3it}}{3i}\right)\Big|_0^{\pi}$$
$$= \tfrac{1}{3}(e^{3\pi i} - e^0)$$
$$= \tfrac{1}{3}(-1-1) = -\tfrac{2}{3}.$$

Note that we used $\frac{e^{3it}}{3i}$ here as an antiderivative of e^{3it} (with respect to t), just as you expect if i were replaced by a real constant.

If were to integrate z^2 over the entire circle $|z| = 1$ (both top and bottom) oriented counterclockwise, we would instead use $z(t) = e^{it},\ 0 \leq t \leq 2\pi$ as a parametrization, and all that would change is the final step when we plug in the bounds of integration:

$$\int_{|z|=1} z^2\,dz = i\left(\frac{e^{3it}}{3i}\right)\Big|_0^{2\pi} = 0$$

since $e^{6\pi i} = 1 = e^0$, so the difference is zero. If C_2 denotes the bottom half of the circle oriented counterclockwise, then we can use the fact that combining C_1 and C_2 gives the full circle to find the integral of C_2 alone:



In general we have

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

where $C_1 + C_2$ denotes the contour obtained by first following C_1 and then C_2 (we can split up integrals by splitting up curves), so

$$\int_{C_2} z^2\,dz = \int_{\text{full circle } = C_1 + C_2} z^2\,dz - \int_{C_1} z^2\,dz = 0 - (-\tfrac{2}{3}) = \tfrac{2}{3}.$$

The integral of z^2 over the bottom half C_2 oriented clockwise instead would be $-\frac{2}{3}$.

The most crucial example of all time. We now determine the value of

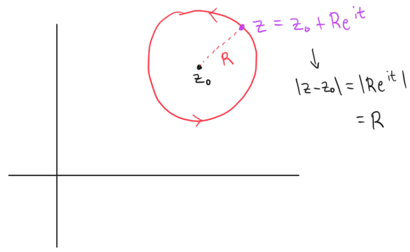
$$\int_{|z|=R} \frac{1}{z}\,dz$$

where the circle $|z| = R$ of radius R is oriented counterclockwise. We claim that this simple example is in fact the most important and crucial example out of all that we will consider as it leads to pretty much all the main applications of contour integration we will see. (I am not exaggerating here—this one example does essentially explain *much* of what we will see going forward!) If we parametrize the circle by $z(t) = Re^{it},\ 0 \leq t \leq 2\pi$, we get

$$\int_{|z|=R} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{Re^{it}}(iRe^{it})\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Such a simple computation, with such broad (to be seen) implications.

We get a similar outcome for circles centered elsewhere (still oriented counterclockwise) after modifying the denominator in $\frac{1}{z}$. The circle of radius R centered at z_0



is parametrized by

$$z(t) = z_0 + Re^{it},\ 0 \le t \le 2\pi$$

(adding z_0 to Re^{it} translates the previous circle centered at 0 so that its center is z_0 instead), so

$$\int_{|z-z_0|=R} \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{(z_0+Re^{it})-z_0}(iRe^{it})\,dt = \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Antiderivatives. If our integrand (i.e., the function we are integrating) has an antiderivative, the usual fundamental theorem of calculus works just the same:

$$\int_C f'(z)\,dz = f(\text{end point}) - f(\text{start point})$$

where $f(z)$ is an antiderivative of $f'(z)$. This comes from applying the real fundamental theorem of calculus after picking a parametrization: we have

$$\int_C f'(z)\,dz = \int_a^b f'(z(t))z'(t)\,dt = \int_a^b (f(z(t))'\,dt = f(z(b)) - f(z(a)) = f(\text{end}) - f(\text{start})$$

where $f'(z(t))z'(t) = (f(z(t))'$ from the chain rule. So for example, z^2 has (holomorphic) antiderivative $\frac{1}{3}z^3$, so for C the line segment from 0 to $1+i$ we have

$$\int_C z^2\,dz = \tfrac{1}{3}z^3\Big|_0^{1+i} = \tfrac{1}{3}(1+i)^3 = \tfrac{1}{3}(-2+2i) = -\tfrac{2}{3} + i\tfrac{2}{3},$$

which agrees with the answer we found before. For the upper-half C_1 of the unit circle oriented counterclockwise, we have

$$\int_{C_1} z^2\,dz = \tfrac{1}{3}z^3\Big|_1^{-1} = \tfrac{1}{3}(-1-1) = -\tfrac{2}{3}$$

just as before. For the full circle $|z|=1$ we get

$$\int_{|z|=1} z^2\,dz = 0$$

since the endpoint and start point are the same. More generally, $\int_{closed} f'(z)\,dz = 0$ for any closed curve and function $f'(z)$ with an antiderivative.

But we should be careful in applying this antiderivative technique. Consider again

$$\int_{|z|=1} \frac{1}{z}\,dz.$$

Since $\frac{1}{z}$ is the derivative of $\log z$, we might expect that

$$\int_{|z|=1} \frac{1}{z}\, dz = \log(\text{end}/\text{start}) - \log(\text{end}/\text{start}) = 0$$

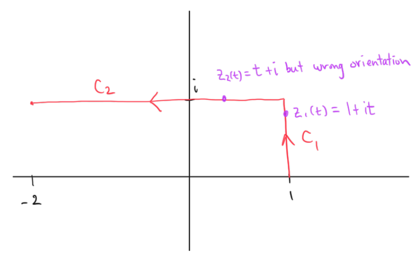
since the ending and starting points are the same, but we know that this is nonsense since we previously computed this integral to be $2\pi i$. The issue is that $\log z$ is not actually a valid antiderivative of $\frac{1}{z}$ on a domain which contains the entirety of the unit circle, since, as we know, there is no branch of log defined on such a domain. If we want a branch of log we have to make a cut, but doing so then necessarily excludes a point on the circle, so $\frac{1}{z}$ has no antiderivative over the entire circle at once. We will see next time, however, that there is a way to use $\log z$ as an antiderivative of $\frac{1}{z}$ in order to get the correct value of

$$\int_{|z|=1} \frac{1}{z}\, dz = 2\pi i,$$

only this will require using *different* branches of $\log z$ on different pieces of the circle.

Lecture 12: More on Integrals

Warm-Up 1. We compute $\int_C (z-1)e^{\overline{z}}\, dz$ where C is the contour consisting of the vertical line segment from 1 to $1+i$ followed by the horizontal line segment from $1+i$ to $-2+i$. We do so by splitting C into its two smooth pieces and use parametric equations for each:



One thing to note here is that trying to find an antiderivative so as to use the fundamental theorem of calculus is a fruitless endeavor: we will see soon enough that derivatives of holomorphic functions are themselves *always* holomorphic, so that the only functions which could potentially have antiderivatives are ones that are holomorphic to begin with; since $(z-1)e^{\overline{z}}$ is not holomorphic (because of the $\overline{z}$ term), it has no antiderivative.

For the vertical segment C_1 we use $z_1(t) = 1 + it$ for $0 \leq t \leq 1$. This gives

$$\begin{aligned}
\int_{C_1} (z-1)e^{\overline{z}}\, dz &= \int_0^1 (z_1(t) - 1)e^{\overline{z_1(t)}} z_1'(t)\, dt \\
&= \int_0^1 (it)e^{1-it} i\, dt \\
&= -e \int_0^1 te^{-it}\, dt \\
&= -e\left(\frac{te^{-it}}{-i}\bigg|_0^1 - \int_0^1 \frac{e^{-it}}{-i}\, dt \right) \\
&= -e\left(ie^{-i} + e^{-it}\bigg|_0^1 \right)
\end{aligned}$$

$$= -ie^{-i+1} - e^{-i+1} + e.$$

(Note we integrated by parts in the fourth step.)

For the horizontal segment C_2 we might try to use $z_2(t) = t + i$ for $-2 \leq t \leq 1$, but this gives the orientation opposite to the one we want. This we can correct for however by changing the sign of the resulting integral, since flipping the orientation changes the sign of the tangent vector $z'(t)$, which changes the sign of the integral. Thus if we denote C_2 with the opposite orientation by $-C_2$, $z_2(t)$ above parametrizes $-C_2$ and we have

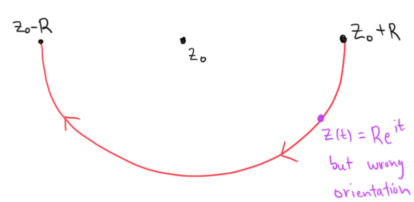
$$\begin{aligned}
\int_{C_2} (z-1)e^{\overline{z}}\,dz &= -\int_{-C_2} (z-1)e^{\overline{z}}\,dz \\
&= -\int_{-2}^{1} (z_2(t)-1)e^{\overline{z_2(t)}} z_2'(t)\,dt \\
&= -\int_{-2}^{1} (t+i-1)e^{t-i}\,dt \\
&= -\int_{-2}^{1} te^{t-i}\,dt - \int_{-2}^{1} (i-1)e^{t-i}\,dt \\
&= -e^{-i}\int_{-2}^{1} te^{t}\,dt - (i-1)e^{-i}\int_{-2}^{1} e^{t}\,dt \\
&= -e^{-i}\left(te^t\Big|_{-2}^{1} - e^t\Big|_{-2}^{1} \right) - (i-1)e^{-i}e^t\Big|_{-2}^{1} \\
&= -e^{-i}(3e^{-2}) - (i-1)e^{-i}(e - e^{-2}).
\end{aligned}$$

Thus altogether we get

$$\begin{aligned}
\int_C (z-1)e^{\overline{z}}\,dz &= \int_{C_1} (z-1)e^{\overline{z}}\,dz + \int_{C_2} (z-1)e^{\overline{z}}\,dz \\
&= -ie^{-i+1} - e^{-i+1} + e - e^{-i}(3e^{-2}) - (i-1)e^{-i}(e - e^{-2}).
\end{aligned}$$

(No need to simplify here to find a simpler value—highlighting the approach to take was the important part.)

Warm-Up 2. For fixed z_0, we determine the value of $\int_C (z-z_0)^n\,dz$ where C is the bottom half of the circle $|z - z_0| = R$ of radius R centered at z_0 oriented clockwise and n is a positive integer:



For a first approach, we note that $(z-z_0)^n$ has antiderivative $f(z) = \frac{(z-z_0)^{n+1}}{n+1}$ everywhere, so

$$\int_C (z-z_0)^n\,dz = f(z_0 - R) - f(z_0 + R) = \frac{(-R)^{n+1} - R^{n+1}}{n+1} = \frac{R^{n+1}[(-1)^n - 1]}{n+1}.$$

Note that this simplifies to 0 when n is even and to $-2R^{n+1}/(n+1)$ when n is odd.

Alternatively we can use a parametrization. We parametrize $-C$ (C with the opposite counterclockwise orientation) by $z(t) = z_0 + Re^{it},\ -\pi \le t \le 0$. Then

$$\begin{aligned}\int_C (z-z_0)^n\,dz &= -\int_{-C}(z-z_0)^n\,dz \\ &= -\int_{-\pi}^0 (Re^{it})^n(iRe^{it})\,dt \\ &= -i\int_{-\pi}^0 R^n e^{int}Re^{it}\,dt \\ &= -iR^{n+1}\int_{-\pi}^0 e^{i(n+1)t}\,dt \\ &= -\frac{iR^{n+1}e^{i(n+1)t}}{i(n+1)}\bigg|_{-\pi}^0 \\ &= -\frac{R^{n+1}-R^{n+1}e^{-i(n+1)\pi}}{n+1}.\end{aligned}$$

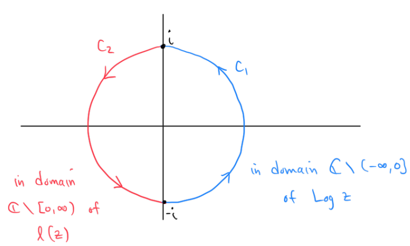
Since $e^{-i(n+1)\pi} = \cos((n+1)\pi) = (-1)^{n+1}$, this agrees with the previous computation.

Integrals and branches. Last time we computed

$$\int_{|z|=1}\frac{1}{z}\,dz = 2\pi i$$

using a parametrization and noted that it is not possible to compute this using an antiderivative as is since $\frac{1}{z}$ does not have an antiderivative on a domain containing the entire circle $|z| = 1$ since $\log z$ has no branch on such a domain. (As a matter of convention, let us assume that when left unspecified we are always taking the orientation of a *closed* curve to be the counterclockwise one, which we refer to as the *positive* orientation. To emphasize that an integral is being taken over a closed curve, we often use the notation $\oint$ for the integral, so the result above is that $\oint_{|z|=1}\frac{1}{z}\,dz = 2\pi i$.)

But it is possible to use antiderivatives to compute the value of this integral as long as we allow ourselves to use different branches of log. Let C_1 denote the right half of the circle and C_2 the left half. Then C_1 does lie within the domain of the principal branch of $\log z$ and C_2 lies within the domain of the branch $\ell(z)$ defined by $0 < \arg z < 2\pi$:



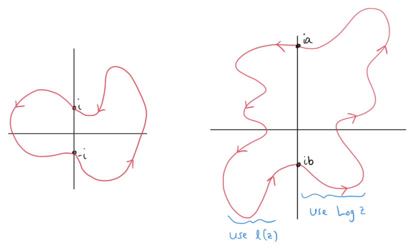
Thus, we can compute the integrals over C_1 and C_2 separately using a branch of log as an antiderivative of $\frac{1}{z}$, and then add the results. We have

$$\begin{aligned}\oint_{|z|=1}\frac{1}{z}\,dz &= \int_{C_1}\frac{1}{z}\,dz + \int_{C_2}\frac{1}{z}\,dz \\ &= [\mathrm{Log}(i) - \mathrm{Log}(-i)] + [\ell(-i) - \ell(i)]\end{aligned}$$

$$= [i\tfrac{\pi}{2} - (-i\tfrac{\pi}{2})] + [i\tfrac{3\pi}{2} - i\tfrac{\pi}{2}]$$
$$= i\pi + i\pi$$
$$= 2\pi i,$$

which is indeed the correct value.

Integrating $\frac{1}{z}$ over other curves. Since the computation above only uses the values of the branches Log and ℓ at $\pm i$, we get the same integral value $2\pi i$ over *any* positively-oriented simple closed contour enclosing 0 which crosses the imaginary axis at $\pm i$:



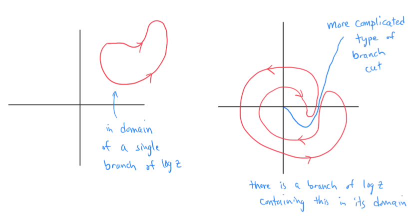
(To be *simple* means that the contour does not intersect itself except for at the point where it closes back up.) More generally, there is nothing special about $\pm i$ and this works regardless of the intersections with the imaginary axis as long as one is on the positive imaginary axis and one on the negative axis: if C is a simple closed contour intersecting the imaginary axis at ia and ib with $a > 0$ and $b < 0$, then

$$\mathrm{Log}(ia) = \ln a + i\tfrac{\pi}{2},\ \mathrm{Log}(ib) = \ln|b| - i\tfrac{\pi}{2} \quad \text{and} \quad \ell(ia) = \ln a + i\tfrac{\pi}{2},\ \ell(ib) = \ln|b| + i\tfrac{3\pi}{2}$$

where Log, ℓ are the same branches as before, so

$$\oint_C \frac{1}{z}\,dz = \int_{\text{left part}} \frac{1}{z}\,dz + \int_{\text{right part}} \frac{1}{z}\,dz$$
$$= [\mathrm{Log}(ia) - \mathrm{Log}(ib)] + [\ell(ib) - \ell(ia)]$$
$$= [\ln a - \ln|b| + i\tfrac{\pi}{2} - (-i\tfrac{\pi}{2})] + [\ln|b| - \ln a + i\tfrac{3\pi}{2} - i\tfrac{\pi}{2}]$$
$$= i\pi + i\pi$$
$$= 2\pi i.$$

If C does not enclose 0, then C lies fully within the domain of a single branch of log, so $\int_C \frac{1}{z}\,dz = 0$ since we take the value of that common branch at the end point minus the same start point. (This works even for more complicated curves that "wrap around" as long as they do not enclose the origin, since it turns out a branch of log *can* be defined everywhere along such a curve by allowing more complicated branch cuts than just rays:



In summary, we have

$$\oint_C \frac{1}{z}\,dz = \begin{cases} 2\pi i & \text{if } C \text{ encloses } 0 \\ 0 & \text{if } C \text{ does not enclose } 0 \end{cases}$$

for any positively-oriented simple closed contour.

Deriving real integrals. It is natural to wonder about what it is that contour integrals actually compute. Do they compute area? volume? something else? In fact, the answer is that we really will not care so much about such interpretations. (However, note at least that the values of $\oint_C \frac{1}{z}\,dz$ above *do* encode some geometric information, namely whether or not C encloses the origin. We will explore this idea at the end of the quarter.) Instead, a main use of contour integrals comes from their use in clarifying and simplifying computations involving *real* integrals, and such computations will play an important role going forward.

For example, we claim that for n a positive integer we have

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t\,dt = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}.$$

Here, $\cos^{2n}(t)$ means $(\cos t)^{2n}$ and the right side uses odd integers in the numerator and even ones in the denominator. The point is that this integral involves a purely real expression, and yet we will compute it by complex-analytic means. Computing this using only real methods is possible using some trig identities, but this ends up being more work and is less enlightening than the approach we take here. We will derive the value of this integral by considering the contour integral

$$\oint_{|z|=1} \frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} dz.$$

After expanding, we get

$$\left(z+\frac{1}{z}\right)^{2n} = z^{2n} + c_{2n-1}z^{2n-1} + c_{2n-2}z^{2n-2} + \cdots + c_1 z + c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots + \frac{1}{z^{2n}}$$

for some coefficients c_i. (Note the notation we use for the index i matches the power of z to which it corresponds.) The point is that every term in the product

$$\left(z+\frac{1}{z}\right)\left(z+\frac{1}{z}\right)\cdots\left(z+\frac{1}{z}\right)$$

involves multiplying a power of z with a power of $\frac{1}{z}$, so that we only get powers of z (including negative powers) as a result with coefficients coming from the number of times each power appears in the end. (For example, there is only way to get z^{2n} from the product above since this comes from picking z instead of $\frac{1}{z}$ from each factor. Actually, there will be no odd powers because there are an even number of factors in the product $(z+\frac{1}{z})^{2n}$, but this will not be so important for us.) This then gives

$$\frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} = z^{2n-1} + c_{2n-1}z^{2n-2} + \cdots + c_1 + \frac{c_0}{z} + c_{-1} + \frac{c_{-2}}{z} + \cdots + \frac{1}{z^{2n+1}},$$

so in order to integrate $\frac{1}{z}(z+\frac{1}{z})$ over the unit circle we simply need to integrate each term above over the circle.

Here's the magic! Each term in the expansion above except for the $\frac{c_0}{z}$ term has an antiderivative over the entire circle since

$$\frac{d}{dz}\left(\frac{z^{k+1}}{k+1}\right) = z^k \text{ for } k \neq -1.$$

Thus, since we are integrating over a closed curve, the integral of each of this terms is zero, so all we are left with is

$$\oint_{|z|=1} \frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} dz = \oint_{|z|=1} \frac{c_0}{z}\, dz,$$

whose value we know: $c_0 \oint_{|z|=1} \frac{1}{z}\, dz = c_0 2\pi i$. (We will come back to the exact value of c_0 in a bit.) On the other hand, using the parametrization $z(t) = e^{it},\ 0 \le t \le 2\pi$ gives

$$\oint_{|z|=1} \frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} dz = \int_0^{2\pi} e^{-it}(e^{it}+e^{-it})^{2n} ie^{it}\, dt$$

where we use $\frac{1}{e^{it}} = e^{-it}$. Since $e^{-it}e^{it} = 1$ and $e^{it}+e^{-it} = 2\cos t$, this simplifies to

$$\oint_{|z|=1} \frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} dz = \int_0^{2\pi} i(2\cos t)^{2n}\, dt = i2^{2n}\int_0^{2\pi} \cos^{2n} t\, dt.$$

Thus, comparing the two values we got for the integral of $\frac{1}{z}(z+\frac{1}{z})^{2n}$ over the unit circle gives

$$i2^{2n}\int_0^{2\pi} \cos^{2n} t\, dt = c_0 2\pi i \implies \frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t\, dt = \frac{c_0}{2^{2n}}.$$

If we backtrack, we see that c_0 is the constant term in the original expansion

$$\left(z+\frac{1}{z}\right)^{2n} = z^{2n} + \cdots + c_1 z + c_0 + \frac{c_{-1}}{z} + + \cdots + \frac{1}{z^{2n}}.$$

This is the coefficient of z^0, which appears whenever we take n z's and n $\frac{1}{z}$'s in

$$\underbrace{\left(z+\frac{1}{z}\right)\left(z+\frac{1}{z}\right)\cdots\left(z+\frac{1}{z}\right)}_{2n \text{ times}}.$$

The number of ways of making such a choice is "$2n$ choose n", which is

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}.$$

(No need to recall what this is, it is something you would be given in the setup to a problem if needed.) Hence,

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t\, dt = \frac{(2n)!}{2^{2n} n!n!}.$$

If we write the denominator as $(2^n n!)(2^n n!)$, each factor is a product of even integers

$$2^n n! = 2^n(1\cdot 2\cdot 3\cdots n) = 2\cdot 4\cdot 6\cdots 2n,$$

so in the fraction

$$\frac{(2n)!}{2^{2n} n!n!} = \frac{(2n)!}{(2^n n!)(2^n n!)}$$

all even terms in the numerator cancel with one factor in the denominator, and we are left with only odd factors in the numerator and even ones in the denominator:

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t\, dt = \frac{1\cdot 3\cdot 5\cdots (2n-1)}{2\cdot 4\cdot 6\cdots 2n}$$

as the desired value. (The manipulations we did to get down to this simplified expression are not important, what matters is the fact that we computed $\oint_{|z|=1} \frac{1}{z}(z+\frac{1}{z})^{2n}\, dz$ in two ways in order to derive a real integral value. This is an idea we will see time and again in the remaining weeks.)

Lecture 13: Cauchy's Theorem

Warm-Up. Let us take it as given for now that

$$\oint_{|z|=1} \frac{z}{z^2+4z+1}\,dz = 2\pi i\left(\frac{-2+\sqrt{3}}{2\sqrt{3}}\right).$$

(We will see how to easily determine this value with minimal computation later this week using what's called *Cauchy's integral formula*; for now just note that $-2+\sqrt{3}$ is one of the roots of z^2+4z+1 via the quadratic formula.) Using this, we derive the values of the real integrals

$$\int_0^{2\pi} \frac{\cos t}{4+2\cos t}\,dt \quad \text{and} \quad \int_0^{2\pi} \frac{\sin t}{4+2\cos t}\,dt.$$

Using the parametrization $z = e^{it}, 0 \le t \le 2\pi$, we get

$$\oint_{|z|=1} \frac{z}{z^2+4z+1}\,dz = \int_0^{2\pi} \frac{e^{it}}{e^{2it}+4e^{it}+1} ie^{it}\,dt.$$

Multiplying by $\frac{e^{-it}}{e^{-it}}$ turns this into

$$\int_0^{2\pi} \frac{e^{it}}{e^{2it}+4e^{it}+1} ie^{it}\,dt = i\int_0^{2\pi} \frac{e^{it}}{e^{it}+4+e^{-it}}\,dt.$$

Now, the denominator is

$$(\cos t + i\sin t) + 4 + (\cos t - i\sin t) = 4 + 2\cos t,$$

so our integral is

$$\oint_{|z|=1} \frac{z}{z^2+4z+1}\,dz = i\int_0^{2\pi} \frac{\cos t + i\sin t}{4+2\cos t}\,dt = i\int_0^{2\pi} \frac{\cos t}{4+2\cos t}\,dt - \int_0^{2\pi} \frac{\sin t}{4+2\cos t}\,dt.$$

Comparing the real and imaginary parts of this integral with those of

$$\oint_{|z|=1} \frac{z}{z^2+4z+1}\,dz = 2\pi i\left(\frac{-2+\sqrt{3}}{2\sqrt{3}}\right)$$

thus gives

$$\int_0^{2\pi} \frac{\cos t}{4+2\cos t}\,dt = 2\pi\left(\frac{-2+\sqrt{3}}{2\sqrt{3}}\right) \quad \text{and} \quad \int_0^{2\pi} \frac{\sin t}{4+2\cos t}\,dt = 0.$$

Cauchy's theorem. We come now to one of the most fundamental results in complex analysis, second only to the Cauchy-Riemann equations in importance: *Cauchy's theorem.* Cauchy's theorem gives the exact value of a wide class of integrals, no computation required. Here is the statement: if f is holomorphic on a domain containing a simple closed contour C and its interior, then

$$\oint_C f(z)\,dz = 0.$$

By the "interior" of C we mean the region enclosed by C, so the result is that as long as the function we are integrating is differentiable on C and the region enclosed by C, the integral will have value zero. A domain D with the property that it contains the interiors of all curves contained within it is said to be *simply-connected*, so we can phrase the statement of Cauchy's theorem as saying that if f is holomorphic on a simply-connected domain, then $\oint_C f(z)\,dz = 0$ for any simple closed contour C in that domain.

For example,

$$\oint_C e^{z^2}\,dz = 0, \text{ and } \oint_C \cos(e^z)\,dz = 0$$

for any simple closed curve C. (These integrands are entire, so the hypothesis of Cauchy's theorem is satisfied everywhere.) Certainly we already know that a function which has an antiderivative will have this integral zero property (since we can evaluate the integral by plugging the common end/start point into the antiderivative and subtracting), but the point is that we get zero even for functions which we do not yet know have antiderivative, such as those above. (The fact that all holomorphic functions do have antiderivatives will be a consequence of Cauchy's theorem.)

The function $\frac{1}{z}$ is not holomorphic on a simply-connected domain containing the unit circle (since it is not differentiable at 0, which is enclosed by the unit circle), so Cauchy's theorem does not apply to $\oint_{|z|=1} \frac{1}{z}\,dz$, which we know has value $2\pi i$ and not zero. (However, Cauchy's theorem *does* apply to $\frac{1}{z}$ over any simple closed contour that does not enclose the origin, such as say a circle of radius 1 centered at $1+i$, since $\frac{1}{z}$ is holomorphic on such contours and their interiors. The integral over such a curve not enclosing the origin is zero by Cauchy's theorem, which matches what we said previously about integrals of $\frac{1}{z}$.) The integral in the Warm-Up

$$\oint_{|z|=1} \frac{z}{z^2+4z+1}\,dz = 2\pi i\left(\frac{-2+\sqrt{3}}{2\sqrt{3}}\right)$$

is another to which Cauchy's theorem does not apply since the integrand fails to be differentiable at $-2+\sqrt{3}$ (a root of the denominator), which is in the unit disk enclosed by the unit circle. (But, Cauchy's theorem still plays a key role in finding the actual value of this integral above, as we will see.) The takeaway is that it is crucial that f be differentiable not only at points on the curve C, but also at all points it encloses.

Proof when derivative is continuous. Here we justify Cauchy's theorem, at least in the case where $f' = u_x + iv_x$ is continuous on the domain in question. We will see later that this assumption is superfluous in that f' is always continuous if f is holomorphic, but this fact will depend on knowing that Cauchy's theorem is true without this assumption beforehand. But, this general setup without assuming continuity of f' takes more work to justify, which is not going to be essential for us to understand in this course. We will say a bit about his general setup after this first proof in the f' continuous case, but this will be purely optional material for the sake of those who are interested in learning more.

If $f' = u_x + iv_x$ is continuous, then so are u_y and v_y by the Cauchy-Riemann equations. If we pick a parametrization $z(t) = x(t) + iy(t), a \le t \le b$ for C, our desired integral is

$$\begin{aligned}\oint_C f(z)\,dz &= \int_a^b (u+iv)(x'+iy')\,dt \\ &= \int_a^b (ux'-vy')\,dt + i\int_a^b (uy'+vx')\,dt.\end{aligned}$$

Here we are suppressing the points at which u, v, x', y' are being evaluated for the sake of clearer notation, and the second line comes from multiplying out $(u+iv)(x'+iy')$ and separating real and imaginary parts. Now, we write the resulting integrands as dot products of a vector with the tangent vector (x', y'):

$$\int_a^b (ux' - vy')\,dt + i\int_a^b (uy' + vx')\,dt = \int_a^b (u, -v)\cdot(x', y')\,dt + i\int_a^b (v, u)\cdot(x', y')\,dt.$$

The point is that the two integrals on the right are examples of *line integrals* from multivariable calculus, with the first being the line integral of the *vector field* $(u, -v)$ over C and the second that of the vector field (v, u). Since C is closed *and* the partial derivatives of u and v are continuous (here is where this assumption is needed), *Green's theorem* from multivariable calculus allows us to write these line integrals as *double integrals* over the region D enclosed by C:

$$\int_a^b (u, -v)\cdot(x', y')\,dt = \iint_D \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right) dA = \iint_D (-v_x - u_y)\,dA$$

and

$$\int_a^b (v, u)\cdot(x', y')\,dt = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA = \iint_D (u_x - v_y)\,dA.$$

But the Cauchy-Riemann equations then give that both of these resulting double integrals are zero: we have $u_x - v_y = 0$ and $-v_x - u_y = 0$, so

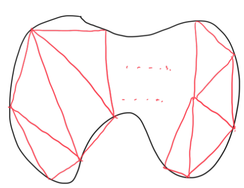
$$\oint_C f(z)\,dz = \iint_D (-v_x - u_y)\,dA + i\iint_D (u_x - v_y)\,dA = \iint_d 0\,dA + i\iint_D 0\,dA = 0$$

as desired.

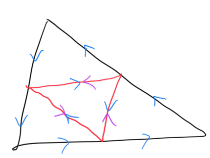
This approach to Cauchy's theorem in the f' continuous case highlights the importance of the Cauchy-Riemann equations in the final step. It also serves to hint at the geometric meaning behind these equations: Green's theorem is ultimately a result about notions of "curl" and "divergence", which have something to do with "rotation" and "scaling" respectively, which is what "holomorphic" (with nonzero derivative) should relate to geometrically.

Proof in general case. For the sake of completeness and independence from the continuous f' assumption, let us talk about why Cauchy's theorem works in general. As mentioned above, this is purely optional material and understanding the details is not something for which you will be responsible—which is why we did not do it in class—but it is important if we care about deriving results from Cauchy's theorem in the most general way possible. A second benefit is that the justification we gave above in the f' continuous case essentially is a "real analytic" approach since in the end we convert to real double integrals and apply the real Green's theorem; the only place holomorphicity really appears is at the end in the form of the Cauchy-Riemann equations. But Cauchy's theorem should really be viewed as truly a "complex analytic" result independent of anything strictly real, so there should be a proof making use of only complex analysis. (The issue, as we will see, is that the purely complex proof is much more involved than what we saw above.)

In fact, we only give the full details in the case where the contour C is a triangle; the general case then comes essentially from approximating general simple closed contours by triangles:



approximate general contour by triangles



divide triangle into 4 smaller pieces by connecting midpoints

Take a triangle T and divide it into four smaller triangles by connecting midpoints of sides, as in the second picture above. The integral of f over T then breaks up as the sum of integrals over these four smaller triangles:

$$\int_T f(z)\,dz = \int_{first} f(z)\,dz + \int_{second} f(z)\,dz + \int_{third} f(z)\,dz + \int_{fourth} f(z)\,dz.$$

(Note that the edges of these smaller triangles lying in the interior of the original T occur twice in the sum above—one for each smaller triangle of which it is an edge—but with opposite orientations, which is why adding the four things on the right together leaves only the contributions from the original T.) At least one of the four integral values on the right must have modulus at least as large as $\frac{1}{4}$ times that of $\int_T f(z)\,dz$, since otherwise adding together the modulus of the four things on the right would give something strictly smaller than $|\int_T f(z)\,dz|$. Denote by T_1 the smaller triangle for which $|\int_{T_1} f(z)\,dz|$ is at least as large as $\frac{1}{4}|\int_T f(z)\,dz|$:

$$\left|\int_{T_1} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_T f(z)\,dz\right|.$$

Note also that the length (i.e., perimeter) of T_1 is exactly half the length/perimeter of T, which follows from the fact that midpoints were used in constructing the smaller triangles.

Now do the same thing with T_1 as the new triangle: divide it into four smaller triangles by connecting midpoints of sides, and pick the smaller triangle T_2 for which $|\int_{T_2} f(z)\,dz|$ is at least as large as $\frac{1}{4}|\int_{T_1} f(z)\,dz|$, so that together with the bound above we get

$$\left|\int_{T_2} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_{T_1} f(z)\,dz\right| \geq \frac{1}{4^2}\left|\int_T f(z)\,dz\right|.$$

The length of T_2 is half that of T_1. Keep going, at each step finding ever smaller triangles T_n such that

$$\left|\int_{T_n} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_{T_{n-1}} f(z)\,dz\right| \geq \frac{1}{4^2}\left|\int_{T_{n-2}} f(z)\,dz\right| \geq \ldots \geq \frac{1}{4^n}\left|\int_T f(z)\,dz\right|$$

and for which

$$\text{length}(T_n) = \frac{1}{2}\,\text{length}(T_{n-1}) = \frac{1}{2^n}\,\text{length}(T_{n-2}) = \cdots = \frac{1}{2^n}\,\text{length}(T).$$

Pick a point z_0 that lies in the interior of all the triangles T_n thus constructed. (There is such a point because the interior of each triangle T_n is contained in the interior of the previous T_{n-1}, so the triangles "shrink down" to a point z_0.) Using the linear approximation approach to differentiability we mentioned awhile back, since f is differentiable at z_0 we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z),$$

where the "error/remainder" term $\epsilon(z)$ satisfies

$$\lim_{z\to z_0} \frac{\epsilon(z)}{z - z_0} = 0.$$

Since $f(z_0)$ (a constant) and $f'(z_0)(z - z_0)$ (constant times z minus constant) each have antiderivatives, they integrate to zero over the closed triangle T_n, so

$$\int_{T_n} f(z)\,z = \int_{T_n} [f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)]\,dz = \int_{T_n} \epsilon(z)\,dz.$$

Denote by M_n the maximum value of $|\frac{\epsilon(z)}{z-z_0}|$ among points on the triangle T_n; since the fraction $\frac{\epsilon(z)}{z-z_0}$ approaches 0 as $z \to z_0$, these maximum values approach 0 as well. We have

$$\left|\frac{\epsilon(z)}{z-z_0}\right| \leq M_n, \text{ so } |\epsilon(z)| \leq M_n|z-z_0| \text{ for } z \text{ on } T_n.$$

We now use that fact that we can bound contour integrals by bounding the integrand:

$$\left|\int_{T_n} \epsilon(z)\,dz\right| \leq \int_{T_n} |\epsilon(z)||dz|.$$

We will say more about this expression after this proof since it is useful in its own right and will be important in various computations we will carry out as we go. In the case at hand, we use the bound on $\epsilon(z)$ in terms of M_n we derived above to get

$$\left|\int_{T_n} \epsilon(z)\,dz\right| \leq \int_{T_n} |\epsilon(z)||dz| \leq \int_{T_n} M_n|z-z_0||dz|.$$

The distance $|z - z_0|$ from any point on T_n to z_0 in the interior of T_n is never larger than the full length/perimeter of T_n, so

$$\left|\int_{T_n} \epsilon(z)\,dz\right| \leq \int_{T_n} M_n|z-z_0||dz| \leq \int_{T_n} M_n \operatorname{length}(T_n)|dz| = M_n \operatorname{length}(T_n) \int_{T_n} |dz|.$$

The remaining integral is the length of T_n (see the discussion on integral bounds that follows this proof), so we finally have

$$\left|\int_{T_n} f(z)\,dz\right| = \left|\int_{T_n} \epsilon(z)\,dz\right| \leq M_n \operatorname{length}(T_n) \int_{T_n} |dz| = M_n \operatorname{length}(T_n)^2.$$

Putting this together with what we had previously, where

$$\left|\int_{T_n} f(z)\,dz\right| \geq \frac{1}{4^n}\left|\int_T f(z)\,dz\right| \quad \text{and} \quad \operatorname{length}(T_n) = \frac{1}{2^n}\operatorname{length}(T),$$

gives

$$\left|\int_T f(z)\,dz\right| \leq 4^n\left|\int_{T_n} f(z)\,dz\right| \leq 4^n M_n \operatorname{length}(T_n)^2 = 4^n M_n \left(\frac{1}{2^n}\right)^2 \operatorname{length}(T) = M_n \operatorname{length}(T).$$

As n increases we have $M_n \to 0$, so the right side approaches 0, and hence so does the left side. But the left side is constant since it is independent of n, so we must have

$$\left|\int_T f(z)\,dz\right| = 0, \text{ so } \int_T f(z)\,dz = 0,$$

which is Cauchy's theorem for a triangle. (Breathe!)

Integral bounds. The argument above is indeed quite involved and optional, but the part about bounding integrals is worth clarifying. The claim is that

$$\left|\int_C f(z)\,dz\right| \leq \int_C |f(z)||dz|.$$

Now, the left side is the modulus of the complex number which equals the value of the integral $\int_C f(z)\,dz$, and on the right side $|f(z)|$ is the modulus of the function we are integrating and $|dz|$ is mean to denote the modulus/length of tangent vectors along the curve. To be precise, once we pick some parametrization $z(t), a \le t \le b$ for C, we get

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right|.$$

A basic fact about integrals like this taken with respect to a real parameter t is that bringing the "absolute value"/modulus term inside can only make the value larger but never smaller, so

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right| \le \int_a^b |f(z(t))||z'(t)|\,dt.$$

(You might have seen this property for real integrals before, but it also holds for complex integrals with a real parameter. We will not give the justification for this here, but it is in in the book if you are interested in seeing them.)

If we bound $|f(z(t))|$ by, say, some constant M, we get

$$\int_a^b |f(z(t))||z'(t)|\,dt \le \int_a^b M|z'(t)|\,dt = M\int_a^b |z'(t)|\,dt.$$

On the right side, $z'(t)$ describes a tangent vector to C and $|z'(t)|$ is its length, and you might recall from multivariable calculus that integrating the length of tangent vectors gives the total length (also called arclength) of the curve:

$$\int_a^b |z'(t)|\,dt = \text{length}(C).$$

This integral is what denote by $\int_C |dz|$ (i.e., it is the integral of the length of the tangent vector piece dz), so

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)||dz|$$

says that we can bound the size (i.e., modulus) of an integral by bounding the integrand $f(z)$ and controlling the length of C. At times we will want to bound $|f(z)|$ by a constant, and at other times by the modulus $|g(z)|$ of a function $g(z)$ that is simpler to work with, but the end result—finding a way to control how large an integral can be—will be the same.

Example. We finish by giving an example of the typical type of computation that Cauchy's theorem and its consequences will now allow us to carry out. This falls within the strategy of using contour integrals to derive results about real integrals, where Cauchy's theorem will tell us what the value of the contour integral should be.

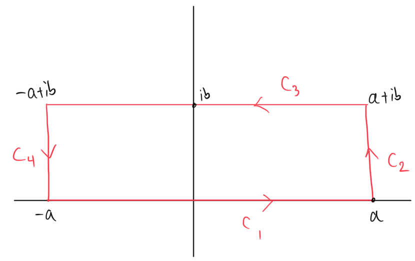
We derive the fact that

$$\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt = \sqrt{\pi}e^{-b^2}$$

for $b > 0$. This is an *improper* real integral, which if you recall is defined as the limit of integrals over bounded intervals as we allow the size of those intervals to increase:

$$\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt = \lim_{a\to\infty}\int_{-a}^{a} e^{-t^2}\cos(2bt)\,dt.$$

The point is that we will obtain this value after taking limits in a contour integral. Specifically, we take C to be the rectangle with vertices at $\pm a, \pm a + ib$ oriented counterclockwise



and consider the contour integral

$$\oint_C e^{-z^2}\,dz.$$

On the one hand, the value of this integral is zero by Cauchy's theorem since e^{-z^2} is entire, so it is differentiable everywhere on C and on the region C encloses. On the other hand, we can split this integral into four pieces taking place over the four sides of C:

$$0 = \oint_C e^{-z^2}\,dz = \int_{C_1} e^{-z^2}\,dz + \int_{C_2} e^{-z^2}\,dz + \int_{C_3} e^{-z^2}\,dz + \int_{C_4} e^{-z^2}\,dz.$$

The integral over the bottom side C_1 lying on the real axis is just the usual real integral of e^{-t^2} from $-a$ to a: parametrizing C_1 by $z(t) = t, -a \le t \le a$ (so that $z'(t) = 1$) gives

$$\int_{C_1} e^{-z^2}\,dz = \int_{-a}^{a} e^{-t^2}\,dt.$$

(Later we will take the limit as our rectangle gets longer using $a \to \infty$, in which case this piece becomes the improper integral $\int_{-\infty}^{\infty} e^{-t^2}\,dt$, which has a known value we will recall at that time.)

Now, consider the integral over the vertical segment C_2. Rather than compute this integral directly, it will instead be enough to determine how large (in modulus) it could be and argue that in the limit as $a \to \infty$ this integral will become zero anyway. (Keep this idea in mind going forward—it is *very* common.) We use the bound

$$\left|\int_{C_2} e^{-z^2}\,dz\right| \le \int_{C_2} |e^{-z^2}||dz|$$

mentioned before. The modulus $|e^{-z^2}|$ among points on the line segment from a to $a+ib$ has maximum value $e^{b^2-a^2}$, which was indeed a problem on the last homework; this comes from parametrizing the segment as $z(t) = a + it, 0 \le t \le b$, computing

$$|e^{-z^2}| = |e^{-(a+it)^2}| = |e^{-(a^2-t^2)}e^{-2ait}| = e^{t^2-a^2},$$

and maximizing the result for $0 \le t \le b$. This gives

$$\left|\int_{C_2} e^{-z^2}\,dz\right| \le \int_{C_2} |e^{-z^2}||dz| \le \int_{C_2} e^{b^2-a^2}|dz| = e^{b^2-a^2}\int_{C_2} |dz|.$$

The remaining integral is just the length of the vertical segment C_2, which is b, so

$$\left|\int_{C_2} e^{-z^2}\,dz\right| \le e^{b^2-a^2}\int_{C_2} |dz| = be^{b^2-a^2}.$$

The point is that once we take $a \to \infty$, this expression will go to zero, so it will not contribute to the overall value of the limit. The same thing happens over the vertical segment C_4 on the left side of the rectangle; the only difference here is that we parametrize by $-a+it$ instead of $a+it$, but this gives the same value for $|e^{-z^2}| = e^{t^2-a^2}$ as C_2 because the a term is squared, so the max value and length terms are the same. So, for neither $\int_{C_2} e^{-z^2}\,dz$ nor $\int_{C_4} e^{-z^2}\,dz$ will we know the definite value, but this will not matter in the limit.

For the integral $\int_{C_3} e^{-z^2}\,dz$ over the remaining horizontal segment, we use the parametrization $z(t) = t+ib$, $-a \le t \le a$, only where we change the sign of the integral to correct for the orientation:

$$\begin{aligned}
\int_{C_3} e^{-z^2}\,dz &= -\int_{-C_3} e^{-z^2}\,dz \\
&= -\int_{-a}^{a} e^{-(t+ib)^2}\,dt \\
&= -\int_{-a}^{a} e^{-[(t^2-b^2)+i2bt]}\,dt \\
&= -\int_{-a}^{a} e^{-t^2}e^{b^2}e^{-2ibt}\,dt \\
&= -e^{b^2}\int_{-a}^{a} e^{-t^2}(\cos(2bt) - i\sin(2bt))\,dt. \\
&= -e^{b^2}\int_{-a}^{a} e^{-t^2}\cos(2bt)\,dt + e^{b^2}\int_{-a}^{a} e^{-t^2}\sin(2bt)\,dt.
\end{aligned}$$

The second integral in the result is zero since the integrand $e^{-t^2}\sin(2bt)$ is an odd function and the interval $[-a,a]$ of integration is symmetric about the origin, but let us not use this simplification and see in a bit that the result (in the limit) will be zero anyway. Note that the first integral above looks suspiciously (or not?) similar to the real integral whose value we seek to determine.

Putting everything together gives

$$\begin{aligned}
0 = \oint_C e^{-z^2}\,dz &= \int_{C_1} e^{-z^2}\,dz + \int_{C_2} e^{-z^2}\,dz + \int_{C_3} e^{-z^2}\,dz + \int_{C_4} e^{-z^2}\,dz \\
&= \int_{-a}^{a} e^{-t^2}\,dt + (\text{something bounded by } be^{b^2-a^2}) \\
&\quad - e^{b^2}\int_{-a}^{a} e^{-t^2}\cos(2bt)\,dt + e^{b^2}\int_{-a}^{a} e^{-t^2}\sin(2bt)\,dt + (\text{bounded by } be^{b^2-a^2}).
\end{aligned}$$

Now we take the limit as $a \to \infty$. Note that in order to do so we have to know that the left side $0 = \oint_C e^{-z^2}\,dz$ remains zero regardless of the length of the rectangle we are using, which is OK by Cauchy's theorem. Since $be^{b^2-a^2} = be^{b^2}e^{-a^2} \to 0$, the integrals over C_2 and C_4 that this bounds also go to zero (this is some kind of squeeze theorem application), so we get

$$0 = \int_{-\infty}^{\infty} e^{-t^2}\,dt + 0 - e^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt + e^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\sin(2bt)\,dt + 0.$$

Note right away that the left side has zero imaginary part, so the imaginary part of the right side must also be zero, and this gives the aforementioned

$$\int_{-\infty}^{\infty} e^{-t^2}\sin(2bt)\,dt = 0.$$

We are left with

$$0 = \int_{-\infty}^{\infty} e^{-t^2}\,dt - e^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt, \text{ so } \int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt = e^{-b^2}\int_{-\infty}^{\infty} e^{-t^2}\,dt.$$

The improper integral $\int_{-\infty}^{\infty} e^{-t^2}\,dt$ (an example of what's called a *Gaussian integral*, heavily used in probability and statistics) has the value $\sqrt{\pi}$. This is often an example computed in a multivariable integral calculus course, where the technique is to write the square of this integral has an improper double integral

$$\left(\int_{-\infty}^{\infty} e^{-t^2}\,dt\right)^2 = \int_{-\infty}^{\infty} e^{-t^2}\,dt \int_{-\infty}^{\infty} e^{-s^2}\,ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(t^2+s^2)}\,dt\,ds,$$

and then to convert to *polar* coordinates to find the value. (We will not go through these details here, but if you have never seen this computation before you should try it for yourself or ask in office hours!) Thus we get

$$\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt = e^{-b^2}\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}e^{-b^2}$$

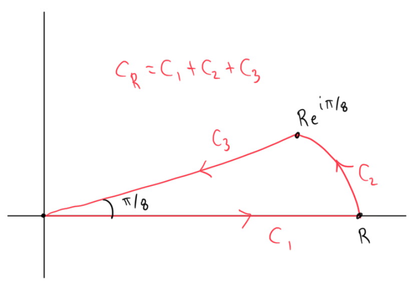
as our desired value. (Phew!) This example is indicative of the types of things we are working towards, where as we find "simple" ways of computing contour integrals to which Cauchy's theorem does not apply, we will expand on the types of real integrals we can determine via this method.

Lecture 14: Cauchy's Integral Formula

Warm-Up. We find the value of

$$\int_0^{\infty} e^{-t^2/\sqrt{2}}\cos\left(\tfrac{t^2}{\sqrt{2}}\right)dt$$

by integrating e^{-z^2} over the contour C_R consisting of the line segment on the real axis from 0 to R, the arc of the circle $|z| = R$ traversing angles from 0 to $\pi/8$, and the line segment from the end of this arc back to the origin:



As with the example at the end of last time, we will obtain the desired integral value above as a limit as $R \to \infty$ in these contour integrals. Since e^{-z^2} is holomorphic everywhere, Cauchy's theorem guarantees that

$$\oint_{C_R} e^{-z^2}\,dz = 0$$

regardless of how large R is, so this will remain true in the limit as $R \to \infty$ as well.

The integral of e^{-z^2} over the bottom segment C_1 of C_R is just the usual real integral

$$\int_{C_1} e^{-z^2}\,dz = \int_0^R e^{-t^2}\,dt.$$

The integral over the circular arc C_2 is not one we will compute directly, but rather we will find a way to bound it using

$$\left|\int_{C_2} e^{-z^2}\,dz\right| \le \int_{C_2} |e^{-z^2}||dz|.$$

The goal is to find such a bound that will go to 0 as $R \to \infty$, so that the integral of e^{-z^2} over C_2 will go to zero as well in the limit. To find a bound we must bound $|e^{-z^2}|$ among points on C_2. If we parametrize C_2 using $z(t) = Re^{it}, 0 \le t \le \frac{\pi}{8}$, we have

$$e^{-z(t)^2} = e^{-R^2 e^{2it}} = e^{-R^2(\cos 2t + i\sin 2t)}, \text{ so } |e^{-z(t)^2}| = e^{-R^2\cos 2t}.$$

If we think of this modulus as $\frac{1}{e^{R^2\cos 2t}}$, then to find a larger bound we must bound the denominator from *below* since making denominators smaller makes fractions larger. Thus, we need a *lower* bound on $\cos 2t$ for $0 \le t \le \frac{\pi}{8}$. For t in this range, $2t$ takes values between 0 and $\frac{\pi}{4}$, and the minimum value that cosine takes on in this range is $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ since cosine is decreasing over this range. So

$$\tfrac{1}{\sqrt{2}} \le \cos 2t \implies \tfrac{1}{e^{R^2\cos 2t}} \le \tfrac{1}{e^{R^2/\sqrt{2}}} \text{ for } 0 \le t \le \tfrac{\pi}{8}.$$

Thus we get

$$\left|\int_{C_2} e^{-z^2}\,dz\right| \le \int_{C_2} |e^{-z^2}||dz| \le \int_{C_2} e^{-R^2/\sqrt{2}}|dz| = e^{-R^2/\sqrt{2}}\textstyle\int_{C_2}|dz| = \frac{\pi R}{8}e^{-R^2/\sqrt{2}}$$

where $\int_{C_2}|dz| = \frac{\pi R}{8}$ is the length of C_2. (The length of an arc on a circle is the radius times the angle the arc subtends.) As $R \to \infty$, this $\frac{\pi R}{8}e^{-R^2/\sqrt{2}}$ bound will go to 0 as we wanted, as can be verified using L'Hopital's rule if nothing else. (The point is that e^{-R^2} goes to zero much more quickly than the rate at which the R term in $\frac{\pi R}{8}$ increases.) Note that we have to be mindful about the bounds we use in order to get this. It is also true that

$$-1 \le \cos 2t \text{ for } 0 \le t \le \tfrac{\pi}{8}, \text{ or } 0 \le \cos 2t \text{ for } 0 \le t \le \tfrac{\pi}{8},$$

but these give $|e^{-z^2}| = e^{-R^2\cos 2t} \le e^{-R^2(-1)} = e^{R^2}$ and $|e^{-z^2}| = e^{-R^2\cos 2t} \le e^0 = 1$ respectively, neither of which will go to zero when multiplied by $\frac{\pi 8}{R}$. We will get used to figuring out what types of bounds to use as we work through more examples.

For the integral over the remaining segment C_3, we use the parametrization $z(t) = te^{i\pi/8}$ (fixed argument but varying modulus) for $0 \le t \le R$, only that we have to correct for the orientation with a negative sign:

$$\begin{aligned}
\int_{C_3} e^{-z^2}\,dz &= -\int_0^R e^{-z(t)^2} z'(t)\,dt \\
&= -\int_0^R e^{-t^2 e^{i\pi/4}} e^{i\pi/8}\,dt \\
&= -e^{i\pi/8}\int_0^R e^{-t^2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})}\,dt \\
&= -e^{i\pi/8}\int_0^R e^{-t^2/\sqrt{2}} e^{-it^2/\sqrt{2}}\,dt \\
&= -e^{i\pi/8}\int_0^R e^{-t^2/\sqrt{2}}(\cos\tfrac{t^2}{\sqrt{2}} - i\sin\tfrac{t^2}{\sqrt{2}})\,dt.
\end{aligned}$$

(Hopefully it is clear now how we will be able to derive the value of an integral involving $e^{-t^2/\sqrt{2}}\cos(\frac{t^2}{\sqrt{2}})$ from this!)

Putting it all together gives

$$\begin{aligned} 0 = \oint_{C_R} e^{-z^2}\,dz = \int_{C_1} e^{-z^2}\,dz + \int_{C_2} e^{-z^2}\,dz + \int_{C_3} e^{-z^2}\,dz \\ = \int_0^R e^{-t^2}\,dt + (\text{something bounded in modulus by } \tfrac{\pi R}{8}e^{-R^2/\sqrt{2}}) \\ - e^{i\pi/8}\int_0^R e^{-t^2/\sqrt{2}}(\cos\tfrac{t^2}{\sqrt{2}} - i\sin\tfrac{t^2}{\sqrt{2}})\,dt \end{aligned}$$

for any $R > 0$. Taking $R \to \infty$ then gives

$$0 = \int_0^\infty e^{-t^2}\,dt + 0 + -e^{i\pi/8}\int_0^\infty e^{-t^2/\sqrt{2}}(\cos\tfrac{t^2}{\sqrt{2}} - i\sin\tfrac{t^2}{\sqrt{2}})\,dt.$$

The first integral has value $\frac{\sqrt{\pi}}{2}$ (it is half of the Gaussian integral $\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$ from last time since the integrand is even), so

$$0 = \tfrac{\sqrt{\pi}}{2} - e^{i\pi/8}\textstyle\int_0^R e^{-t^2/\sqrt{2}}(\cos\tfrac{t^2}{\sqrt{2}} - i\sin\tfrac{t^2}{\sqrt{2}})\,dt,$$

and thus

$$\int_0^\infty e^{-t^2/\sqrt{2}}(\cos\tfrac{t^2}{\sqrt{2}} - i\sin\tfrac{t^2}{\sqrt{2}})\,dt = \tfrac{\sqrt{\pi}}{2}e^{-i\pi/8} = \tfrac{\sqrt{\pi}}{2}(\cos\tfrac{\pi}{8} - i\sin\tfrac{\pi}{8}).$$

Taking real parts gives

$$\int_0^\infty e^{-t^2/\sqrt{2}}\cos(\tfrac{t^2}{\sqrt{2}})\,dt = \tfrac{\sqrt{\pi}}{2}\cos\tfrac{\pi}{8}$$

as our desired value! (Taking imaginary parts will give you the value of $\int_0^\infty e^{t^2/\sqrt{2}}\sin(\frac{t^2}{\sqrt{2}})\,dt$.) If you happen to know the value of $\cos\frac{\pi}{8}$, you can write this as

$$\int_0^\infty e^{-t^2/\sqrt{2}}\cos(\tfrac{t^2}{\sqrt{2}})\,dt = \tfrac{\sqrt{\pi}}{2}\cos\tfrac{\pi}{8} = \tfrac{\sqrt{\pi}}{2}(\tfrac{1}{2}\sqrt{2+\sqrt{2}}) = \tfrac{\sqrt{\pi}}{4}\sqrt{2+\sqrt{2}}.$$

Making a change of variables $u = \frac{t}{2^{1/4}}$ turns this into

$$\int_0^\infty e^{-u^2}\cos(u^2)\,du = \tfrac{\sqrt{\pi}}{2\cdot 2^{1/4}}\sqrt{2+\sqrt{2}} = \tfrac{\sqrt{\pi}}{2^{1/4}}\sqrt{1+\sqrt{2}},$$

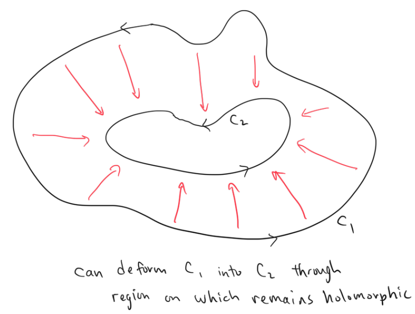
which is more often the way in which such an integral is expressed.

Deformation theorem. The fact that functions which are holomorphic on and interior to a simple closed contour integrate to zero is useful, but perhaps just as useful is the following result known as the *deformation theorem*. Indeed, the deformation theorem plays the key role in deriving what's called *Cauchy's integral formula*, which we will discuss in a bit, which itself is the holy grail of all integration results.

The statement (of the deformation theorem) is that if C_1 and C_2 are simple closed contours with one, say C_2, lying fully interior to C_1, then

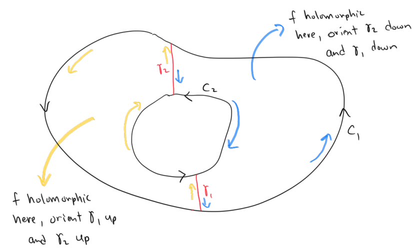
$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

for any f that is holomorphic on a domain containing C_1, C_2 and the region between them. The point is that if we can "deform" one contour into the other through a region on which f remains holomorphic, the value of the integral does not change:



For example, the integral of $\frac{1}{z}$ over the circle $|z| = R$ is $2\pi i$, and thus so is the integral of $\frac{1}{z}$ over any simple closed contour enclosing 0 since any such contour can be deformed into a circle; we knew this already via a branches of log argument, but this gives a more geometric justification.

To prove the deformation theorem, take the contours C_1 and C_2, and "cut" the region between them into two pieces by putting in segments γ_1 and γ_2 as follows:



The left of this region is then enclosed by the contour we will write as

$$(\text{left } C_1) + (\text{up } \gamma_1) + (\text{left } -C_2) + (\text{up } \gamma_2),$$

where we traverse the left half of C_1, then move up γ_1, then traverse the left half of C_2 but in the *clockwise* orientation, and then move up γ_2 to close up. Similarly, the right half of the region between C_1 and C_2 is enclosed by

$$(\text{right } C_1) + (\text{down } \gamma_1) + (\text{right } -C_2) + (\text{down } \gamma_2).$$

The integral of f over each of these region boundaries is zero (!!!) by Cauchy's theorem since f is holomorphic on the the left and right half regions they enclose:

$$\oint_{(\text{left } C_1)+(\text{up } \gamma_1)+(\text{left } -C_2)+(\text{up } \gamma_2)} f(z)\,dz = 0 = \oint_{(\text{right } C_1)+(\text{down } \gamma_1)+(\text{right } -C_2)+(\text{down } \gamma_2)} f(z)\,dz.$$

So, adding these two integrals together should also give 0. But when adding, we get two terms like

$$\int_{\text{left } C_1} f(z)\,dz + \int_{\text{right } C_1} f(z)\,dz,$$

which combine to give $\oint_{C_1} f(z)\,dz$. We also get

$$\int_{\text{left } -C_2} f(z)\,dz + \int_{\text{right } -C_2} f(z)\,dz = \oint_{-C_2} f(z)\,dz,$$

and all that remains cancels out: we get the integral of f over "up γ_1" plus the integral over "down γ_1", which gives 0, and similar to the pieces over up/down γ_2. Thus, adding the terms in

$$\oint_{(\text{left } C_1)+(\text{up } \gamma_1)+(\text{left } -C_2)+(\text{up } \gamma_2)} f(z)\,dz = 0 = \oint_{(\text{right } C_1)+(\text{down } \gamma_1)+(\text{right } -C_2)+(\text{down } \gamma_2)} f(z)\,dz$$

leaves only

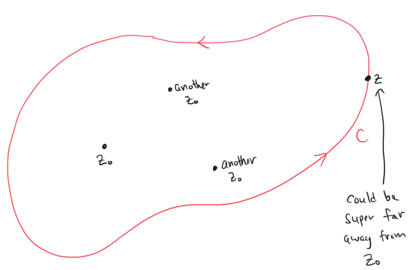
$$\oint_{C_1} f(z)\,dz + \oint_{-C_2} f(z)\,dz = 0, \text{ so } \oint_{C_1} f(z)\,dz = -\oint_{-C_2} f(z)\,dz = \oint_{C_2} f(z)\,dz,$$

which is the desired deformation result.

Cauchy's integral formula. *Cauchy's integral formula* gives a way to express values of holomorphic functions as integrals. We cannot oversell how crucial having such a representation is, and all of the "big" results we will see in the coming week arise from this one fact. The claim is that if f is holomorphic on a domain containing a simple closed contour C and its interior (satisfied if the domain is simply-connected, for example), then

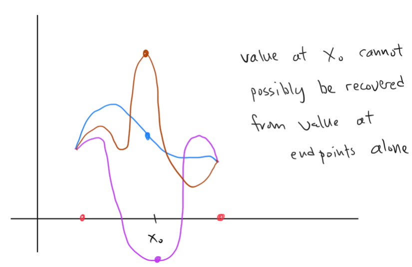
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$$

for any z_0 interior to C. The point is that the right side only uses the values of f along the contour C itself, and yet from these alone we can recover the value of f at any point within C:



No matter how large C is, nor how far away from z_0 the points on it are, we always have enough information to determine $f(z_0)$ exactly.

We will save the justification (which will depend on the deformation theorem, which in turn depended on Cauchy's theorem) for next time. But we emphasize for now just how different this result is from anything in *real* calculus. In the purely real setting, the analogous thing would be trying to obtain the value $f(x_0)$ for x_0 in some interval solely from the values on the "boundary" (i.e., endpoints) of that interval:



But there is no way in which such a thing can happen for real differentiable functions, since there are many many many functions we can come up with that have the same values at the endpoints but wildly different values at "interior" points. The value $f(x_0)$ in general has nothing to do with the value of f at points far enough away, but in the complex setting there is an intimate connection between the value of f at one point z_0 and its values even very far away.

Example. Here is a first (silly) example. The constant function $f(z) = 1$ is holomorphic everywhere, so the assumption in Cauchy's integral formula is satisfied for all simple closed contours C

enclosing a given z_0 and we get that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \oint_C \frac{1}{z - z_0}\, dz.$$

The left side is 1, so after rearranging we get

$$\oint_C \frac{1}{z - z_0}\, dz = 2\pi i$$

as expected. This example is "silly" because we do not need the integral formula to obtain this value (it follows from examples we have seen before), but perhaps more so because the justification of Cauchy's integral formula will *use* the fact that this integral has this value, so it is circular reasoning to obtain this value from the integral formula.

Here is a better example. Consider

$$\oint_{|z|=2} \frac{\cos z}{z - \frac{\pi}{2}}\, dz.$$

The function $\cos z$ is holomorphic everywhere, so we can apply the integral formula to any simple closed contour. With $z_0 = \frac{\pi}{2}$, we get

$$\oint_{|z|=2} \frac{\cos z}{z - \frac{\pi}{2}}\, dz = 2\pi i(\cos \tfrac{\pi}{2}) = 0.$$

(Compared the form of the integral formula we originally gave, here we have just put the $2\pi i$ in a different place.) Similarly, we have

$$\oint_{|z|=2} \frac{\sin z}{z - \frac{\pi}{2}}\, dz = 2\pi i(\sin \tfrac{\pi}{2}) = 2\pi i.$$

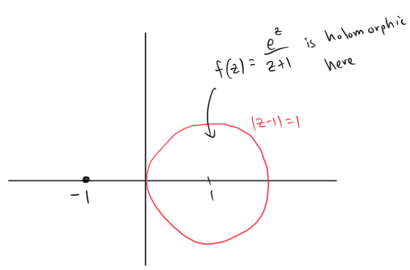
Another example. Consider

$$\oint_{|z-1|=1} \frac{e^z}{z^2 - 1}\, dz.$$

(Note the contour of integration is a circle centered at 1, not 0.) As written, it is not in the form required of the integral formula yet since the denominator does not look like $z - z_0$. But since $z^2 - 1 = (z-1)(z+1)$, we can write this as

$$\oint_{|z-1|=1} \frac{e^z}{z^2 - 1}\, dz = \oint_{|z-1|=1} \frac{e^z}{(z-1)(z+1)}\, dz = \oint_{|z-1|=1} \frac{e^z/(z+1)}{z - 1}\, dz.$$

That is, we incorporate the $z+1$ in the denominator into the function $f(z) = \frac{e^z}{z+1}$, which *is* holomorphic on a domain containing the circle of radius 1 centered at 1 and its interior:



The function $f(z) = \frac{e^z}{z+1}$ fails to be differentiable only at $z = -1$, but this falls outside the interior of the given contour, so we are good to go. We thus get

$$\oint_{|z-1|=1} \frac{e^z}{z^2-1}\,dz = \oint_{|z-1|=1} \frac{e^z/(z+1)}{z-1}\,dz = 2\pi i \left(\frac{e^z}{z+1}\bigg|_{z=1} \right) = 2\pi i(\tfrac{e}{2}) = \pi i e.$$

If we considered a contour whose interior included both -1 and 1, such as in

$$\oint_{|z|=2} \frac{e^z}{z^2-1}\,dz = \oint_{|z|=2} \frac{e^z}{(z-1)(z+1)}\,dz,$$

Cauchy's integral formula would now no longer be applicable. We will near the end of the quarter, however, that such integrals can still be easily computed using the method of *residues*, of which the integral formula is the simplest non-trivial case.

Back to Warm-Up from last time. In the Warm-Up from last time, we began by taking the value

$$\oint_{|z|=1} \frac{z}{z^2+4z+1}\,dz = 2\pi i \left(\frac{-2+\sqrt{3}}{2\sqrt{3}} \right)$$

for granted, which now we can justify. As in the previous example, in order to apply the integral formula we must factor the denominator. By the quadratic formula, the roots of $z^2 + 4z + 1$ are

$$\frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3},$$

so $z^2 + 4z + 1 = (z - [-2+\sqrt{3}])(z - [-2-\sqrt{3}])$. Note that of these two roots, only $-2+\sqrt{3}$ lies inside $|z| = 1$, which is what makes the integral formula applicable since $f(z) = \frac{z}{z-(-2-\sqrt{3})}$ is then holomorphic on and interior to $|z| = 1$. We get

$$\oint_{|z|=1} \frac{z}{z^2+4z+1}\,dz = \oint_{|z|=1} \frac{z/(z-[-2-\sqrt{3}])}{z-(-2+\sqrt{3})}\,dz = 2\pi i \left(\frac{z}{z-(-2-\sqrt{3})}\bigg|_{z=-2+\sqrt{3}} \right).$$

After plugging in $z = -2+\sqrt{3}$ and simplifying we do get the value we claimed was correct.

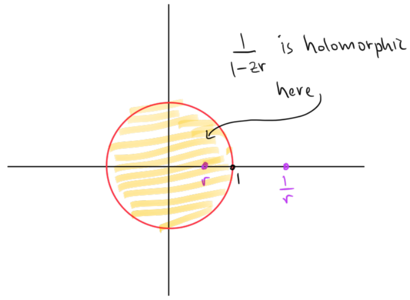
Lecture 15: More on Cauchy's Formula

Warm-Up 1. We compute the contour integrals

$$\oint_{|z|=1} \frac{1}{(z-r)(1-rz)}\,dz \quad \text{and} \quad \oint_{C_R} \frac{e^{iz}}{z^2+1}\,dz$$

where $0 < r < 1$ in the first integral and C_R in the second is the contour enclosing the upper half of the disk $|z| \leq R$ for $R > 1$, so that C_R consists of a line segment on the real axis and the top half of a circle of radius $R > 1$. We compute both using the Cauchy integral formula.

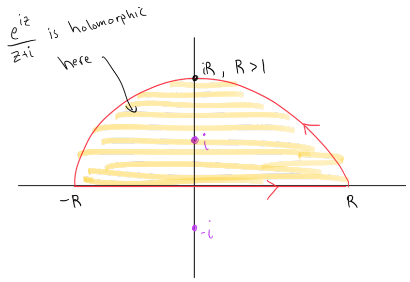
First, note that $(z-r)(1-rz)$ is zero when $z = r$ or $z = \frac{1}{r}$, but the only one of these that lies within $|z| = 1$ is $z = r$ (recall $0 < r < 1$) since $\frac{1}{r} > 1$:



The function $f(z) = \frac{1}{1-rz}$ is then holomorphic on and interior to $|z| = 1$, so Cauchy's formula applies and we get

$$\oint_{|z|=1} \frac{1}{(z-r)(1-rz)}\,dz = \oint_{|z|=1} \frac{1/(1-rz)}{z-r}\,dz = 2\pi i \left(\frac{1}{1-rz}\bigg|_{z=r} \right) = \frac{2\pi i}{1-r^2}.$$

For the second integral, we have $z^2+1 = (z-i)(z+i)$, and of the two roots $\pm i$ of this expression only i lies within C_R:



(This is where we need the $R > 1$ assumption; for $0 < R < 1$ the second integral is zero by Cauchy's theorem since $\frac{e^{iz}}{z^2+1}$ is then holomorphic within the circle $|z| = R$.) Thus $g(z) = \frac{e^{iz}}{z+i}$ is holomorphic on and interior to C_R, so we get

$$\oint_{C_R} \frac{e^{iz}}{z^2+1}\,dz = \oint_{C_R} \frac{e^{iz}/(z+i)}{z-i}\,dz = 2\pi i \left(\frac{e^{iz}}{z+i}\bigg|_{z=i} \right) = 2\pi i \left(\frac{e^{-1}}{2i} \right) = \frac{\pi}{e}.$$

Warm-Up 2. We determine the values of

$$\int_0^{2\pi} \frac{1}{1-2r\cos t + r^2}\,dt \ (\text{for } 0 < r < 1) \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\cos t}{1+t^2}\,dt.$$

The point is that these are not some random standalone integrals, but rather they actually follow directly from the computations in the first Warm-Up! Indeed, go back to the first integral we computed in the first Warm-Up and instead use the parametrization $z = e^{it}, 0 \leq t \leq 2\pi$. We get

$$\begin{aligned}
\oint_{|z|=1} \frac{1}{(z-r)(1-rz)}\,dz &= \oint_{|z|=1} \frac{1}{z - rz^2 - r + r^2 z}\,dz \\
&= \int_0^{2\pi} \frac{1}{e^{it} - re^{2it} - r + r^2 e^{it}} (ie^{it})\,dt \\
&= i \int_0^{2\pi} \frac{1}{1 - re^{it} - re^{-it} + r^2}\,dt
\end{aligned}$$

where in the last step we multiplied numerator and denominator by e^{-it} in order to simplify. But then $-re^{it} - re^{-it} = -2r\cos t$, so the final integral above is precisely i times the one listed first in this Warm-Up, so by taking the value for the first contour integral in the first Warm-Up we get

$$i\int_0^{2\pi} \frac{1}{1-2r\cos t + r^2}\,dt = \frac{2\pi i}{1-r^2}, \text{ so } \int_0^{2\pi} \frac{1}{1-2r\cos t + r^2}\,dt = \frac{2\pi}{1-r^2}.$$

For the second integral, we consider

$$\oint_{C_R} \frac{e^{iz}}{z^2+1}\,dz = \frac{\pi}{e}$$

from the first Warm-Up take the limit as $R \to \infty$. As we take this limit the value $\frac{\pi}{e}$ does not change since C_R remains a simple closed contour enclosing i, at least for $R > 1$ which is a fine assumption since in the limit $R \to \infty$ we only care about large values of R anyway. If we denote the bottom segment of C_R by C_1 (oriented left to right) and the top circular piece by C_2 (oriented counterclockwise) we have

$$\frac{\pi}{e} = \oint_{C_R} \frac{e^{iz}}{z^2+1}\,dz = \int_{C_1} \frac{e^{iz}}{z^2+1}\,dz + \int_{C_2} \frac{e^{iz}}{z^2+1}\,dz.$$

The integral over C_1 just becomes the integral with respect to the real parameter $z = t$ on the interval $[-R, R]$:

$$\int_{C_1} \frac{e^{iz}}{z^2+1}\,dz = \int_{-R}^{R} \frac{e^{it}}{t^2+1}\,dt.$$

Note that the real part of this integral uses $\cos t$ in the numerator, which is what will give us the value of the integral we desire after taking the limit.

For the integral over the circular piece C_2, we argue that this will go to zero in the limit. Indeed, we have $e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix}$, so

$$\left|\frac{e^{iz}}{z^2+1}\right| = \frac{e^{-y}}{|z^2+1|} \leq \frac{1}{|z^2+1|}$$

since $y \geq 0$ for points along C_2. For the denominator we use what's called the *reverse triangle inequality*, which says that $|z| - |w| \leq |z+w|$. (The *triangle inequality* gives the other direction $|z+w| \leq |z| + |w|$, so the takeaway is that the modulus of a sum is always bounded form below by what we get when subtracting individual moduli and is bounded from above by adding individual moduli.) In our case this gives $|z^2+1| \geq |z^2| - |1| = |z|^2 - 1$, so

$$\left|\frac{e^{iz}}{z^2+1}\right| = \frac{e^{-y}}{|z^2+1|} \leq \frac{1}{|z^2+1|} \leq \frac{1}{|z|^2-1} = \frac{1}{R^2-1}.$$

Note that we needed to bound $|z^2+1|$ from *below* not above since it occurs in the denominator of a fraction, and the fact that $|z| = R$ in the last step just comes from the fact that z lies on the circular arc C_2, which is the top half of the circle $|z| = R$. With this bound we get

$$\left|\int_{C_2} \frac{e^{iz}}{z^2+1}\,dz\right| \leq \int_{C_2} \left|\frac{e^{iz}}{z^2+1}\right| |dz| \leq \int_{C_2} \frac{1}{R^2-1}|dz| = \frac{1}{R^2-1}\int_{C_2} |dz|.$$

The remaining integral is the length of C_2, which is πR (half the circumference of a full circle), so

$$\left|\int_{C_2} \frac{e^{iz}}{z^2+1}\,dz\right| \leq \frac{\pi R}{R^2-1}.$$

The right side goes to 0 as $R \to \infty$, so, as we claimed, so to does the integral over C_2.

Taking $R \to \infty$ in

$$\frac{\pi}{e} = \oint_{C_R} \frac{e^{iz}}{z^2+1}\, dz = \int_{C_1} \frac{e^{iz}}{z^2+1}\, dz + \int_{C_2} \frac{e^{iz}}{z^2+1}\, dz$$

thus gives

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{e^{it}}{1+t^2}\, dt + 0 = \int_{-\infty}^{\infty} \frac{\cos t + i \sin t}{1+t^2}\, dt.$$

Taking real parts then gives

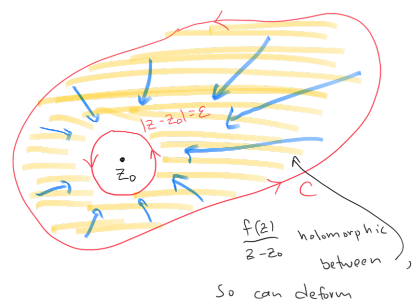
$$\int_{-\infty}^{\infty} \frac{\cos t}{1+t^2}\, dt = \frac{\pi}{e}$$

as our desired value. (Both of the integral in this Warm-Up are, yet again, instances of phrasing a real integral in terms of a complex integral and then applying results about contour integrals to make the resulting values simple to determine.)

Proof of integral formula. We now give a justification for the integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz.$$

Recall here that $f(z)$ is meant to be holomorphic on and interior to the simple closed contour C containing z_0 in its interior. The first observation to make is that, by the deformation theorem from last time, we can replace C more concretely by a small circle centered at z_0:



Indeed, we can deform C into any other simple closed contour around z_0 as long as we do so through a region on which the integrand $\frac{f(z)}{z-z_0}$ remains holomorphic; this integrand fails to be differentiable only at z_0 (when the denominator is zero), which does not lie between C and $|z - z_0| = \epsilon$, so we are good to go. Eventually we will take a limit as $\epsilon \to 0$ and the circle around z_0 shrinks, and the point is that doing so does not change the value of any of our integrals by deformation.

Consider then

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz - f(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=\epsilon} \frac{f(z)}{z-z_0}\, dz - f(z_0).$$

Using the fact that

$$\oint_{|z-z_0|=\epsilon} \frac{1}{z-z_0}\, dz = 2\pi i,$$

we can write the constant $f(z_0)$ as

$$f(z_0) = f(z_0) \underbrace{\left(\frac{1}{2\pi i} \oint_{|z-z_0|=\epsilon} \frac{1}{z-z_0}\, dz \right)}_{1} = \frac{1}{2\pi i} \oint_{|z-z_0|=\epsilon} \frac{f(z_0)}{z-z_0}\, dz.$$

(In the final step we can bring $f(z_0)$ into the integral simply because it is constant with respect to the variable of integration z.) We should note that when first introducing complex integrals a few times ago we made a point to highlight how important (crucial, in fact) the value

$$\oint_{|z-z_0|=\epsilon} \frac{1}{z-z_0}\, dz = 2\pi i$$

was going to be, and now we can see why: it plays a key role in writing $f(z_0)$ as an integral, and thus plays a key role in the proof of the integral formula and in all the (amazing) consequences we will see in the next week! None of this would work without knowing this one specific integral value.

With this integral expression for $f(z_0)$, we can write

$$\begin{aligned}\frac{1}{2\pi i}\oint_{|z-z_0|=\epsilon} \frac{f(z)}{z-z_0}\, dz - f(z_0) &= \frac{1}{2\pi i}\oint_{|z-z_0|=\epsilon} \frac{f(z)}{z-z_0}\, dz - \frac{1}{2\pi i}\oint_{|z-z_0|=\epsilon} \frac{f(z_0)}{z-z_0}\, dz \\ &= \frac{1}{2\pi i}\oint_{|z-z_0|=\epsilon} \frac{f(z)-f(z_0)}{z-z_0}\, dz.\end{aligned}$$

The strategy is to now argue that this resulting integral will approach 0 as $\epsilon \to 0$; if so, then our original

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\, dz - f(z_0)$$

that this equaled by deformation will approach 0 as well, but since this remains constant (since it uses the original contour C and not the small circle around z_0) as $\epsilon \to 0$, it must have the value zero to begin with, giving

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\, dz - f(z_0) = 0,$$

which is the Cauchy integral formula after rearranging. To show that

$$\frac{1}{2\pi i}\oint_{|z-z_0|=\epsilon} \frac{f(z)-f(z_0)}{z-z_0}\, dz \to 0 \text{ as } \epsilon \to 0$$

we find some bounds. Denote by M_ϵ the maximum value of $|f(z)-f(z_0)|$ among points z on the circle $|z-z_0| = \epsilon$; we do not care about what this maximum equals exactly, but we care that $M_\epsilon \to 0$ as $\epsilon \to 0$, which is true because $f(z) \to f(z_0)$ as $z \to z_0$ by continuity, meaning that the difference $f(z) - f(z_0)$, and hence maximum, will approach zero as well.

We have

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \frac{|f(z)-f(z_0)|}{|z-z_0|} \le \frac{M_\epsilon}{\epsilon},$$

where we use the fact that $|z-z_0|$ in the denominator has the value ϵ for points on the circle $|z-z_0| = \epsilon$. The length of this circle is $2\pi\epsilon$, so

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_{|z-z_0|=\epsilon} \frac{f(z)-f(z_0)}{z-z_0}\, dz\right| &\le \frac{1}{|2\pi i|}\oint_{|z-z_0|=\epsilon} \left|\frac{f(z)-f(z_0)}{z-z_0}\right| |dz| \\ &\le \frac{1}{2\pi}\oint_{|z-z_0|=\epsilon} \frac{M_\epsilon}{\epsilon}|dz| \\ &= \frac{M}{2\pi\epsilon}(2\pi\epsilon) = M_\epsilon.\end{aligned}$$

As mentioned above, $M_\epsilon \to 0$ as $\epsilon \to 0$, so we get

$$\frac{1}{2\pi i}\oint_{|z-z_0|=\epsilon} \frac{f(z)-f(z_0)}{z-z_0}\,dz \to 0 \text{ as } \epsilon \to 0$$

as desired, which completes our proof, as explained above. (Remark: The Cauchy integral formula is the thing that will lead to important consequences, but its proof depends—in addition to the integral of $\frac{1}{z-z_0}$ over a circle– on the ability to deform contours, which depends on Cauchy's theorem. This is why Cauchy's theorem is considered to be the more fundamental result in this subject!)

Differentiating under the integral sign. So we have

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw$$

for f holomorhpic on and and interior to a simple closed contour C for any z enclosed by C. Here we have written z instead of z_0 since we now want to think of z itself as a *variable* rather than a fixed point in the interior, which then requires us to use something other than z—say w—as notation for the variable of integration. Thus we think of the right side above as a function of z, and ask whether or not this function is differentiable with respect to z. In fact it is, as guaranteed by what is called the method of *differentiation under the integral sign*, which says that the derivative of the right side exists and is given by what we get if we differentiate inside the integral instead:

$$\frac{d}{dz}\left(\oint_C \frac{f(w)}{w-z}\,dw\right) = \oint_C \frac{d}{dz}\left(\frac{f(w)}{w-z}\right)\,dw.$$

The book has a justification for this using the limit definition of the derivative, but instead we will postpone the reason for why this works until later where we will come at it from a different—and simpler—perspective.

Assuming that this is valid for now, since

$$\frac{d}{dz}\left(\frac{f(w)}{w-z}\right) = \frac{f(w)}{(w-z)^2}$$

by writing $\frac{1}{w-z}$ as $(w-z)^{-1}$ as usual, we get

$$\frac{d}{dz}\left(\frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw\right) = \frac{1}{2\pi i}\oint_C \frac{d}{dz}\left(\frac{f(w)}{w-z}\right)\,dw = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z)^2}\,dw.$$

But

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw,$$

so the derivative of the right side is the derivative of f, and thus

$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z)^2}\,dw,$$

giving an expression for $f'(z)$ as an integral as well! So, both $f(z)$ and $f'(z)$ are expressible as integrals, with the only difference being the power of $w-z$ that occurs in the denominator of the integral.

Cauchy's formula for higher derivatives. We can keep going. Differentiation under the integral sign applied to the right side of

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^2}\, dw$$

gives

$$\frac{d}{dz}\left(\frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^2}\, dw\right) = \frac{1}{2\pi i} \oint_C \frac{d}{dz}\left(\frac{f(w)}{(w-z)^2}\right) dw = \frac{1}{2\pi i} \oint_C \frac{2f(w)}{(w-z)^3}\, dw,$$

where we use the fact that $(w-z)^{-2}$ differentiates (with respect to z) to $2(w-z)^{-3}$. Part of the result here is that this derivative exists in the first place, so since

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^2}\, dw,$$

we get that the derivative of the left side must exist as well with

$$f''(z) = \frac{1}{2\pi i} \oint_C \frac{2f(w)}{(w-z)^3}\, dw.$$

In particular, we get that when f is holomorphic, f' is holomorphic as well with derivative $f''(z)$ given by the integral expression above. (Again, we will come back to see why this work in a way that avoids differentiation under the integral sign, so there will be no leap in logic.) This is big, as this says that the derivative of a holomorphic function is itself always holomorphic, which is very far from what happens in the case of real calculus, where there are many functions that have a first derivative but not a second derivative. This is why only holomorphic functions can hope to have antiderivatives (a fact we've mentioned before), and why f' is always continuous when f is holomorphic: f' is differentiable and differentiable implies continuous. Moreover, this is why u_x and v_x in $f' = u_x + iv_x$—and hence u_y and v_y as well by the Cauchy-Riemann equations—are always continuous, which we had said before was a "superfluous" assumption. All of this follows from the fact that values of holomorphic functions are expressible as integrals, and thus ultimately from Cauchy's theorem. (The proof we first gave from Cauchy's theorem assumed f' was differentiable, but we can avoid the circular reasoning by arguing along the lines of the second, more involved, argument we gave without assuming continuity of f' in the case of integrating over a triangle.)

We then get that the third derivative of f exists by differentiating both sides of

$$f''(z) = \frac{1}{2\pi i} \oint_C \frac{2f(w)}{(w-z)^3}\, dw \rightsquigarrow f'''(z) = \frac{1}{2\pi i} \oint_C \frac{3\cdot 2f(w)}{(w-z)^4}\, dw,$$

and then the fourth derivative, and so on without end. The conclusion is that a holomorphic f is *infinitely* differentiable with its derivatives given by

$$f^{(n)}(z) = \frac{1}{2\pi i} \oint_C \frac{n!f(w)}{(w-z)^{n+1}}\, dw, \text{ or commonly written as } f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

Each time we differentiate we pick up an extra coefficient contributing to $n!$ overall, and we increase the power of $w-z$ in the denominator by 1.

Example. For example, let us finish by computing

$$\oint_{|z|=1} \frac{\sin z}{z^4}\, dz.$$

By thinking of the denominator as $z^4 = (z-0)^4$ with 0 enclosed by the circle $|z| = 1$, we see that this is in the form required of Cauchy's integral formula for higher-order derivatives, in particular a third derivative in this case. After moving the $\frac{n!}{2\pi i}$ in the integral expression for $f^{(n)}(z)$ given above to the other side, we have

$$\oint_{|z|=1} \frac{\sin z}{z^4}\, dz = \frac{2\pi i}{3!}\left(\frac{d^3}{dz^3}(\sin z)\Big|_{z=0}\right),$$

where $\frac{d^3}{dz^3}$ indicates taking the third derivative. The third derivative of $\sin z$ is $-\cos z$, so

$$\oint_{|z|=1} \frac{\sin z}{z^4}\, dz = \frac{2\pi i}{3!}(-\cos 0) = -\frac{2\pi i}{6} = -\frac{\pi i}{3}.$$

Lecture 16: Liouville's Theorem

Warm-Up 1. We compute

$$\oint_{|z-i|=1} \frac{z^{10}+e^z}{(z-i)^n}\, dz$$

for all positive integers n. For $n = 1$ this is just the usual Cauchy integral formula for function $f(z) = z^{10} + e^z$, which is holomorphic on a domain containing the circle $|z - i| = 1$ and its interior. So for $n = 1$ the value is

$$\oint_{|z-i|=1} \frac{z^{10}+e^z}{z-i}\, dz = 2\pi i\left(z^{10}+e^z\right)\Big|_{z=i} = 2\pi i(i^{10}+e^i) = 2\pi i(e^i - 1)$$

since $i^{10} = i^2 = -1$. For $n \geq 2$ we can use the higher-order derivative form of Cauchy's formula:

$$\oint_{|z-i|=1} \frac{z^{10}+e^z}{(z-i)^n}\, dz = \frac{2\pi i}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}(z^{10}+e^z)\right)\Bigg|_{z=i},$$

which by relabeling n for $n \geq 2$ as $n+1$ for $n \geq 1$ so as to better match the usual integral formula, we can write as

$$\oint_{|z-i|=1} \frac{z^{10}+e^z}{(z-i)^{n+1}}\, dz = \frac{2\pi i}{n!}\left(\frac{d^{n}}{dz^{n}}(z^{10}+e^z)\right)\Bigg|_{z=i}.$$

For $n \geq 11$ the n-th derivative of z^{10} is zero and the n-th derivative of e^z is e^z, so for these we get

$$\frac{2\pi i}{n!}\left(\frac{d^{n}}{dz^{n}}(z^{10}+e^z)\right)\Bigg|_{z=i} = \frac{2\pi i}{n!}e^i.$$

For $1 \leq n \leq 10$, the n-th derivative of z^{10} is $10 \cdot 9 \cdots (10-n+1)z^{10-n}$, so for these n we have

$$\frac{2\pi i}{n!}\left(\frac{d^{n}}{dz^{n}}(z^{10}+e^z)\right)\Bigg|_{z=i} = \frac{2\pi i}{n!}[10 \cdot 9 \cdots (10-n+1)i^{10-n} + e^i].$$

Thus in summary, and after relabeling $n+1$ back as n, we have

$$\oint_{|z-i|=1} \frac{z^{10}+e^z}{(z-i)^n}\, dz = \begin{cases} \frac{2\pi i}{(n-1)!}e^i & \text{if } n \geq 12 \\ \frac{2\pi i}{(n-1)!}[10 \cdot 9 \cdots (10-(n-1)+1)i^{10-(n-1)} + e^i] & \text{if } 1 \leq n \leq 11. \end{cases}$$

(The latter case includes the $n = 1$ case where no derivatives of $f(z)$ are needed and we obtained $2\pi i(e^i - 1)$ as the value above.)

Warm-Up 2. We compute

$$\int_{-\infty}^{\infty} \frac{\cos t}{(1+t^2)^2}\, dt.$$

Last time we computed

$$\int_{-\infty}^{\infty} \frac{\cos t}{1+t^2}\, dt = \frac{\pi}{e}$$

by integrating $\frac{e^{iz}}{1+z^2}$ over the contour forming the boundary of the top half of the disk $|z| \leq R$ and taking the limit $R \to \infty$, and in fact the computation of this new improper integral proceeds in exactly the same way only with a slightly different holomorphic function.

Consider

$$\frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z+i)^2(z-i)^2}.$$

With the same contour C_R as with the example last time, only the "singularity" i lies interior to C_R once $R > 1$, so

$$\frac{e^{iz}}{(z+i)^2}$$

remains holomorphic on and interior to C_R. Thus by Cauchy's integral formula (for first derivatives), we get

$$\oint_{C_R} \frac{e^{iz}}{(1+z^2)}\, dz = \oint_{C_R} \frac{e^{iz}}{(z-i)^2(z+i)^2}\, dz = 2\pi i \left(\frac{d}{dz} \frac{e^{iz}}{(z+i)^2} \right) \Bigg|_{z=i}.$$

This derivative can be computed using the quotient rule, and after doing so and evaluating at i we get $\frac{\pi}{e}$ as the value of this integral. (Just happens to be the same as what you get for $\oint_{C_R} \frac{e^{iz}}{1+z^2}\, dz$ from last time!)

The integral over the bottom segment of C_R becomes

$$\int_{-R}^{R} \frac{e^{it}}{(1+t^2)^2}\, dt,$$

so taking the real part of the limit as $R \to \infty$ will give the value we need. For the integral over the circle piece of C_R, we proceed just as last time with the only difference being that we bound

$$\left| \frac{e^{iz}}{(z^2+1)^2} \right| \leq \frac{e^{-y}}{(|z|^2-1)^2} \leq \frac{1}{(R^2-1)^2}$$

instead of with just $\frac{1}{R^2-1}$ at the end as we had last time. This gives

$$\left| \int_{\text{circular piece}} \frac{e^{iz}}{(z^2+1)^2}\, dz \right| \leq \int_{\text{circular piece}} \left| \frac{e^{iz}}{(z^2+1)^2} \right| |dz| \leq \frac{1}{(R^2-1)^2}(\pi R)$$

as a bound, which goes to 0 as $R \to \infty$.

Thus we get

$$\frac{\pi}{e} = \int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\, dz = \int_{-R}^{R} \frac{e^{it}}{(1+t^2)^2}\, dt + (\text{something that will go to } 0),$$

so after taking $R \to \infty$ and taking real parts we get

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{\cos t}{(1+t^2)^2}\,dt$$

as the desired value. (With the same argument and same contour, you can find the value of $\int_{-\infty}^{\infty} \frac{\cos t}{(1+t^2)^n}\,dt$ for any positive integer n.)

Liouville's theorem. Cauchy's integral formulas place big restrictions on behaviors of holomorphic functions. The first such restriction we will look at is known as *Liouville's theorem*, and implies that, in a sense, functions which are *entire*—meaning holomorphic on all of $\mathbb{C}$—can often be described fairly explicitly. (Next time we will look at what's called the *maximum modulus principle*, which is the second main restriction we get on behaviors of holomorphic functions.)

Liouville's theorem states that an entire function which is bounded must be constant. Here, to be *bounded* means that there is a restriction on how large the modulus values $|f(z)|$ can be, so that $|f(z)| \leq M$ for some $M > 0$ and all z. This is very different than in the real setting, where there are tons of non-constant differentiable functions which are bounded: $\cos x$, $\sin x$, and $\arctan x$ for example. Not so in the complex case, where all we get are constant functions. (Of course, Liouville's theorem does not apply to $\cos z$ and $\sin z$ since these are no longer bounded when taking imaginary directions into account.) Here is the reason. Start with Cauchy's integral formula (for first derivatives):

$$f'(z) = \frac{1}{2\pi i} \oint_{|w-z|=R} \frac{f(w)}{(w-z)^2}\,dw$$

where $R > 0$ is an arbitrary radius. Using the bound $|f(w)| \leq M$ that holds regardless of what R is, we get

$$\begin{aligned} |f'(z)| &\leq \frac{1}{2\pi} \oint_{|w-z|=R} \frac{|f(w)|}{|w-z|^2}\,|dw| \\ &= \frac{1}{2\pi} \oint_{|w-z|=R} \frac{M}{R^2}\,|dw| \\ &= \frac{M}{2\pi R^2} \oint_{|w-z|=R} |dw| \\ &= \frac{M}{2\pi R^2}(2\pi R) = \frac{M}{R}. \end{aligned}$$

This holds for all $R > 0$, so taking $R \to \infty$ (possible since f is entire) gives

$$|f'(z)| \leq 0, \text{ so } f'(z) = 0.$$

The point z was arbitrary, so f has derivative zero on all of $\mathbb{C}$, so f is constant.

Example. The fact that bounded entire functions are all constant may not seem so useful on its own, in particular since when checking the "bounded" condition would we not know that our function was constant already? But the true power of Liouville's theorem lies in its clever uses.

For example, suppose $f = u + iv$ is entire with negative real part u at all points. Geometrically, this says that the image of all of $\mathbb{C}$ under the transformation $f(z)$ lies in the left-half plane:

The claim is that such a function must in fact be constant! (In particular, the picture above, which suggests the image is the entire left-half plane, cannot happen; as soon as the image lies in the left-hale plane, the image will be a single point and nothing else.) We do not know at the outset that f must be bounded, so the strategy is not to apply Liouville's theorem to f itself but instead to a well-chosen entire function constructed from f.

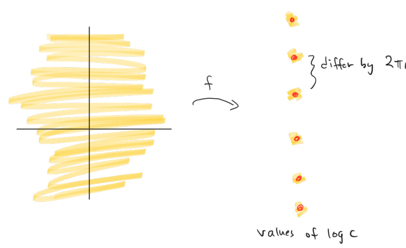
Consider the function $e^{f(z)}$. This is entire by the chain rule (composition of entire functions), and with $f = u + iv$ we have

$$|e^f| = |e^{u+iv}| = e^u.$$

Since u is always negative, e^u is always bounded by $e^0 = 1$, so e^f is bounded. Thus Liouville's theorem says that $e^{f(z)}$ must be constant, say with constant value c:

$$e^{f(z)} = c.$$

To get from here to f, note that this equation says that $f(z)$ is a value of $\log c$ for all z. But values of $\log c$ all differ from one another by a multiple of $2\pi i$, so the values of $f(z)$ for varying z must all differ from one another by a multiple of $2\pi i$ as well. The image of f must be connected as the domain $\mathbb{C}$ is connected and f is continuous, so this image cannot consist of more than one value of $\log c$ since as soon as two values are attained the image is no longer connected:

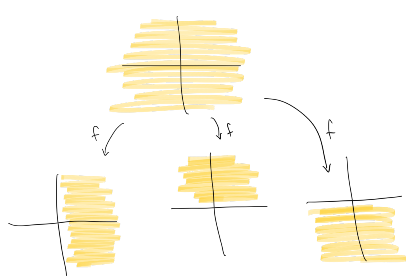


Thus $f(z)$ is a single value of $\log c$ for all z, meaning that f is constant.

With slight modifications, we can also rule out other types of half-plane images for an entire function. For example, if instead the real part of u was positive, applying Liouville's theorem to $e^{-f(z)}$ would give a way to show that f must be constant. If the imaginary part v of $f = u + iv$ was positive, then we use the entire function $e^{if(z)}$ instead since

$$|e^{if}| = |e^{i(u+iv)}| = |e^{-v+iu}| = e^{-v},$$

and if v was negative we use e^{-if}. Thus no picture like



is possible for the image of an entire function.

Another example. For another type of application of Liouville's theorem, suppose f is entire and is bounded by e^z in the sense that

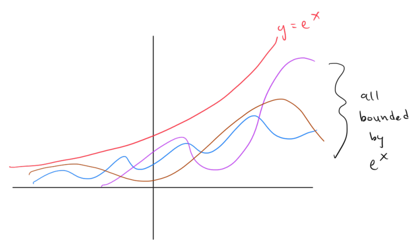
$$|f(z)| \leq |e^z|.$$

We claim that we can very explicitly say what function f has to be. Indeed, after dividing by $|e^z|$, which is never zero, we get

$$\left|\frac{f(z)}{e^z}\right| \leq 1.$$

But then $\frac{f(z)}{e^z}$ is entire (by the quotient rule) and bounded, so it must be constant $\frac{f(z)}{e^z} = c$, and then $f(z) = ce^z$ is simply a constant multiple of e^z. Being bounded by the entire function e^z thus places severe restrictions on f itself, so that we have no choice as to what f must look like. If then $f(z)$ and e^z happened to agree at even one point, maybe $z = 0$, we would get that $c = 1$ so $f(z) = e^z$ would agree everywhere.

Note again that this is very different than the case of real calculus, where there are numerous functions we can draw which are bounded by e^x and yet are not multiples of e^x:



Fundamental theorem of algebra. Finally we give a standard use of Liouville's theorem, namely to prove the *fundamental theorem of algebra*. The fundamental theorem of algebra is the claim that all nonconstant polynomials with complex coefficients have at least one complex root. Of course this is not true if we stick only with real numbers and roots, since, for example, $x^2 + 1$ does not have a real root. The claim is that as soon we allow complex roots, we always get existence, even if we allow complex coefficients as well. For small degree polynomials we might have explicit formulas for the roots (such as the quadratic formula in the degree 2 case), but here we are saying that even without an explicit form of a root, we can always guarantee one exists regardless of the degree.

Say that $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is our polynomial, and let us suppose that $p(z)$ did not in fact have any complex roots. Then the denominator of $\frac{1}{p(z)}$ is never zero, so $\frac{1}{p(z)}$ is entire. The intuition is that this reciprocal should be bounded since

$$|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|$$

should go to ∞ as $|z| \to \infty$, meaning that $\frac{1}{p(z)}$ should decrease in modulus and hence be bounded. If so, then Liouville's theorem implies that $\frac{1}{p(z)}$ must be constant, and hence $p(z)$ must be constant as well. Thus the only way in which a complex polynomial can fail to have a root is for that polynomial to be constant, which is just a rephrasing of the claim made by the fundamental theorem of algebra: if the polynomial was not constant to begin with, there must have been a root.

To be more precise in the "bounding" part, we use the reverse triangle inequality:

$$|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \geq |a_n||z|^n - |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|.$$

As $|z|$ increases, the n-th power $|z|$ will grow more rapidly than any lower order power, so the first term $|a_n||z|^n$ on the right side above will eventually get much much larger than the term $|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|$ being subtracted. In particular, for large enough $|z|$ we have that

$$\tfrac{1}{2}|a_n||z|^n \text{ will be at least as large as } |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|,$$

so

$$|a_n||z|^n - |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \geq |a_n||z|^n - \tfrac{1}{2}|a_n||z|^n = \tfrac{1}{2}|a_n||z|^n$$

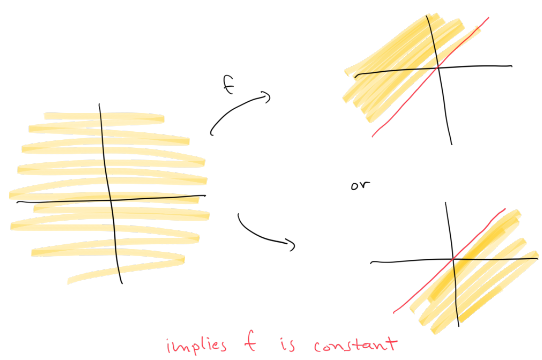
once $|z|$ is large enough. This then gives

$$\frac{1}{|a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|} \leq \frac{1}{|a_n||z|^n - |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|} \leq \frac{1}{\frac{1}{2}|a_n||z|^n}$$

once $|z|$ is large enough, so $\frac{1}{p(z)}$ is bounded outside some large enough circle $|z| = R$, and since it is also bounded within $|z| \leq R$, it is bounded everywhere as needed. (The fact that $\frac{1}{p(z)}$ is bounded within $|z| \leq R$ comes from the *extreme value theorem* you would have seen in a multivariable calculus course, but we will not be concerned with the details here.)

Lecture 17: Maximum Modulus Principle

Warm-Up 1. Suppose f is entire and has image lying fully on one side of the line $y = x$. We show that f must be constant:



The point is that regardless whether the image is above $y = x$ or below, we can rotate to put us into the scenario of having negative real part, which we argued last time forces an entire function to be constant.

If the image of f lies above $y = x$, we consider

$$g(z) = e^{i\pi/4} f(z).$$

This g is still entire and has image lying in the left-half plane $x < 0$ since multiplying by $e^{i\pi/4}$ rotates by $\pi/4$. Thus g is constant by a result from last time, so

$$\text{constant} = e^{i\pi/4} f(z) \rightsquigarrow (\text{constant})e^{-i\pi/4} = f(z),$$

meaning that f is constant as well. (Recall that the argument showing that g is constant is a consequence of Liouville's theorem and uses the fact that $\operatorname{Re} g < 0$ implies e^g has bounded modulus.) If instead the image of f were below $y = x$, we would use

$$g(z) = e^{-i\pi/4} f(z)$$

to get an entire function with negative real part and proceed similarly.

Warm-Up 2. Suppose f is entire and satisfies $|f^{(5)}(z)| \leq 3|e^{iz}|$ for all z. We derive what f has to look like fairly explicitly. From the given inequality we have

$$\left| \frac{f^{(5)}(z)}{e^{iz}} \right| \leq 3.$$

The function $f^{(5)}(z)/e^{iz}$ is thus entire (denominator is never zero) and bounded, so it is constant by Liouville's theorem. This means that $f^{(5)}(z)$ is a multiple of e^{iz}:

$$\frac{f^{(5)}(z)}{e^{iz}} = c \rightsquigarrow f^{(5)}(z) = ce^{iz}.$$

From here we can take antiderivatives to recover f. Anti-differentiating once gives

$$f^{(4)}(z) = \tfrac{c}{i} e^{iz} + a_0$$

for some constant a_0. Let us relabel $\frac{c}{i}$ as c since $\frac{c}{i}$ is still just some arbitrary constant. Anti-differentiating again gives

$$f^{(4)}(z) = ce^{iz} + a_0 \rightsquigarrow f^{(3)}(z) = ce^{iz} + a_0 z + a_1$$

where again we relabel the constant in front of e^{iz} as c. Continuing eventually gives

$$f(z) = ce^{iz} + a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

for some constants $c, a_0, a_1, a_2, a_3, a_4$.

Mean value theorem. Let us return to Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz$$

where f is holomorphic on some simply-connected domain containing z_0 and the circle $|z - z_0| = r$. Parametrize the circle using $z = z_0 + re^{it}, 0 \leq t \leq 2\pi$ to write this integral formula as

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{(z_0 + re^{it}) - z_0} ire^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} re^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

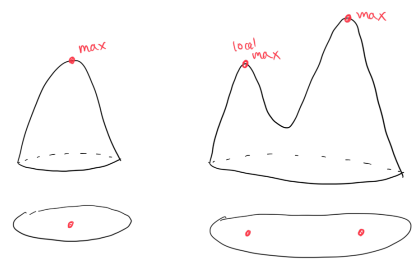
This final identity

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

is called the *mean value theorem* for contour integration, and expresses the value of a holomorphic f at the center z_0 of a circle as the average of its values $f(z_0 + re^{it})$ along the circle itself. Indeed, the right side "adds" up the values of f along the circle, and "averages" them out by dividing by the length 2π of the interval of integration. The fact that the value of f at *any* point can be expressed as such an average value is, as usual, in stark contrast to what happens for real functions. Note as well that it does not matter how large or how small the circle is—we always get the value at the center when averaging the values on the circle.

Maximum modulus principle. The mean value theorem has many consequences in complex analysis, but we will only give perhaps the most important one. Intuitively, if $f(z_0)$ is meant to be the average of values on a circle, then the "size" of $f(z_0)$ should relate directly to the "sizes" of this circular values, and in particular it should not be true that $f(z_0)$ is "larger" than the values $f(z_0 + re^{it})$ along the circle since $f(z_0)$ cannot be an average of values which are all "smaller".

We make the notion of "size" precise by talking about moduli, so the upshot is that there should be a restriction on how large $|f(z_0)|$ can be in relation to $|f(z_0 + re^{it})|$. The *maximum modulus principle* is thus the claim that, if f is nonconstant, the maximum of $|f(z)|$ on a disk $|z - z_0| \leq r$ cannot occur at z_0 (or, in fact, any interior point), and must occur along the boundary circle $|z - z_0| = r$. More generally, this holds for regions other than disks: if R is a bounded closed region in the domain on which (nonconstant) f is holomorphic, then the maximum modulus of f on R must occur on the boundary of R. This, again, is in stark contrast to what happens for real functions since it says that the typical type of "maximum" pictures from a multivariable calculus course like



cannot happen in the complex setting. In fact, the true takeaway is that the notion of "local maximum" does not exist at all in the complex case, or rather that the only way a local maximum can exist is for f to be constant everywhere.

To see this, suppose f has a local maximum at an interior point z_0 of the region R. (If a global maximum exists at an interior point, then that is in particular also a local maximum, so this local maximum case covers the original statement of the maximum modulus principle we gave above.) To be a local maximum means that

$$|f(z_0)| \geq |f(z)| \text{ for } z \text{ in some disk } |z - z_0| \leq r \text{ centered at } z_0.$$

From the mean value theorem we know that

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt, \text{ so } |f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|\,dt.$$

But the local maximum property $|f(z_0)| \geq |f(z_0 + re^{it}|$ then gives

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,dt = |f(z_0)|.$$

Hence we must have equality throughout:

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|\,dt.$$

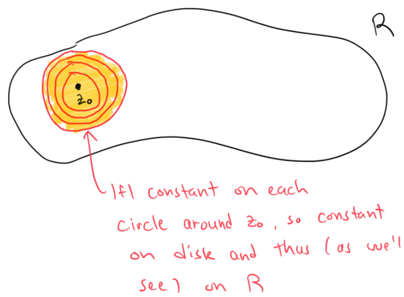
Write the left side as the integral $\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,dt$ and subtract one side from the other to get

$$\frac{1}{2\pi}\int_0^{2\pi} [|f(z_0)| - |f(z_0 + re^{it})|]\,dt = 0.$$

Here $|f(z_0)|-|f(z_0+re^{it}|$ is a nonnegative (because of the local maximum property) continuous function, and the only way the integral of such a function can be zero is for the function to be zero itself, so

$$|f(z_0)| - |f(z_0 + re^{it}| = 0, \text{ or } |f(z_0)| = |f(z_0 + re^{it})|.$$

But the radius r could have been anything (as long as we remain within R), so by shrinking this radius we get that f must have constant modulus on the original disk $|z - z_0| \leq r$:



Constant modulus implies constant function based on an early homework problem we did, so we get that if f has a local maximum at the interior point z_0, it must be constant on a disk around z_0, and this implies that f is constant on the entire region R. (The fact that constant on a disk implies constant everywhere comes from the *identity theorem* we will state in a few days. The book has an argument which avoids the identity theorem, but the identity theorem gives a cleaner approach.)

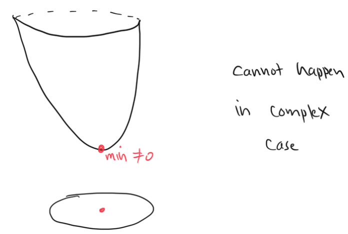
Example. We determine the points at which $|z^2+3z-1|$ has a maximum value on the disk $|z| \leq 1$. The function $f(z) = z^2 + 3z - 1$ is not constant, so the maximum modulus principle tells us that the maximum must occur along the boundary circle $|z| = 1$. (In similar problems in a multivariable calculus course, checking for maxima along the boundary is not enough and interior points must also be considered, so view the maximum modulus principle here as a tool for narrowing down where the maximum must occur.) Thus we need only consider the value of f at points $z = e^{it}$:

$$\begin{aligned}|z^2 + 3z - 1| &= |e^{2it} + 3e^{it} - 1| \\ &= |e^{it}||e^{it} + 3 - e^{-it}| \\ &= |3 + i2\sin t| \\ &= \sqrt{9 + 4\sin^2 t},\end{aligned}$$

where in the second step we factored out e^{it} in order to be left with the $e^{it}-e^{-it}$ piece that simplifies to a sine expression.

The value $\sqrt{9+4\sin^2 t}$ is maximized when $\sin^2 t = 1$, so at $t = \pm\frac{\pi}{2}$. Thus $|z^2+3z-1|$ attains it maximum on the disk $|z| \leq 1$ at $z = i$ and $z = -i$, and this maximum value is $\sqrt{9+4} = \sqrt{13}$.

Minimum modulus. If $f(z_0) = 0$ at some z_0, then 0 is the minimum value of $|f(z)|$ since a modulus can never be negative. But if f is never zero, then the minimum modulus f will be some positive number. If f is never zero, then $\frac{1}{f}$ is holomorphic, and the minimum value of $|f|$ becomes the maximum value of $|\frac{1}{f}|$. Thus the maximum modulus principle applied to $\frac{1}{f}$ gives what we call the *minimum modulus principle* for f: if f is non-constant and never zero, then the minimum modulus of f in some bounded region R must occur on the boundary of R. In particular, "local minimums" do not exist in the complex case for nonzero nonconstant functions:



Lecture 18: Power/Taylor Series

Warm-Up 1. We find the maximum and minimum values of $|e^{z^2}|$ on $|z| \leq 2$. Note that e^{z^2} is never zero, so the desired minimum value is positive. The maximum and minimum moduli principles imply that these extreme values occur on the boundary circle $|z| = 2$. Parametrizing this as $z = 2e^{it}$ gives

$$|e^{z^2}| = |e^{4e^{2it}}| = |e^{4\cos 2t + i4\sin 2t}| = e^{4\cos 2t}.$$

The maximum value thus occurs when $\cos 2t = 1$, so when $t = 0, \pi$, and the minimum value occurs when $\cos 2t = -1$, so when $t = \pm\frac{\pi}{2}$. Hence the maximum of $|e^{z^2}|$ on $|z| \leq 2$ occurs at $z = \pm 2$ and has value e^4 and the minimum occurs at $z = \pm 2i$ and has value e^{-4}.

Warm-Up 2. We derive the fundamental theorem of algebra as a consequence of the minimum modulus principle. Recall that the fundamental theorem of algebra says that if $p(z)$ is a nonzero complex polynomial, then $p(z)$ has a root, which we previously justified using Liouville's theorem.

Suppose $p(z) = a_n z^n + \cdots + a_1 z + a_0$ has no root, so that the minimum modulus principle applies. In particular then, the minimum of $|p(z)|$ on any disk centered at 0 does not occur in the interior of that disk unless $p(z)$ were constant. Using the inequalitiy

$$|p(z)| \geq |a_n||z|^n - (|a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|)$$

derived from the reverse triangle inequality, since $|z|^n$ grows much more quickly than any other power above as $|z| \to \infty$ we see that

$$|p(z)| \to \infty \text{ as } |z| \to \infty.$$

But then on some large enough circle $|z| = R$ we would have

$$|p(z)| > |a_0| = |p(0)|$$

since $|p(z)|$ can get arbitrarily large, which would say that the minimum of $|p(z)|$ on $|z| \leq R$ would not occur on the boundary circle since the modulus of values on the boundary are larger than that at the interior point $z = 0$. This can only happen if $p(z)$ is constant, so the only complex polynomials without roots are the nonzero constant ones as claimed.

Integrals and series. Let us come back to Cauchy's integral formula. Fix z_0 in the domain of f and take a circle centered at z_0 as a contour, so that

$$f(z) = \frac{1}{2\pi i} \oint_{|w-z_0|=R} \frac{f(w)}{w - z}\, dw.$$

for z in the disk $|z - z_0| < R$. We now use this to derive a nice way of expressing $f(z)$. Write the $\frac{1}{w-z}$ term in the integrand above as

$$\frac{1}{w-z} = \frac{1}{(w-z_0) - (z-z_0)} = \frac{1}{w-z_0}\left(\frac{1}{1 - \frac{z-z_0}{w-z_0}}\right),$$

were in the first step we subtract and add z_0 and in the second step we factor $w - z_0$ out of the denominator.

The point is that the final term in parentheses we are left with is of the form $\frac{1}{1-y}$ for $y = \frac{z-z_0}{w-z_0}$, and such expressions can be expanded by means of a *geometric* series:

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \text{ for } |y| < 1.$$

Recall that this series identity means that $\sum_{n=0}^{\infty} y^n$ *converges* to $\frac{1}{1-y}$ for $|y| < 1$, which in turns means that the *partial sums*

$$1 + y + y^2 + \cdots + y^n$$

approach $\frac{1}{1-y}$ as $n \to \infty$ for $|y| < 1$. This comes from the fact that

$$1 + y + y^2 + \cdots + y^n = \frac{1 - y^{n+1}}{1-y} \text{ for } y \neq 1,$$

which is an identity we saw back on the first homework, and the fact that powers of y approach 0 when $|y| < 1$. This all works the same way as what you would have seen in a previous calculus course for a real geometric series, where now the only difference is that we get convergence on a disk $|y| < 1$ of radius 1 instead of just an interval as in the real case.

With this we can write our expression $\frac{1}{w-z}$ above as

$$\frac{1}{w-z} = \frac{1}{w-z_0}\left(\frac{1}{1-\frac{z-z_0}{w-z_0}}\right) = \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty}\frac{1}{(w-z_0)^{n+1}}(z-z_0)^n$$

when $|\frac{z-z_0}{w-z_0}| < 1$, or equivalently $|z - z_0| < |w - z_0|$. Multiplying through by $f(w)$ then gives

$$\frac{f(w)}{w-z} = \sum_{n=0}^{\infty}\frac{f(w)}{(w-z_0)^{n+1}}(z-z_0)^n \text{ when } |z-z_0| < |w-z_0|.$$

Holomorphic implies analytic. We substitute this series expression for $\frac{f(w)}{w-z}$ into the right side of the integral formula

$$f(z) = \frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{w-z}\,dw$$

and manipulate a bit to get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\oint_{|w-z_0|=R}\left(\sum_{n=0}^{\infty}\frac{f(w)}{(w-z_0)^{n+1}}(z-z_0)^n\right)dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n. \end{aligned}$$

(If you have seen the notion of *uniform convergence* before, note that we are using the fact that power series always converge uniformly on closed regions within their disks of convergence in order to be able to swap summation and integration in the final step above. If you have not seen this notion before, no worries as it is not something we will develop in this course.)

What we get as a result is thus a *power series* representation for f on the disk $|z - z_0| < R$. Functions which are expressible as power series centered at any point in their domains are said to be *analytic*, so the upshot (this is what's called *Taylor's theorem* in complex analysis) is that holomorphic functions are thus always analytic as a consequence of the integral formula. This is big, as it is not true in the real case that differentiable functions are always real analytic (take the second quarter of the real analysis sequence to see why). Morever, power series themselves always have a first derivative, a second derivative, a third derivative, and so on, so we get as a consequence that holomorphic functions are always infinitely differentiable; we had previously alluded to this when discussing "differentiation under the integral sign", but now we have the definitive reason as to why this is true.

Recall that the book uses the term "analytic" to mean just "holomorphic", and the point is that we now know the two terms are equivalent. However, as mentioned earlier, we prefer to use the term holomorphic to mean differentiable at all points in a domain as opposed to analytic, since the true meaning of analytic has to do with the property of being representable as a power series. Yes the two notions end up meaning the same thing in the setting of complex analysis, but not in the setting of real analysis, so it makes sense to make a slight distinction between how these terms are used. The fact that holomorphic functions are analytic is really a fundamental fact in complex analysis, and using the term "analytic" to mean "holomorhpic" does not quite get this across.

Coefficients and convergence radii. Recall (from a previous course) that the coefficients in a usual power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

are the *Taylor coefficients* of f at z_0:

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

In the power series expansion we derived above the coefficients were

$$a_n \frac{1}{2\pi i} \oint_{|w-z_0|=R} \frac{f(w)}{(w - z_0)^{n+1}}\, dw,$$

but we can now recognize this as precisely the integral that appears in Cauchy's formula for higher-order derivatives:

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{|w-z_0|=R} \frac{f(w)}{(w - z_0)^{n+1}}\, dw.$$

Thus the expression we derived above *is* indeed the Taylor series of f centered at z_0. To be more precise, it is this derivation which justifies the higher-order Cauchy formula obtained before without having to worry about whether "differentiation under the integral sign" is actually valid—it *is* valid, as a consequence of the holomorhpic implies analytic result.

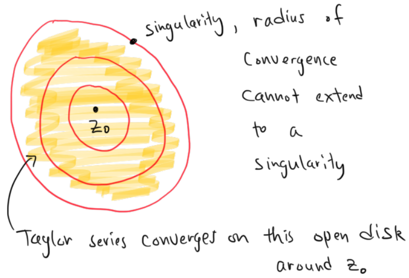
The convergence of the Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

is also easy to determine. Recall that this series expansion was derived from Cauchy's integral formula when integrating over a circle $|w - z_0| = R$ centered at z_0. This integral formula holds for any such circle on which f remains holomorphic, so we get a valid convergent series for

$$|z - z_0| < |w - z_0| = R$$

for as large a radius R as we can take while maintaining holomorphicity. This breaks down once we hit a "singuarlity" of f, which we take to mean a point at which f is not differentiable:



Thus, the upshot is that the Taylor series

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{|w-z_0|=R} \frac{f(w)}{(w-z_0)^{n+1}}\, dw \right) (z-z_0)^n = f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

has radius of convergence equal to the distance from z_0 to the *nearest* singularity of f. Whereas in a previous calculus course you had different tools available to find radii of convergence—such as the ratio test or the root test—no such things are necessary in the complex setting, where radii are easy to find!

Examples. The geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

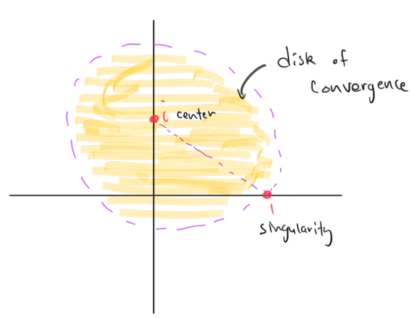
converges for $|z| < 1$, so has radius of convergence 1. This makes sense from our discussion above, as 1 *is* the distance from the center 0 to $z = 1$, which is the only singularity of $\frac{1}{1-z}$.

The Taylor series of e^z centered at 0 is the usual one

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

as can be found by noting that all derivatives of e^z are e^z, and evaluating these at 0 always gives 1. Since e^z is entire, it has no singularities, so the power series above has infinite radius of convergence since we never hit a singularity when moving away from the center 0. The same is true of the Taylor expansions of $\sin z$, $\cos z$, or any other entire function.

More examples. The Taylor series of $\frac{1}{1-z}$ centered at i, say, has radius of convergence $\sqrt{2}$ since $\sqrt{2}$ is the distance from i to the singularity 1 of $\frac{1}{1-z}$. We thus get a disk of convergence for this series which looks like



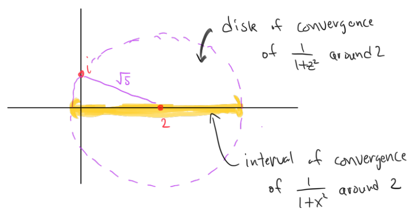
This particular Taylor series can be found using the same type of geometric series manipulation we used in the "holomorphic implies analytic" result:

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i}\left(\frac{1}{1-\frac{z-i}{1-i}}\right) = \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n = \sum_{n=0}^{\infty}\frac{1}{(1-i)^{n+1}}(z-i)^n.$$

The Taylor series of $\frac{1}{1+z^2}$ centered at 2 has radius of convergence

$$\text{distance from 2 to } \pm i = |2 \pm i| = \sqrt{5}$$

since the only singularities are at $z = \pm i$, which are of equal distance to 2. Restricting this disk of convergence to the real axis gives the *interval* of convergence of the real Taylor series of $\frac{1}{1+x^2}$ centered at 2, which thus has real radius of convergence $\sqrt{5}$ as well:



Trying to determine this real radius of convergence using only real methods is likely to be a fruitless (or at least very challenging) endeavor, with the main reason being that finding the desired real Taylor series using real methods only is nearly impossible. There is no discernible pattern to the derivatives of $\frac{1}{1+x^2}$ at 2 that makes finding the general Taylor coefficient feasible, and without these explicit Taylor coefficients no ratio nor root test is applicable. And yet, we see that by viewing this real Taylor series as the restriction of a complex Taylor series, the radius is simple to find.

Lecture 19: More on Power Series

Warm-Up 1. We represent $\frac{z-2}{(i+z)^2}$ as a power series centered at 2 and determine the radius of convergence of this series. This latter question is one we can easily answer now: the radius will equal the distance from the center 2 to the nearest singularity of $\frac{z-2}{(i+z)^2}$, which in this case is i, so the radius of convergence is $|2-i| = \sqrt{5}$.

To obtain the desired series expansion we start with the same type of geometric series manipulation we saw in the derivation of Taylor's theorem (holomorphic implies analytic) last time. We have

$$\frac{1}{i+z} = \frac{1}{(i+2)+(z-2)} = \frac{1}{i+2}\left(\frac{1}{1+\frac{z-2}{i+2}}\right) = \frac{1}{i+2}\left(\frac{1}{1-(-\frac{z-2}{i+2})}\right).$$

Using the standard geometric series $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ for $|y| < 1$, we thus get

$$\frac{1}{i+z} = \frac{1}{i+2}\left(\frac{1}{1-(-\frac{z-2}{i+2})}\right) = \frac{1}{i+2}\sum_{n=0}^{\infty}\left(-\frac{z-2}{i+2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{(i+2)^{n+1}}(z-2)^n,$$

which is valid when $|\frac{z-2}{i+2}| < 1 \iff |z-2| < |i+2| = \sqrt{5}$, as expected.

Recall that real power series are always differentiable and can be differentiated term-by-term, and in fact the same applies to complex power series. We can thus differentiate both sides of the expression obtained above to get

$$-\frac{1}{(i+z)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{(i+2)^{n+1}}(z-2)^{n-1},$$

and then multiply through by $-(z-2)$ to get

$$\frac{z-2}{(i+z)^2} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(i+2)^{n+1}}(z-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(i+2)^{n+1}}(z-2)^n,$$

which is the desired expansion of $\frac{z-2}{(i+z)^2}$ centered at 2.

Warm-Up 2. Suppose f is entire and satisfies $|f(z)| \leq 10|z|^4$ for $|z| > 100$. We justify the fact that f must be a polynomial of degree at most 4. For this we use Cauchy's integral formula for higher-order derivatives:

$$f^{(n))}(0) = \frac{n!}{2\pi i} \oint_{|z|=R} \frac{f(w)}{w^{n+1}}\, dw$$

where we take the radius R to be larger than 100. Using the bound $|f(z)| \leq 10|z|^4$ for $|z| > 100$, we then have

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi} \oint_{|z|=R} \frac{|f(w)|}{|w|^{n+1}} |dw| \leq \frac{n!}{2\pi} \oint_{|z|=R} \frac{10R^4}{R^{n+1}} |dw| = \frac{n!}{2\pi} \frac{10R^4}{R^{n+1}} (2\pi R) = n! 10 R^{4-n}.$$

If $n > 4$, we are left with a negative exponent of R here, so the limit as $R \to \infty$ is zero and thus we get that

$$f^{(n)}(0) = 0 \text{ for } n > 4.$$

Now, since f is entire is must equal its Taylor series centered at 0 at all points, so

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \text{ for all } z.$$

But the coefficient in this series are 0 for $n > 4$ since $f^{(n)}(0) = 0$ for this values, so the series is just a polynomial and

$$f(z) = f(0) + f'(0)z + \tfrac{1}{2} f''(0)z^2 + \tfrac{1}{3!} f'''(0)z^3 + \tfrac{1}{4!} f^{(4)}(0)z^4$$

s a polynomial of degree at most 4 as claimed.

Isolated zeroes. The fact that holomorphic functions are always expressible as power series leads to more restrictions on their behavior. Suppose f is not constant and has a zero at z_0. Expand f as a power series centered at z_0:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \text{ on some disk } |z-z_0| < R.$$

The constant term above is $c_0 = f(z_0)$, which is zero by assumption. If all coefficients were zero then the power series and hence f would be the constant zero function, but we are assuming f is not constant. So, not all coefficients in the expansion above are zero; say the first nonzero coefficient is c_k, so that the expansion above looks like

$$f(z) = c_k(z - z_0)^k + c_{k-1}(z - z_0)^{k+1} + \cdots$$

where $k \geq 1$. Factor out $(z - z_0)^k$ to get

$$f(z) = (z - z_0)^k \underbrace{[c_k + c_{k-1}(z - z_0) + \cdots]}_{h(z)} = (z - z_0)^k h(z),$$

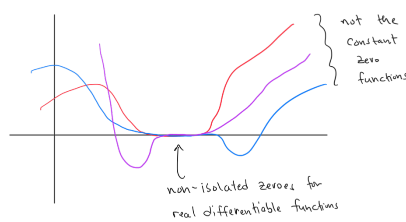
where $h(z)$ is the name we give to the holomorphic function defined by the series in brackets above.

Since $c_k = h(z_0)$ is nonzero (because c_k was supposed to be the first nonzero coefficient in the expansion of f), $h(z)$ is also nonzero on some disk around z_0. Thus, on this disk the only way in which

$$f(z) = (z - z_0)^k h(z)$$

could be zero is for the $(z - z_0)^k$ factor be zero, but his happens only at z_0. The upshot is that there is a disk around z_0 on which the only zero of f is z_0 itself, so z_0 is what we call an *isolated* zero of f. (Being isolated means that there is some positive distance between it and the next closest zero.) For example, the zeroes of $\sin z$ are $z = n\pi$ for n an integer, which are indeed isolated.

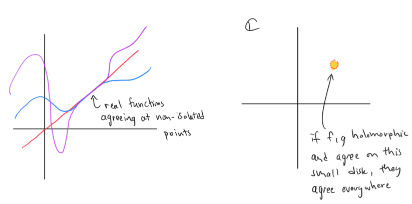
As a consequence, the only way in which a holomorphic can have non-isolated zeroes is for f to be the constant zero function. In the real case this is not true, as a real differentiable function can be zero an entire interval without being zero everywhere:



But in the complex case, if a holomorphic function were zero on an interval (or on a disk), it would necessarily be zero everywhere as it would have non-isolated zeroes.

Identity theorem. This leads to the fact that if two holomorphic functions agree on non-isolated points, then they must be exactly the same function. Indeed, if f and g agree at non-isolated points (such as all points within some disk or on some line segment), then $f - g$ has non-isolated zeroes, so $f - g$ must be the constant zero function, meaning that $f = g$ everywhere on their domains. We will call this result the *identity theorem*.

Again this is in stark contrast to the real case, where we can have differentiable functions like



all agreeing on some interval while being different elsewhere. If f and g are holomorphic and all we know that is they agree even on some incredibly small disk (say the size of an electron!), we will then know that they agree everywhere. Good stuff.

Uniqueness of analytic continuations. With the identity theorem we can now go back and justify various definitions we gave previously and easily justify various identities. For example, ages ago we defined e^z as

$$e^z = e^x \cos y + ie^x \sin y$$

where $z = x + iy$. At the time we motivated this by using the idea that e^{x+iy} "should" equal $e^x e^{iy}$ and by using the definition we gave on the second day of class for e^{iy}; we stated at the time that we would eventually see we really had no choice in matter, and now we can understand why. The point is that if f is any entire function which gives the values $f(x) = e^x$ when x is real, then f must necessarily be given by

$$f(z) = e^x \cos y + ie^x \sin y.$$

Indeed, both f and the right side here are entire functions (we verified the right side was entire previously using the Cauchy-Riemann equations) which agree on the real axis, so since points on the real axis are not isolated, f and the right side above must be the same function everywhere. We call the right side above the *analytic continuation* of e^x for x real, since it is the only way we can extend $f(x) = e^x$ for x real so as to be holomorphic for z complex. Analytic continuations, if they exist, are unique as a consequence of the identity theorem.

This then is the reason why the definition $e^{i\theta} = \cos\theta + i\sin\theta$ from the start of the quarter was the only possible thing we could have used. It is also the reason why the definitions we gave for

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) \quad \text{and} \quad \sin z = \tfrac{1}{2}(e^{iz} + e^{-iz})$$

were the only one possible: both right sides here are holomorphic (entire) and agree with $\cos x$ and $\sin x$ respectively for x real, so they must be the analytic continuations of $\cos x$ and $\sin z$ to $\cos z$ and $\sin z$ for z complex.

We can use this idea to give a quick justification for the identity

$$\sin^2 z + \cos^2 z = 1 \text{ for } z \text{ complex},$$

at least taking for granted the fact that this holds when $z = x$ is real: both $\sin^2 z + \cos^2 z$ and the constant function 1 are entire and agree on the real axis (which consists of non-isolated points), so $\sin^2 z + \cos^2 z$ and 1 agree everywhere. This identity can of course be verified algebraically using the definitions of $\sin z$ and $\cos z$, by the identity theorem approach gives a nice clean approach. Similarly, if we fix w and think of both

$$e^{z+w} \quad \text{and} \quad e^z e^w$$

as entire functions of the variable z, then since e^{z+w} and $e^z e^w$ agree when z, w are real, they must agree everywhere so that $e^{z+w} = e^z e^w$ for all z, w.

Lecture 20: Laurent Series

Warm-Up. Suppose f is entire and real-valued on an interval (a, b) on the real axis. We justify the fact that f is real-valued on the entire real axis. This applies to functions like $e^z, \sin z$ and $\cos z$ for example, where if we only they knew they were real-valued on some small real interval, we would immediately get that they were real-valued at all real inputs. We use the fact that if $f(z)$ is entire then so is the function

$$\overline{f(\overline{z})},$$

which was a problem back on the set of practice problems for the first midterm. (The point is that $\overline{f(\overline{z})}$ will satisfy the Cauchy-Riemann equations if f itself does.)

To say that $f(x)$ is real-valued when x is in (a, b) means that $f(x)$ equals its own conjugate, so

$$f(x) = \overline{f(x)} = \overline{f(\overline{x})} \text{ for } x \text{ in } (a, b).$$

(The conjugate on x in the final term comes from $x = \overline{x}$ being real.) This says that $f(z)$ and $\overline{f(\overline{z})}$ are entire functions that agree at the non-isolated points of (a, b), so these functions must agree everywhere:

$$f(z) = \overline{f(\overline{z})} \text{ for all } z.$$

For x any real number (not just in (a, b)), this gives $f(x) = \overline{f(x)}$, which means that $f(x)$ is real for real x as claimed.

Series expansions with singularities. Power series can be used to express holomorphic functions as series, but often we want to be able to handle functions which have singularities. (Think about the integrands $\frac{f(z)}{z-z_0}$ that appear in Cauchy's integral formula for example.) How can we come up with series expansions that incorporate singularities?

For example, take

$$\frac{e^z}{(z-2)^3}.$$

The numerator is entire so we can expand it as a power series centered at 2 using the usual $e^w = \sum_{n=0}^{\infty} w^n/n!$:

$$e^z = e^{2+z-2} = e^2 e^{z-2} = e^2 \sum_{n=0}^{\infty} \frac{(z-2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^n.$$

Dividing by $(z-2)^3$ then gives

$$\frac{e^z}{(z-2)^3} = \frac{1}{(z-2)^3} \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^{n-3}.$$

The resulting series is *almost* a power series, only that some of the exponents of $z-2$ are negative, meaning that we get $z-2$ terms in denominators:

$$\frac{e^z}{(z-2)^3} = \frac{e^2}{(z-2)^3} + \frac{e^2}{(z-2)^2} + \frac{e^2}{2!(z-2)} + \frac{e^2}{3!} + \frac{e^2}{4!}(z-2) + \cdots.$$

This is an example of what is called a *Laurent series*, in this case centered at 2. The nonnegative exponent terms

$$\frac{e^2}{3!} + \frac{e^2}{4!}(z-2) + \frac{e^2}{5!}(z-2)^2 + \cdots$$

in

$$\frac{e^z}{(z-2)^3} = \frac{e^2}{(z-2)^3} + \frac{e^2}{(z-2)^2} + \frac{e^2}{2!(z-2)} + \underbrace{\sum_{n=3}^{\infty} \frac{e^2}{n!}(z-2)^{n-3}}_{\text{power series}}$$

define an ordinary power series, so this piece converges on some disk centered at 2. The first three terms on the right above above exist everywhere except at $z = 2$, so the full Laurent series converges on a *punctured* disk.

Another example. For another example take

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

We can certainly expand this as a power series centered at 0 on some disk around 0 since this function is holomorphic on such a disk. (The radius of the disk is 1 as this is the distance from the center 0 to the nearest singularity $z = 1$.) But what if we want to expand this as a series centered at 0 on a region that goes beyond $z = 1$? A power series will not suffice due to the singularity at 1, but if we allow negative exponents of z we can make it work.

In fact, we claim that we can express this function as a Laurent series centered at 0 on the *annulus* $1 < |z| < 2$, which is the region between the circles $|z| = 1$ and $|z| = 2$. Indeed, take the $\frac{1}{z-2}$ piece and manipulate as

$$\begin{aligned} \frac{1}{z-2} &= \frac{1}{-2(1-\frac{z}{2})} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \\ &= -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n \end{aligned}$$

where along the way we use a geometric series. By geometric series stuff, this converges when $|\frac{z}{2}| < 1$, so far $|z| < 2$.

For the $\frac{1}{z-1}$ piece we do something similar, only by factoring out z at the start of our manipulation:

$$\begin{aligned} \frac{1}{z-1} &= \frac{1}{z(1-\frac{1}{z})} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}. \end{aligned}$$

This converges when $|\frac{1}{z}| < 1$, so for $1 < |z|$, which describes the region outside of a circle. Thus, putting everything together gives

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1} = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}},$$

which is a Laurent series centered at 0. This converges on the overlap of $|z| < 2$ and $1 < |z|$ (overlap between we need each piece to converge separately), so we get convergence on the annulus $1 < |z| < 2$ as claimed.

Laurent series. Let us be more precise. A *Laurent series* centered at z_0 is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n,$$

where we allow negative exponents of $z - z_0$. In general, a Laurent series will converge on an annulus, which is the Laurent analog of how a power/Taylor series converges on a disk. To see why, it is useful to break up the sum above into the negative exponent terms and everything else:

$$\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

The second sum is a usual power series, so it converges on a disk $|z - z_0| < R_2$. For the first sum, we consider instead the power series which has the same a_{-n} as its coefficients:

$$\sum_{n=1}^{\infty} a_{-n} w^n.$$

This converges on some disk $|w| < r_1$, so by substituting $w = \frac{1}{z-z_0}$ we see that the negative-exponent piece of our Laurent series converges when $|\frac{1}{z-z_0}| < r_1$, so for $\frac{1}{r_1} < |z - z_0|$. Thus the full Laurent series

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

converges on the overlap $R_1 < |z - z_0| < R_2$, where $R_1 = \frac{1}{r_1}$, which is an annulus. (In the case where $R_1 = 0$, the annulus is actually a punctured disk as in the $\frac{e^z}{(z-2)^3}$ example. When $R_2 = \infty$, we get convergence on the unbounded outside a circle, and when $R_1 = 0$ and $R_2 = \infty$ we have convergence on a punctured plane.)

As another example, consider $e^{1/z}$. Using the usual $e^w = \sum_{n=0}^{\infty} w^n/n!$ when $w = \frac{1}{z}$, we get

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!z^n},$$

which is a Laurent series centered at 0 that converges on the punctured plane $\mathbb{C}^*$. This series looks like

$$e^{1/z} = \cdots + \frac{1}{3!z^3} + \frac{1}{2z^2} + \frac{1}{z} + 1$$

with infinitely many negative exponent terms—with increasing powers of z in the denominator—heading towards the left.

Deriving the coefficients. Recall that in the usual Taylor series case, the coefficients of

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

are explicitly given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

What about the coefficients of a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n?$$

What are they given by? To start, we first integrate both sides around some circle centered at z_0 and exchange integration and summation:

$$\oint_{|z-z_0|=r} f(z)\,dz = \oint_{|z-z_0|=r} \left(\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n\right) dz = \sum_{n=-\infty}^{\infty}\left(\oint_{|z-z_0|=r} a_n(z-z_0)^n\,dz\right).$$

Every $(z-z_0)^n$ term at the end except for the $n=-1$ term has an antiderivative $\frac{1}{n+1}(z-z_0)^{n+1}$, so these terms integrate (over a closed contour) to zero by the fundamental theorem of calculus. (We saw this same idea in an example when integrating $\frac{1}{z}(z+\frac{1}{z})^{2n}$ a while back.) Thus the only potentially nonzero term in the final sum above comes from the $n=-1$ term, so

$$\oint_{|z-z_0|=r} f(z)\,dz = \sum_{n=-\infty}^{\infty}\left(\oint_{|z-z_0|=r} a_n(z-z_0)^n\,dz\right) = \oint_{|z-z_0|=r} \frac{a_{-1}}{z-z_0}\,dz.$$

But the integral of $\frac{1}{z-z_0}$ over the circle $|z-z_0|=r$ is $2\pi i$ (the most crucial integral of all time!), so we get

$$\oint_{|z-z_0|=r} f(z)\,dz = \oint_{|z-z_0|=r} \frac{a_{-1}}{z-z_0}\,dz = 2\pi i a_{-1},$$

and thus

$$a_{-1} = \frac{1}{2\pi i}\oint_{|z-z_0|=r} f(z)\,dz.$$

(This specific coefficient a_{-1} is called the *residue* of f at z_0, where the name comes from the fact that it is what's "left behind" after integrating. We will discuss residues and their uses in detail in a few days.)

The other negative exponent terms can be derived similarly by essentially "shifting" the exponent that is left behind when integrating. For example, from

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

we have

$$(z-z_0)f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^{n+1}.$$

When integrating the only potential nonzero term comes when $n+1=-1$, so when $n=-2$. Thus

$$\oint_{|z-z_0|=r} (z-z_0)f(z)\,dz = 2\pi i a_{-2}, \text{ so } a_{-2} = \frac{1}{2\pi i}\oint_{|z-z_0|=r} (z-z_0)f(z)\,dz.$$

To get a_{-3} we would integrate $(z-z_0)^2 f(z)$ instead, and in general we get

$$a_{-n} = \frac{1}{2\pi i}\oint_{|z-z_0|=r} (z-z_0)^{n-1} f(z)\,dz.$$

To derive formulas for the nonnegative-exponent terms, we instead divide by powers of $z-z_0$: we have

$$\frac{f(z)}{(z-z_0)^k} = \frac{1}{(z-z_0)^k}\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^{n-k},$$

so that the term left when integrating occurs when $n-k=-1$, so for $n=k-1$. Thus

$$\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^k}\,dz = 2\pi i a_{k-1}, \text{ so } a_{k-1} = \frac{1}{2\pi i}\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^k}\,dz.$$

The upshot is that we can derive expressions for the coefficients of a Laurent series in terms of integrals, analogously to how the coefficients of a Taylor series can be expressed in terms of derivatives. (Note that a power series is a special case of a Laurent series, namely the case where all negative exponent terms have coefficient zero. Thus the Laurent coefficients for a_{k-1} when $k \geq 0$ should literally match what we expect from usual Taylor series, and indeed they do: the integral formula

$$\frac{1}{2\pi i}\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^k}\,dz$$

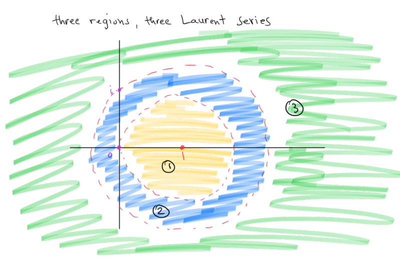
for a_{k-1} derived above does give $\frac{f^{(k-1)}(z_0)}{(k-1)!}$ when f is holomorphic by the higher-order derivative form of Cauchy's integral formula.) We caution, however, that these integral formulas for the Laurent coefficients will not be a useful took for actually computing Laurent series, where other simpler methods (such as ones we used in the examples we saw before) are usually available; rather, these integral formula are more conceptually useful as a way to highlight the connection between Laurent series and integration, which we will develop more next time.

Lecture 21: More on Laurent Series

Warm-Up. We expand

$$\frac{2z-i}{z(z-i)} = \frac{1}{z} + \frac{1}{z-i}$$

as a Laurent series centered at 1 in three different regions, namely in the disk $|z-1|<1$, in the annulus $1<|z-1|<\sqrt{2}$, and in the region $\sqrt{2}<|z-1|$ outside the circle $|z-1|=\sqrt{2}$, which we think of as an annulus with infinite "outer" radius:



In the first region, the Laurent expansion of this function is actually an ordinary power/Taylor series expansion since the function is holomorphic on $|z-1|<1$. We use a manipulation with a geometric series:

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{n=0}^{\infty}(-(z-1))^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$\frac{1}{z-i} = \frac{1}{(1-i)+(z-1)} = \frac{1}{(1-i)(1+\frac{z-1}{1-i})} = \frac{1}{1-i}\sum_{n=0}^{\infty}\left(-\frac{z-1}{1-i}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{(1-i)^{n+1}}(z-1)^n.$$

The first series converges for $|z-1|<1$ and the second for $|\frac{z-1}{1-i}|<1 \iff |z-1|<\sqrt{2}$, so their sum converges for $|z-1|<1$:

$$\frac{2z-i}{z(z-i)}=\frac{1}{z}+\frac{1}{z-i}=\sum_{n=0}^{\infty}(-1)^n(z-1)^n+\sum_{n=0}^{\infty}\frac{(-1)^n}{(1-i)^{n+1}}(z-1)^n.$$

To get convergence in the annulus $1<|z-1|<\sqrt{2}$, we can keep the second series the same as it already converges for $|z-1|<\sqrt{2}$ as is, but we must "invert" the $z-1$ term in the first series in order to get convergence for $1<|z-1|$. We do this again via a geometric series manipulation:

$$\frac{1}{z}=\frac{1}{1+(z-1)}=\frac{1}{(z-1)(\frac{1}{z-1}+1)}=\frac{1}{z-1}\sum_{n=0}^{\infty}\left(-\frac{1}{z-1}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}}.$$

This converges when $|\frac{1}{z-1}|<1$, so for $|z-1|>1$ as desired. Thus we have

$$\frac{2z-i}{z(z-i)}=\frac{1}{z}+\frac{1}{z-i}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(1-i)^{n+1}}(z-1)^n$$

on $1<|z-1|<\sqrt{2}$. Note that the inner and outer radius of this annulus are precisely the distances from the center 1 to the nearest singularity at 0 and the further singularity at i, respectively.

Finally, to get convergence on $\sqrt{2}<|z-1|$ we leave the first term above as is and invert the second using

$$\frac{1}{z-i}=\frac{1}{(1-i)+(z-1)}=\frac{1}{(z-1)(\frac{1-i}{z-1}+1)}=\frac{1}{z-1}\sum_{n=0}^{\infty}\left(-\frac{1-i}{z-1}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n(1-i)^n}{(z-1)^{n+1}},$$

which converges when $|frac1-iz-1<1$, which is the same as $\sqrt{2}<|z-1|$. Thus we have

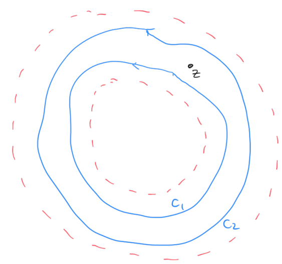
$$\frac{2z-i}{z(z-i)}=\frac{1}{z}+\frac{1}{z-i}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n(1-i)^n}{(z-1)^{n+1}}$$

on $\sqrt{2}<|z-1|$.

Cauchy's formula for an annulus. Laurent series converge on annuli in general, and indeed our aim is to now show that holomorphic functions on annuli do always have Laurent series representations. The fact that holomorphic functions can always be represented by power series (centered at a point at which the function is differentiable) was a consequence of the Cauchy integral formula, so in order to derive Laurent series expansions (centered at a point at which the function might not be differentiable) we need an analog of the integral formula for annuli. The result is that if f is holomorphic on an annulus, then for any z in that annulus we have

$$f(z)=\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw-\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw$$

where C_2, C_1 are simple closed contours in the annulus which lie exterior to and interior to z respectively:

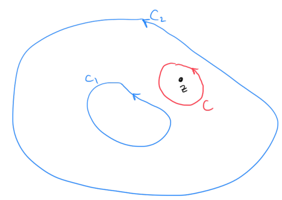


If f were holomorphic on the region interior to C_1, the integral over C_1 above would be zero by Cauchy's theorem (note that $\frac{1}{w-z}$ is holomorphic on this interior region since z lies exterior to C_1), so the formula above reduces to

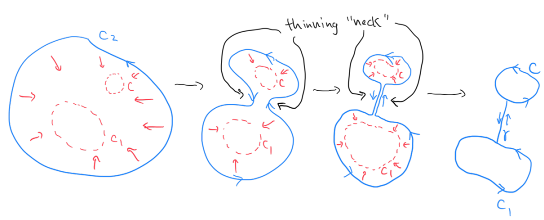
$$f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z}\, dw,$$

which is the usual Cauchy integral formula. This new version is thus more general in that it handles the case where f is not holomorphic everywhere inside C_1.

This justification for this annular integral formula is an application of the deformation theorem. Take a small circle C around z between C_1 and C_2:



We can deform the outer contour C_2 into C *combined with* C_1 as follows:



That is, we deform C_2 by "wrapping" it around C and around C_1 with an ever-shrinking neck as we go, so that at the end we end up with C, C_1, and a line segment γ connecting them. (This resulting contour is no longer simple, but rather consists of three simple pieces.) These deformations never pass through the interior of C_1 nor that of the circle C, so $\frac{f(w)}{w-z}$ remains holomorphic throughout the deformation and thus the integral of $\frac{f(w)}{w-z}$ over the original C_2 equals its integral over the deformed contour consisting of C_1, C, and the line segment γ:

$$\oint_{C_2} \frac{f(w)}{w-z}\, dw = \oint_{C_1} \frac{f(w)}{w-z}\, dw + \oint_{C} \frac{f(w)}{w-z}\, dw + \oint_{\gamma} \frac{f(w)}{w-z}\, dw.$$

But actually, the line segment γ occurs twice in deformed contour, as can be seen by looking at the "neck" during the deformation: one side of the neck approaches γ, but so does the other side, so we get two copies of γ at the end only with *opposite* orientations! Thus, the final integral above is really two integrals over the same γ but with opposite orientations, which means that this contribution is zero and we are left with

$$\oint_{C_2} \frac{f(w)}{w-z}\, dw = \oint_{C_1} \frac{f(w)}{w-z}\, dw + \oint_{C} \frac{f(w)}{w-z}\, dw.$$

Since C is a circle around z, the usual integral formula applies to the second integral above:

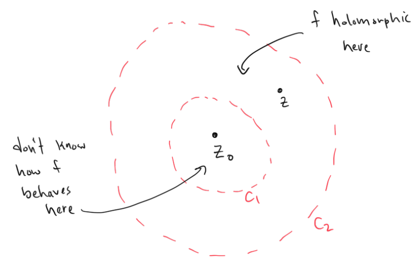
$$\oint_C \frac{f(w)}{w-z}\,dw = 2\pi i f(z).$$

Thus we get

$$\oint_{C_2} \frac{f(w)}{w-z}\,dw = \oint_{C_1} \frac{f(w)}{w-z}\,dw + 2\pi i f(z),$$

which after solving for $f(z)$ gives the desired integral formula for an annulus.

Laurent's theorem. With Cauchy's theorem for an annulus we can now derive a Laurent series expansion. Take C_2 and C_1 to be circles in the annulus on which f is holomorphic with C_1 interior to C_2:



For z between C_1 and C_2 we thus have

$$f(z) = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw.$$

Now, take z_0 to be any point interior to C_1, as in the picture above. We expand around z_0 by doing the same type of geometric series manipulations we have seen plenty of times (precisely what we did for Taylor's theorem before, which was the result that holomorphic functions are expressible as power series), where for the first integral above we expand in powers of $z - z_0$ while for the second we "invert" to expand in powers of $\frac{1}{z-z_0}$:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-z_0)-(z-z_0)}\,dw - \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-z_0)-(z-z_0)}\,dw \\ &= \frac{1}{2\pi i}\oint_{C_2} \frac{1}{w-z_0}\left(\frac{f(w)}{1-\frac{z-z_0}{w-z_0}}\right)dw - \frac{1}{2\pi i}\oint_{C_1} \frac{1}{z-z_0}\left(\frac{f(w)}{\frac{w-z_0}{z-z_0}-1}\right)dw. \end{aligned}$$

Using

$$\frac{1}{1-\frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^n} \quad \text{and} \quad \frac{1}{\frac{w-z_0}{z-z_0}-1} = -\sum_{n=0}^{\infty} \frac{(w-z_0)^n}{(z-z_0)^n}$$

and integrating term by term then produces a Laurent series expansion for $f(z)$ centered at z_0:

$$f(z) = \sum_{n=0}^{\infty}(\text{some integral coefficient})(z-z_0)^n + \sum_{n=0}^{\infty} \frac{(\text{some integral coefficient})}{(z-z_0)^n}.$$

(The expressions for the "integral coefficients" are precisely the ones we derived a few days ago for a general Laurent series.) The resulting Laurent expansion converges on the largest annulus

$R_1 < |z - z_0| < R_2$ on which we can take circles C_1 and C_2 so that the annular integral formula is valid with f remaining holomorphic. (This is why in previous examples we get annuli that reach no further than singularities.)

Isolated singularities. As a special case of Laurent's theorem, we obtain Laurent expansions centered at *isolated singularities*. We say that z_0 is an isolated singularity of f is f is holomorphic on a punctured disk $0 < |z - z_0| < R$ centered at z_0. (We make no assumption about the behavior of f at z_0 itself.) By treating a punctured disk as an annulus with zero inner radius, f then has a Laurent expansion centered at z_0.

For example, $\frac{e^z}{(z-2)^3}$ has an isolated singularity at 2, and we previously derived the Laurent expansion

$$\frac{e^z}{(z-2)^3} = \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^{n-3} = \frac{e^2}{(z-2)^3} + \frac{e^2}{(z-2)^2} + \frac{e^2}{2(z-2)} + \sum_{n=3}^{\infty} \frac{e^2}{n!}(z-2)^{n-3},$$

which is valid on the punctured plane $0 < |z - 2|$. Here is some terminology: we say that the isolated singularity of 2 in this case is a *pole* of $\frac{e^z}{(z-2)^3}$, which means that there are only finitely many (nonzero) negative exponent terms in the Laurent expansion; the fact that the largest degree that occurs as a denominator is 3 means that this is a pole of *order* 3.

The expansion

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}$$

we computed last time occurs around the isolated singularity 0 of $e^{1/z}$. In this case we call 0 an *essential* singularity since the Laurent expansion contains infinitely many negative exponent terms. We will say something about the properties different types of singularities have next time.

Example. For a final example, consider

$$\frac{1}{(z-2)(z-1)},$$

which we previously expanded as a Laurent series centered at 0 in the annulus $1 < |z| < 2$. But we can also expand around one of the singularities of this function, say 2. Note that $\frac{1}{z-1}$ is holomorphic on a disk around 2, so we can expand this as a power series centered at 2:

$$\frac{1}{z-1} = \frac{1}{1+(z-2)} = \sum_{n=0}^{\infty} (-1)^n (z-2)^n.$$

Thus dividing through by $z - 2$ gives

$$\frac{1}{(z-2)(z-1)} = \sum_{n=0}^{\infty} (-1)^n (z-2)^{n-1} = \frac{1}{z-2} - 1 + (z-2) - (z-2)^2 + \cdots .$$

Thus 2 is a pole of the given function, this time of order 1, which is what we call a *simple pole.*

Lecture 22: Singularities and Residues

Warm-Up. We find the Laurent expansions of each of the following functions at the given isolated singularity:

$$\frac{e^z-1}{z} \text{ at } 0, \qquad \frac{1}{(z^2+1)^2} \text{ at } i, \qquad (z-1)\cos(\tfrac{1}{z-1}) \text{ at } 1.$$

First, we can expand e^z as a power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \cdots .$$

Subtracting 1 gets rid of the constant term in this expansion:

$$e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z + \frac{z^2}{2} + \cdots .$$

Finally we can divide through by z to get the desired Laurent expansion of $\frac{e^z-1}{z}$ around 0:

$$\frac{e^z-1}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = 1 + \frac{z}{2} + \frac{z^2}{3!} + \cdots .$$

In fact, the Laurent series in this case is just a power series with no negative exponent terms, which is what it means for 0 to be a *removable singularity* of $\frac{e^z-1}{z}$. The point is that the "singularity" at 0 is not actually present in the sense that we can give the function a value at 0 so as to make it holomorphic, thereby "removing" the singularity. To be precise, the function

$$f(z) = \begin{cases} \frac{e^z-1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

is actually entire since it is given by the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = 1 + \frac{z}{2} + \frac{z^2}{3!} + \cdots$$

for all z. The value $1 = f(0)$ we assign to $\frac{e^z-1}{z}$ at zero to make it holomorphic comes from the constant term in this power series expansion. The fact that 0 initially appeared to a singularity of $\frac{e^z-1}{z}$ is actually just an artifact of the way in which this function was written, and by looking at the Laurent/power series we see that the singularity isn't really there.

Next we look at

$$\frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2}.$$

To expand as a Laurent series around i we can note that $\frac{1}{(z+i)^2}$ is holomorphic on a disk around i, so it should be possible to expand this as a power series centered at i and then divide through by $(z-i)^2$ to get the Laurent expansion we want. To find the power series expansion of $\frac{1}{(z+i)^2}$ we will differentiate the expansion for $\frac{1}{z+i}$ centered at i:

$$\frac{1}{z+i} = \frac{1}{2i+(z-i)} = \frac{1}{2i(1+\frac{z-i}{2i})} = \frac{1}{2i}\sum_{n=0}^{\infty}\left(-\frac{z-i}{2i}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n$$

so taking derivatives gives

$$-\frac{1}{(z+i)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{(2i)^{n+1}} (z-i)^{n-1}.$$

Thus the desired expansion of $\frac{1}{(z^2+1)^2}$ is

$$\frac{1}{(z^2+1)^2} = -\frac{1}{(z-i)^2} \sum_{n=1}^{\infty} \frac{(-1)^n n}{(2i)^{n+1}} (z-i)^{n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(2i)^{n+1}} (z-i)^{n-3}.$$

This sum looks like

$$\frac{1}{(2i)^2 (z-i)^2} - \frac{2}{(2i)^3 (z-i)} + \text{(higher-order terms)},$$

so i is a pole of order 2 of $\frac{1}{(z^2+1)^2}$.

Finally, we start with the usual Taylor expansion of $\cos w$:

$$\cos w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n}.$$

Setting $w = \frac{1}{z-1}$ gives

$$\cos\left(\frac{1}{z-1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n!)} \frac{1}{(z-1)^{2n}}$$

and multiplying through by $z-1$ yields

$$(z-1)\cos\left(\frac{1}{z-1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n!)} \frac{1}{(z-1)^{2n-1}}$$

as the desired expansion of $(z-1)\cos(\frac{1}{z-1})$ centered at 1. Since this expansion has infinitely many terms with negative exponents (or in other words infinitely terms with $z-1$ in the denominator), 1 is an essential singularity of $(z-1)\cos(\frac{1}{z-1})$.

Fun with removable singularities. The presence of removable singularities allows us to apply previous results that at first glance might not seem applicable. For example, suppose f is entire and satisfies $|f(z)| \leq |z|$ for all z. We claim that this forces $f(z)$ to be a constant multiple of z. We saw a similar example earlier where $|f(z)| \leq |e^z|$ forces $f(z) = ce^z$ for some constant c, where in that case the justification used the fact that e^z was never zero to say that $\frac{f(z)}{e^z}$ is entire and bounded by the $|f(z)| \leq |e^z|$ condition, so that Liouville's theorem is applicable. In this case, the condition $|f(z)| \leq |z|$ gives

$$\left|\frac{f(z)}{z}\right| \leq 1,$$

but at first glance only for nonzero z since otherwise the left side is not defined. The point is that $z = 0$ is a singularity of $\frac{f(z)}{z}$ here, so Liouville's theorem does not seem applicable because it appears that $\frac{f(z)}{z}$ is not entire.

However, the point is that the singularity at 0 is actually removable (!!!), so that Liouville's theorem will apply once we remove it. Since 0 is an isolated singularity of $\frac{f(z)}{z}$, we can expand this quotient as a Laurent series centered at 0:

$$\frac{f(z)}{z} = \cdots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots.$$

If any of the negative exponent terms above were actually present here, then we would have that $|\frac{f(z)}{z}|$ grows without restriction as z approaches 0 since the negative exponent terms $\frac{1}{z^k}$ "blow up" to infinity in modulus as z approaches 0. But

$$\left|\frac{f(z)}{z}\right| \leq 1 \text{ for } z \neq 0,$$

so $\frac{f(z)}{z}$ is bounded near 0 and thus cannot increase in modulus without restriction. This means that the negative exponent terms in the Laurent expansion cannot not actually be there, or in other words that all of the negative exponent coefficients c_{-n} must actually be zero. Thus the Laurent expansion above must actually look like

$$\frac{f(z)}{z} = c_0 + c_1 z + c_2 z^2 + \cdots,$$

which is a normal power series, so the singularity at 0 of $\frac{f(z)}{z}$ is removable. Thus we can give this function a value at 0 (namely, whatever the constant term c_0 is) so as to make it entire. This entire function is still bounded by 1, so Liouville's theorem implies that $\frac{f(z)}{z}$ is constant, so $f(z) = cz$ for some c. The upshot is that things like Liouville's theorem and the maximum modulus principle still apply even in the presence of removable singularities.

The same reasoning shows that if $|f(z)| \leq |\sin z|$, then $f(z) = c \sin z$ for some constant c since all singularities of $\frac{f(z)}{\sin z}$ will be removable. More generally, if $|f(z)| \leq |g(z)|$ for *any* entire functions f, g, then $f(z) = cg(z)$ by the same logic—the only way one entire function can bound another is if they were multiples of one another, which means that entire functions are in a sense "unique" when it comes to bounding.

Poles vs essential singularities. Suppose f has a pole at z_0, so that the Laurent expansion of f around z_0 looks like

$$f(z) = \frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-n+1}}{(z-z_0)^{n-1}} + \cdots.$$

Then $(z-z_0)^n f(z)$ has the expansion

$$(z-z_0)^n f(z) = c_{-n} + c_{-n+1}(z-z_0) + c_{-n+2}(z-z_0)^2 + \cdots.$$

This means that the singularity of $(z-z_0)^n f(z)$ at z_0 is actually removable, so $(z-z_0)^n f(z)$ is holomorphic; if we call this function $g(z)$, then

$$(z-z_0)^n f(z) = g(z) \rightsquigarrow f(z) = \frac{g(z)}{(z-z_0)^n}.$$

Thus, to say that f has a pole at z_0 means that, the singularity is not itself immediately removable, but that it *becomes* removable after multiplying by some polynomial $(z-z_0)^n$ of largest enough degree. As a consequence of the manipulation above, near a pole z_0 a function can be written as the quotient $\frac{g(z)}{(z-z_0)^n}$ of a holomorphic function by some large degree factor $(z-z_0)^n$, so that poles are basically the singularities that arise when dividing by polynomials that have roots.

If f has an essential singularity at z_0, there there is no large enough degree factor $(z-z_0)^n$ we can multiply in order to remove the singularity, so essential singularities remain no matter what we do. (This is where the term "essential" comes from—they are an *essential* aspect of the function that cannot be gotten rid of by manipulating.) We will not discuss the behavior of a function near an essential singularity in this course, but the behavior can get quite wild. Look up the *Casorati-Weierstrass theorem* and *Picard's theorem* to learn more.

Residues. Among all the terms in a Laurent expansion of f around an isolated singularity z_0, the main one we will care about is the first negative exponent term:

$$f(z) = \cdots + \underbrace{\frac{c_{-1}}{z - z_0}}_{\text{this one}} + \cdots .$$

The reason for this is, as we saw when deriving an expression for the Laurent coefficients in terms of integrals, this is the only term that matters when integrating f along a closed contour around z_0. As we mentioned before, because of this property we call the coefficient c_{-1} the *residue* of f at z_0, and we denote it by

$$\operatorname{Res}_{z_0} f = c_{-1} = \text{coefficient of } \frac{1}{z - z_0}.$$

We will talk about the importance of residues and their uses next time, but for now we focus on computing them.

For example, in the Warm-Up we found that

$$\frac{1}{(z^2+1)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(2i)^{n+1}} (z-i)^{n-3} = \frac{1}{(2i)^2(z-i)^2} - \frac{2}{(2i)^3(z-i)} + (\text{higher-order terms}).$$

The residue of $f(z) = \frac{1}{(z^2+1)^2}$ at i is thus

$$\operatorname{Res}_i f = -\frac{2}{(2i)^3} = -\frac{2}{8i^3} = \frac{1}{4i} = -\frac{i}{4}.$$

But actually, we do not need to know the full Laurent expansion in order to compute the residue in this case, and more generally this holds true for poles. (For essential singularities you really do need the full expansion, which is part of what makes them tougher to work with outside of basic examples.) Being able to compute residues without computing full Laurent series is what makes their use in computing integrals (next time) actually worthwhile since Laurent series in general can be actually tough to compute.

In this case, we note that

$$\frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2} = \frac{1/(z+i)^2}{(z-i)^2},$$

which is the quotient of a holomorphic function (on a disk around i) with some power of $z - i$. (Recall that this is the type of behavior you can expect near poles in general.) Consider the general setup of such a quotient $\frac{g(z)}{(z-i)^2}$ with $g(z)$ holomorphic. Being holomorphic, $g(z)$ is expressible as a power series centered at i:

$$g(z) = a_0 + a_1(z-i) + a_2(z-i)^2 + \cdots .$$

Dividing by $(z-i)^2$ then gives

$$\frac{g(z)}{(z-i)^2} = \frac{a_0}{(z-i)^2} + \frac{a_1}{z-i} + a_2 + a_3(z-z_0) + \cdots ,$$

so that $\operatorname{Res}_i f = a_1$. But this coefficient comes from the $(z-i)^1$ term in the original Taylor expansion of $g(z)$, so we know that this coefficient is given by the derivative of $g(z)$ at i:

$$\operatorname{Res}_i \frac{g(z)}{(z-i)^2} = a_1 = g'(i).$$

Thus, we are able to determine this residue solely from knowing this one derivative, without knowing full Laurent expansion of $\frac{g(z)}{(z-i)^2}$.

In the case of $f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2}$, we thus have

$$\operatorname{Res}_i f = \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\bigg|_{z=i} = -\frac{2}{(z+i)^3}\bigg|_{z=i} = -\frac{2}{(2i)^2} = -\frac{i}{4},$$

just as we computed before.

Example. We find the residue of

$$f(z) = \frac{z^5+z^4+z^3+z^2+z+1}{(z-1)^4}$$

at 1, which is a pole of order 4. For a general holomorphic $g(z)$, we have an expansion

$$g(z) = a_0 + a_1(z-1) + a_2(z-1)^2 + a_3(z-1)^3 + \cdots,$$

so that the $\frac{1}{z-1}$ term in $\frac{g(z)}{(z-1)^4}$ occurs at the degree 3 term in the Taylor series of g. The coefficient a_3 in this Taylor expansion is $g^{(3)}(1)/3!$, so this is the residue of $\frac{g(z)}{(z-1)^4}$ at 1. For the given $f(z)$, we thus get

$$\operatorname{Res}_1 f = \frac{1}{3!}\frac{d^3}{dz^3}(z^5+z^4+z^3+z^2+z+1)\Big|_{z=1} = \frac{1}{3!}(60z^2+24z+6)\Big|_{z=1} = \frac{90}{3!} = 15.$$

Lecture 23: Residue Theorem

Warm-Up 1. We find the residues of

$$\frac{\cos(iz)}{(z^2-4)(z+2)}$$

at each of its poles. (We will see why we get poles in the course of computing these residues.) If we factor $z^2-4=(z-2)(z+2)$, we see that our function is

$$\frac{\cos(iz)}{(z-2)(z+2)^2},$$

so there are singularities at 2 and -2. For the singularity at 2, we think of our function as

$$\frac{\cos(iz)/(z+2)^2}{z-2}$$

with a numerator that is holomorphic on a disk around 2. If we expand this numerator as a power series centered at 2 and then divide by $z-2$ to get the Laurent series of the given function at 2, we see that the $\frac{1}{z-2}$ term will come from constant term in the power series expansion of the numerator:

$$f(z) = a_0 + a_1(z-2) + a_2(z-2)^2 + \cdots \rightsquigarrow \frac{f(z)}{z-2} = \underbrace{\frac{a_0}{z-2}}_{\text{residue term}} + a_1 + a_2(z-2) + \cdots.$$

(So we see that 2 is indeed a pole of order 1.) This constant term in the Taylor expansion of $f(z)$ is the value $f(2)$ of f at the singularity, so in our case this is the value of $\frac{\cos(iz)}{(z+2)^2}$ at 2. Thus

$$\operatorname{Res}_2\left(\frac{\cos(iz)}{(z-2)(z+2)^2}\right) = \left(\frac{\cos(iz)}{(z+2)^2}\right)\Bigg|_{z=2} = \frac{\cos(2i)}{16}.$$

For the residue at $z=-2$ we use a similar idea. Our function is

$$\frac{\cos(iz)/(z-2)}{(z+2)^2}$$

with a numerator holomorphic on a disk around -2, so expanding this numerator as a power series at -2 and dividing by $(z+2)^2$ gives the Laurent series. The difference is that now it is the $(z+2)^1$ (first power) term in the Taylor expansion that gives the residue:

$$f(z) = a_0 + a_1(z+2) + a_2(z+2)^2 + \cdots \rightsquigarrow \frac{f(z)}{(z+2)^2} = \frac{a_0}{(z+2)^2} + \frac{a_1}{z+2} + a_2 + \cdots.$$

(So we have a pole of order 2.) The coefficient a_1 in the Taylor expansion of f is now the first derivative $f'(-2)$ evaluated at the pole, so we have that

$$\begin{aligned}
\operatorname{Res}_{-2}\left(\frac{\cos(iz)}{(z-2)(z+2)^2}\right) &= \frac{d}{dz}\left(\frac{\cos(iz)}{z-2}\right)\Bigg|_{z=-2} \\
&= \left(\frac{-(z-2)i\sin(iz) - \cos(iz)}{(z-2)^2}\right)\Bigg|_{z=-2} \\
&= \frac{4i\sin(-2i) - \cos(-2i)}{16}.
\end{aligned}$$

In general, a function of the form

$$\frac{f(z)}{(z-z_0)^n}$$

where $f(z)$ is holomorphic on a disk around z_0 has a pole of order n at z_0 with residue given by

$$\operatorname{Res}_{z_0} \frac{f(z)}{(z-z_0)^n} = \frac{f^{(n-1)}(z_0)}{(n-1)!},$$

which is the coefficient of $(z-z_0)^{n-1}$ in the Taylor expansion of f around z_0.

Warm-Up 2. Now we find the residues of

$$\frac{e^z}{\sin z}$$

at each of its (as we will see) poles. The singularities come from points where the denominator is zero, so let us consider the singularity at 0 first. A key observation here is that 0 is a *simple zero* of $\sin z$ (or a zero of "order 1"), which means that it is zero of $g(z) = \sin z$ but not of its derivative $g'(z) = \cos z$.

In fact, residues for quotients $\frac{f(z)}{g(z)}$ where g has a simple zero at z_0 are given by a formula that works in general. Indeed, we start by expanding $g(z)$ as a Taylor series:

$$g(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots.$$

The value $a_0 = g(z_0)$ is zero by the simple zero assumption, but $a_1 = g'(z_0)$ is not zero. Thus the expansion above can be written as

$$g(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \cdots = (z - z_0)[a_1 + a_2(z - z_0) + a_3(z - z_0)^2 + \cdots]$$

where we factor $z - z_0$ out of every term. If we denote the sum $a_1 + a_2(z - z_0) + \cdots$ in brackets simply by $h(z)$, then we have

$$g(z) = (z - z_0)h(z).$$

Moreover, $h(z_0)$ is the constant term a_1 in the expansion of $h(z)$, but this came from the $(z - z_0)^1$ term in the expansion of g, which is $a_1 = g'(z_0) \neq 0$. The upshot is that the function $h(z)$ in

$$g(z) = (z - z_0)h(z)$$

is nonzero at z_0, which reflects the fact that z_0 as a zero of order 1 of $g(z)$ and not of a higher order. If instead z_0 were a zero of order 2, meaning that $g(z_0) = 0$ and $g'(z_0) = 0$ but $g''(z_0) \neq 0$, we would be able to factor $(z - z_0)^2$ out of the Taylor expansion of $g(z)$ (but not a third power) to get

$$g(z) = (z - z_0)^2 h(z) \text{ with } h(z_0) \neq 0.$$

For a zero of "order n" we would have $g(z) = (z - z_0)^n h(z)$ with $h(z_0) \neq 0$.

So, in the simple zero case we have $g(z) = (z - z_0)h(z)$ with $h(z_0) \neq 0$, so that

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z - z_0)h(z)} = \frac{f(z)/h(z)}{z - z_0}.$$

This is now in the form where $\frac{f(z)}{g(z)}$ has a pole of order 1 at z_0 since $\frac{f(z)}{h(z)}$ is holomorphic on a disk around z_0. (This is why we need $h(z_0) \neq 0$, since otherwise $1/h(z)$ would not be differentiable at z_0.) Thus, simple zeros of $g(z)$ correspond to simple poles of $\frac{f(z)}{g(z)}$ (actually, if $f(z)$ is also zero at z_0, then you get a removable singularity instead of a simple pole, but the residue computation gives the correct value regardless so we will not dwell on this fact), and more generally (by similar reasoning) zeros of order n of $g(z)$ correspond to poles of order n (at least when the numerator is not zero at the singularity) of $\frac{f(z)}{g(z)}$. In the simple case, we then have that the residue of $\frac{f(z)/h(z)}{z - z_0}$ at 0 is just the value of the holomorphic numerator at 0

$$\operatorname{Res}_{z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{h(z_0)}.$$

But if we go back to the definition of $h(z)$, the value $h(z_0)$ came from $g'(z_0)$, so the conclusion is that

$$\operatorname{Res}_{z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)} \text{ when } g \text{ has a simple zero at } z_0.$$

Note that, once again, we are able to compute the residue (albeit with more effort in this case) without knowing the full Laurent expansion. Formulas for the residues of $\frac{f}{g}$ in the cases where g has zeroes of higher-orders can be found by the same methods, although the formulas get complicated fairly quickly; we will only need a few examples of this later. Functions which are holomorphic except for the presence of poles are often called *meromorphic* functions, and these are essentially the types of things you get by taking quotients $\frac{f}{g}$ of holomorphic functions.

If we apply this to our original $\frac{e^z}{\sin z}$, where $\sin z$ has a simple zero at 0, we thus get

$$\operatorname{Res}_0 \frac{e^z}{\sin z} = \left.\frac{e^z}{\frac{d}{dz}(\sin z)}\right|_{z=0} = \frac{e^0}{\cos 0} = 1.$$

In fact, all zeroes $n\pi$ of $\sin z$ are simple as well, so the residues of $\frac{e^z}{\sin z}$ at its poles are

$$\operatorname{Res}_{n\pi} \frac{e^z}{\sin z} = \frac{e^z}{\frac{d}{dz}(\sin z)}\bigg|_{z=n\pi} = \frac{e^{n\pi}}{\cos(n\pi)} = (-1)^n e^{n\pi}.$$

Residue theorem (simplest version). Residues were supposed to be the things "left behind" when integrating a Laurent series, and now we will make this precise in what's called the *residue theorem.* The simplest version of the residue theorem states that

$$\oint_C f(z)\,dz = 2\pi i(\operatorname{Res}_{z_0} f)$$

where C is a simple closed contour enclosing z_0 and no other singularity of f. Indeed, this is precisely what we obtained when deriving the integral formula for the Laurent coefficient a_{-1}. We call this the "simplest" version of the residue theorem since soon we will consider a more general version which allows for the presence of multiple singularities.

For example, $e^{1/z}$ has a singularity at 0 (an essential one), and from the Laurent expansion

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = \cdots + \frac{1}{2z^2} + \frac{1}{z} + 1$$

we derived before we see that the residue of $e^{1/z}$ at 0 is 1. Thus the residue theorem gives

$$\oint_{|z|=1} e^{1/z}\,dz = 2\pi i(\operatorname{Res}_0 e^{1/z}) = 2\pi i.$$

Examples. Now we compute

$$\oint_{|z|=1} \frac{\sin z}{2z^2 - 5z + 1}\,dz.$$

The denominator factors as

$$2z^2 - 5z + 1 = (2z-1)(z-2) = 2(z - \tfrac{1}{2})(z-2),$$

so we have singularities at $\frac{1}{2}$ and 2, but only $\frac{1}{2}$ falls within $|z| = 1$. Thus

$$\oint_{|z|=1} \frac{\sin z}{2z^2 - 5z + 1}\,dz = 2\pi i \operatorname{Res}_{1/2}\left(\frac{\sin z}{2z^2 - 5z + 1}\right)$$

by the residue theorem. To compute this residue, we can view our function as

$$\frac{\sin z}{2z^2 - 5z + 1} = \frac{(\sin z)/(2(z-2))}{z - \frac{1}{2}}$$

with holomorphic numerator, in which the residue is the value of the holomorphic piece at $\frac{1}{2}$:

$$\operatorname{Res}_{1/2}\left(\frac{\sin z}{2z^2 - 5z + 1}\right) = \left(\frac{\sin z}{2(z-2)}\right)\bigg|_{z=\frac{1}{2}} = \frac{\sin\frac{1}{2}}{2(\frac{1}{2} - 2)} = -\frac{\sin\frac{1}{2}}{3}.$$

Alternatively, we can view $\frac{1}{2}$ as a simple zero of the denominator and use

$$\operatorname{Res}_{1/2}\left(\frac{\sin z}{2z^2 - 5z + 1}\right) = \frac{\sin z}{\frac{d}{dz}(2z^2 - 5z + 1)}\bigg|_{z=\frac{1}{2}} = \frac{\sin z}{4z - 5}\bigg|_{z=\frac{1}{2}} = -\frac{\sin\frac{1}{2}}{3}.$$

Thus our desired integral value is

$$\oint_{|z|=1} \frac{\sin z}{2z^2 - 5z + 1}\, dz = 2\pi i \operatorname{Res}_{1/2}\left(\frac{\sin z}{2z^2 - 5z + 1}\right) = -\frac{2\pi i \sin \frac{1}{2}}{3}.$$

If we integrate over the circle $|z - 2| = 1$, which encloses 2 but not $\frac{1}{2}$, we shift to using the residue at 2 instead, which is still a simple zero of the denominator:

$$\oint_{|z-2|=1} \frac{\sin z}{2z^2 - 5z + 1}\, dz = 2\pi i \operatorname{Res}_{1/2}\left(\frac{\sin z}{2z^2 - 5z + 1}\right) = 2\pi i \left(\frac{\sin z}{4z - 5}\right)\bigg|_{z=2} = \frac{2\pi i \sin 2}{3}.$$

Now let us invert our function and integrate over the unit circle again:

$$\oint_{|z|=1} \frac{2z^2 - 5z + 1}{\sin z}\, dz.$$

We have singularities $z = n\pi$, where $\sin z$ is zero, and of these only 0 lies within the contour. This is a simple zero of the denominator, so

$$\operatorname{Res}_0\left(\frac{2z^2 - 5z + 1}{\sin z}\right) = \frac{2z^2 - 5z + 1}{\frac{d}{dz}\sin z}\bigg|_{z=0} = \frac{2z^2 - 5z + 1}{\cos z}\bigg|_{z=0} = 1,$$

and thus

$$\oint_{|z|=1} \frac{2z^2 - 5z + 1}{\sin z}\, dz = 2\pi i \operatorname{Res}_0\left(\frac{2z^2 - 5z + 1}{\sin z}\right) = 2\pi i.$$

Relation to integral formulas. For something like

$$\oint_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz$$

where f is holomorphic on a simply-connected domain containing $|z - z_0| = r$, the residue theorem gives

$$\oint_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz = 2\pi i \left(\operatorname{Res}_{z_0} \frac{f(z)}{z - z_0}\right).$$

We have a simple pole so the residue is the value of the numerator at z_0, so that

$$\oint_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).$$

But this is precisely what we had before in Cauchy's integral formula, so the upshot is that the integral formula is just a special case of the residue theorem. More generally, $\frac{f(z)}{(z-z_0)^{n+1}}$ has a pole of order $n+1$ at z_0, so the residue comes from the coefficient of degree n term in the Taylor expansion of f, which is $f^{(n)}(z_0)/n!$. Thus the residue theorem gives

$$\oint_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz = 2\pi i \left(\operatorname{Res}_{z_0} \frac{f(z)}{(z - z_0)^{n+1}}\right) = 2\pi i \frac{f^{(n)}(z_0)}{n!},$$

which is nothing but Cauchy's integral formula for higher-order derivatives.

If $f(z)$ has no singularities, or in other words if we can treat any singularity as removable, then every residue is zero (since the Laurent series is just a power series with no negative exponent terms at all), so the residue theorem gives $\oint_C f(z)\,dz = 0$, which was the original Cauchy's theorem. The upshot is that the residue theorem subsumes every main integration result we saw before, so it is the ultimate integration theorem. (Of course, though, the residue theorem in the end is itself a consequence of Cauchy's theorem, since the residue theorem depends on existence of Laurent series which depends on Cauchy's integral formula for an annulus which depends on the deformation theorem which depends on Cauchy's theorem!)

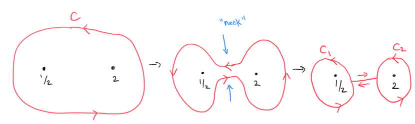
Multiple singularities. Now consider

$$\oint_C \frac{\sin z}{2z^2 - 5z + 1}\,dz$$

where C is a simple closed contour that encloses *both* of the singularities $\frac{1}{2}$ and 2 we worked with before. What does the residue theorem say in the presence of two (or more) singularities? The fact is that all we need to do is include more terms, one for each additional singularity:

$$\oint_C \frac{\sin z}{2z^2 - 5z + 1}\,dz = 2\pi i \left(\operatorname{Res}_{1/2} \frac{\sin z}{2z^2 - 5z + 1} + \operatorname{Res}_2 \frac{\sin z}{2z^2 - 5z + 1} \right).$$

To see why this works consider (as in the justification of the annular integral formula) a deformation, where we deform C into two small circles C_1, C_2 each enclosing only one of the singularities:



(As in the annulus case, we actually get a line segment connecting our two small circles as well, but the contribution to the integral from this vanishes since the line segment occurs twice—once for each side of the "neck" being squeezed into the segment—with opposite orientations.) After deforming we thus get

$$\oint_C \frac{\sin z}{2z^2 - 5z + 1}\,dz = \oint_{C_1} \frac{\sin z}{2z^2 - 5z + 1}\,dz + \oint_{C_2} \frac{\sin z}{2z^2 - 5z + 1}\,dz.$$

By the simpler version of the residue theorem, since C_1 only encloses $\frac{1}{2}$, the integral over C_1 uses only the residue at $\frac{1}{2}$, and similarly the integral over C_2 uses only the residue at 2. Thus

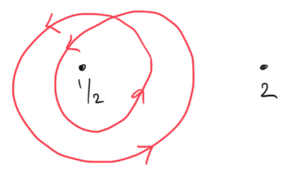
$$\begin{aligned}
\oint_C \frac{\sin z}{2z^2 - 5z + 1}\,dz &= \oint_{C_1} \frac{\sin z}{2z^2 - 5z + 1}\,dz + \oint_{C_2} \frac{\sin z}{2z^2 - 5z + 1}\,dz \\
&= 2\pi i(\text{residue at } \tfrac{1}{2}) + 2\pi i(\text{residue at } 2) \\
&= 2\pi i[(\text{residue at } \tfrac{1}{2}) + (\text{residue at } 2)].
\end{aligned}$$

The same idea works when C (still a simple closed contour) encloses even more singularities, so in general the integral is $2\pi i$ times a sum of even more residues.

For the example at hand, using the residues we computed previously we get

$$\begin{aligned}
\oint_C \frac{\sin z}{2z^2 - 5z + 1}\,dz &= 2\pi i \left(\operatorname{Res}_{1/2} \frac{\sin z}{2z^2 - 5z + 1} + \operatorname{Res}_2 \frac{\sin z}{2z^2 - 5z + 1} \right) \\
&= 2\pi i \left(-\frac{\sin \frac{1}{2}}{3} + \frac{\sin 2}{3} \right).
\end{aligned}$$

Residue theorem (general version). We can get even more general with the residue theorem beyond multiple singularities alone and consider non-simple closed contours as well. For example, say with the same function as above with singularities at $\frac{1}{2}$ and 2, for a contour like



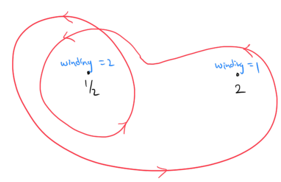
we get *two* copies of what we would have for a single simpler contour enclosing $\frac{1}{2}$ only: the "inner" loop of the contour above gives $2\pi i$ times the residue value, but then the "outer" loop gives the same value, so over all we would get

$$\oint_C f(z)\,dz = 2\oint_{\text{simple contour}} f(z)\,dz = 2 \text{ times } 2\pi i(\text{residue}).$$

We call 2 here the *winding number* of C at $\frac{1}{2}$ since it counts how many times C "winds" or "wraps around" $\frac{1}{2}$. So the integral is

$$\oint_C f(z)\,dz = 2\pi i(\text{winding number})(\text{residue}).$$

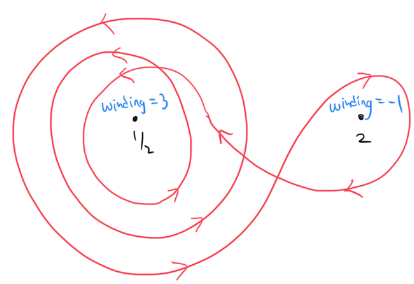
The contour



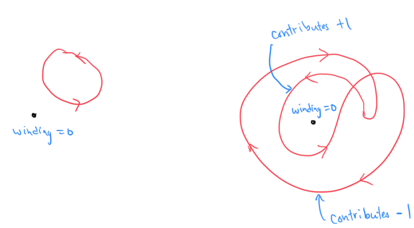
has winding number 2 at $\frac{1}{2}$ and winding number 1 at 2 since it wraps around $\frac{1}{2}$ twice but only once around 2. The integral is

$$2\pi i[2(\text{residue at } \tfrac{1}{2}) + 1(\text{residue at } 2)].$$

The contour



has winding number 3 at $\frac{1}{2}$ and winding number -1 (note the negative) at 2: counterclockwise windings count as positive and clockwise windings as negative since clockwise orientations change the sign of an integral. It is even possible to have winding number zero, either because the contour does not wind around a point at all or because the number of counterclockwise and clockwise windings cancel each other out:



The general version of the residue theorem, which allows for non-simple closed contours, is thus

$$\oint_C f(z)\,dz = \sum_{\substack{\text{singularities}\\ \text{within } C}} 2\pi i(\text{winding number})(\text{residue}).$$

Our book does not consider the case of general winding numbers, and only considers the positively-oriented simple contour case where all winding numbers are 1. Indeed, this is really the only setup we will care about as well, but it is good to know that more general versions are available.

Lecture 24: More on Residue Theorem

Warm-Up 1. We compute

$$\oint_{|z|=1} \frac{z^2+z+4}{e^{iz}-1}\,dz \quad \text{and} \quad \oint_{|z-1|=2} \frac{z^2+z+4}{(z-1)(e^{iz}-1)}\,dz.$$

In the first case, 0 is the only singularity and it is a simple zero of $e^{iz}-1$, so

$$\operatorname{Res}_0 \frac{z^2+z+4}{e^{iz}-1} = \left.\frac{z^2+z+4}{\frac{d}{dz}(e^{iz}-1)}\right|_{z=0} == \left.\frac{z^2+z+4}{ie^{iz}}\right|_{z=0} = \frac{4}{i} = -4i.$$

Thus the residue theorem (note the winding number is 1) gives

$$\oint_{|z|=1} \frac{z^2+z+4}{e^{iz}-1}\,dz = 2\pi i(\text{residue}) = 2\pi i(-4i) = 8\pi.$$

For the second integral, there are now two singularities within $|z-1|=2$, namely 0 and 1. So we need both residues to get the value of the integral. If we write our function as holomorphic/$(e^{iz}-1)$, we can again use the simple zero formula to get

$$\operatorname{Res}_0 \frac{(z^2+z+4)/(z-1)}{e^{iz}-1} = \left.\left(\frac{(z^2+z+4)/(z-1)}{\frac{d}{dz}(e^{iz}-1)}\right)\right|_{z=0} = \frac{-4}{i} = 4i.$$

For the residue at 1 we write our function as holomorphic/$(z-1)$ to get

$$\operatorname{Res}_1 \frac{(z^2+z+4)/(e^{iz}-1)}{z-1} = \left.\left(\frac{z^2+z+4}{e^{iz}-1}\right)\right|_{z=1} = \frac{6}{e^i-1}.$$

Thus (both winding numbers are 1)

$$\begin{aligned}\oint_{|z-1|=2} \frac{z^2+z+4}{(z-1)(e^{iz}-1)}\,dz &= 2\pi i[(\text{residue at } 0) + (\text{residue at } 1)]\\ &= 2\pi i\left(4i + \frac{6}{e^i-1}\right)\\ &= -8\pi + \frac{12\pi i}{e^i-1}.\end{aligned}$$

Warm-Up 2. Now we compute

$$\oint_{|z-1|=2\pi} \frac{1+z}{1-\cos z}\,dz.$$

Singularities occur when $\cos z = 1$, so at $z = 2n\pi$. The only two of these within the circle $|z-1| = 2\pi$ are 0 and 2π, so we get two residue contributions to the integral. However, neither of these are simple zeroes of $g(z) = 1 - \cos z$, but in fact they are zeroes of order 2, meaning that $g(z)$ and $g'(z) = \sin z$ are both zero at the singularity but $g''(z) = \cos z$ is not. The simple zero formula from before thus does not apply, so we need something new.

Let us consider a general $\frac{f(z)}{g(z)}$ with z_0 a zero of order 2 of $g(z)$. We proceed using the same method as in the simple zero case: expand $g(z)$ as a power series

$$g(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots ,$$

where now $a_0 = g(z_0)$ and $a_1 = g'(z_0)$ are both zero, but $a_2 = g''(z_0)/2$ is not. Then

$$g(z) = a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots = (z - z_0)^2(a_2 + a_3(z - z_0) + \cdots) = (z - z_0)^2 h(z)$$

where $h(z) = a_2 + a_3(z - z_0) + \cdots$ with $h(z_0) = a_2 \neq 0$. This gives

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z - z_0)^2 h(z)} = \frac{f(z)/h(z)}{(z - z_0)^2}.$$

This is now the setting of a pole of order 2, where the residue comes from the derivative of the holomorphic numerator:

$$\operatorname{Res}_{z_0} \frac{f(z)/h(z)}{(z - z_0)^2} = \frac{d}{dz}\left(\frac{f(z)}{h(z)}\right)\bigg|_{z=z_0} = \frac{h(z_0)f'(z_0) - f(z_0)h'(z_0)}{h(z_0)^2},$$

with the last step coming from the quotient rule. Now, the values $h(z_0)$ and $h'(z_0)$ can be determined from those of the original $g(z)$ by using the Taylor expansion definition of $h(z)$ built from the original expansion of $g(z)$, where

$$h(z_0) = a_2 = \frac{g''(z_0)}{2} \quad \text{and} \quad h'(z_0) = a_3 = \frac{g'''(z_0)}{3!}.$$

The upshot is that

$$\operatorname{Res}_{z_0} \frac{f(z)}{g(z)} = \frac{\frac{1}{2}g''(z_0)f'(z_0) - \frac{1}{6}f(z_0)g'''(z_0)}{\frac{1}{4}g''(z_0)^2} \text{ when } g(z) \text{ has a zero of order 2 at } z_0.$$

In general, it is possible to find similar formulas for the residues of $\frac{f(z)}{g(z)}$ no matter the order of the pole, but these formulas get crazy quickly, as we can already see in the second-order pole case above. The simple pole (i.e., simple zero of the denominator) case and the general

$$\operatorname{Res}_{z_0} \frac{f(z)}{(z - z_0)^n} = \frac{f^{(n-1)}(z_0)}{(n-1)!}$$

one are the only formulas worth knowing by heart. For all other types of residues it will be simpler to derive a formula as needed using power series manipulations. (In particular, no need to memorize the order 2 zero/pole formula derived above!)

For the example at hand, with $f(z) = 1 + z$ and $g(z) = 1 - \cos z$ we have

$$f(0) = 1,\ f'(0) = 1,\ g''(0) = \cos 0 = 1, \text{ and } g'''(0) = -\sin 0 = 0.$$

The residue of $\frac{f(z)}{g(z)}$ is thus

$$\operatorname{Res}_0 \frac{1+z}{1-\cos z} = \frac{\frac{1}{2}g''(0)f'(0) - \frac{1}{6}f(0)g'''(0)}{\frac{1}{4}g''(0)^2} = \frac{\frac{1}{2}-0}{\frac{1}{4}} = 2.$$

Since $z = 2\pi$ is also a zero of order 2 of $g(z) = 1 - \cos z$, the same formula gives

$$\operatorname{Res}_{2\pi} \frac{1+z}{1-\cos z} = \frac{\frac{1}{2}g''(2\pi)f'(2\pi) - \frac{1}{6}f(2\pi)g'''(2\pi)}{\frac{1}{4}g''(2\pi)^2} = \frac{\frac{1}{2}-0}{\frac{1}{4}} = 2.$$

(The only different value here from the $z = 0$ case is $f(2\pi) = 1 + 2\pi$ instead of $f(0) = 1$, but this ends up being multiplied by $g'''(2\pi) = 0$ so it does not matter in the end.) Thus

$$\oint_{|z-1|=2\pi} \frac{1+z}{1-\cos z}\, dz = 2\pi i(2+2) = 8\pi i.$$

Real integral example. Let us now compute the real integral

$$\int_0^{2\pi} \frac{2\cos\theta}{4-2\cos\theta}\, d\theta,$$

which we do by expressing this as a contour integral over the unit circle. Since $2\cos\theta = e^{i\theta} + e^{-i\theta}$, we have

$$\int_0^{2\pi} \frac{2\cos\theta}{4-2\cos\theta}\, d\theta = \int_0^{2\pi} \frac{e^{i\theta}+e^{-i\theta}}{4-(e^{i\theta}+e^{-i\theta})}\, d\theta.$$

We make another algebraic manipulation by multiplying numerator and denominator by $e^{i\theta}$ (so as to get rid of negative exponents) to get

$$\int_0^{2\pi} \frac{2\cos\theta}{4-2\cos\theta}\, d\theta = \int_0^{2\pi} \frac{e^{2i\theta}+1}{4e^{i\theta}-(e^{2i\theta}+1)}\, d\theta.$$

The point is that this is precisely the type of integral that arises when integrating over $|z| = 1$ using the parametrization $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$, except that such a parametrization would also introduce a $z' = ie^{i\theta}$ term which is so far missing from our integral. So, we introduce this term by putting $ie^{i\theta}$ in the numerator and denominator to get

$$\begin{aligned}\int_0^{2\pi} \frac{2\cos\theta}{4-2\cos\theta}\, d\theta &= \int_0^{2\pi} \frac{e^{2i\theta}+1}{ie^{i\theta}[4e^{i\theta}-(e^{2i\theta}+1)]}\, ie^{i\theta}\, d\theta \\ &= \oint_{|z|=1} \frac{z^2+1}{iz(4z-z^2-1)}\, dz \\ &= i\oint_{|z|=1} \frac{z^2+1}{z(z^2-4z+1)}\, dz.\end{aligned}$$

(In the last step we factored out $\frac{1}{i} = -i$ and used the negative to change some signs in the denominator.)

To compute the resulting contour integral we use residues. The roots of $z^2 - 4z + 1$ are

$$z = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

Of these two roots, only $2-\sqrt{3}$ lies within the circle $|z|=1$, so only $z=0$ and $z=2-\sqrt{3}$ contribute residues of $\frac{z^2+1}{z(z^2-4z+1)}$ that we care about. If we write

$$\frac{z^2+1}{z(z^2-4z+1)}=\frac{z^2+1}{z(z-[2-\sqrt{3}])(z-[2+\sqrt{3}])},$$

we get that

$$\text{residue at } 0=\left.\frac{z^2+1}{(z-[2-\sqrt{3}])(z-[2+\sqrt{3}])}\right|_{z=0}=\frac{1}{(2-\sqrt{3})(2+\sqrt{3})}=1$$

and

$$\text{residue at } 2-\sqrt{3}=\left.\frac{z^2+1}{z(z-[2+\sqrt{3}])}\right|_{z=2-\sqrt{3}}=\frac{(2-\sqrt{3})^2+1}{(2-\sqrt{3})(-2\sqrt{3})}=\frac{4-2\sqrt{3}}{3-2\sqrt{3}}.$$

Thus

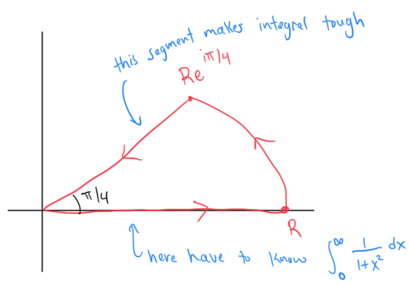
$$\begin{aligned}\int_0^{2\pi}\frac{2\cos\theta}{4-2\cos\theta}\,d\theta&=i\oint_{|z|=1}\frac{z^2+1}{z(z^2-4z+1)}\,dz\\&=i(2\pi i)(\text{first residue}+\text{second residue})\\&=-2\pi\left(1+\frac{4-2\sqrt{3}}{3-2\sqrt{3}}\right),\end{aligned}$$

assuming my computations were done correctly.

Improper integral example. Finally we compute the real improper integral

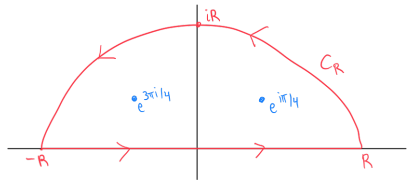
$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx.$$

This was actually computed on a previous homework, where the strategy was to integrate $\frac{1}{1+z^2}$ over the circle sector bounding an eight of a circle of radius R and then taking a limit:



(Actually, the homework problem asked about the integral $\int_0^\infty \frac{x^2}{1+x^4}\,dx$ starting at 0, but this integral is half of the one we are now looking at here since $\frac{x^2}{1+x^4}$ is even with respect to x, meaning that its integral over $(-\infty,0]$ is the same as its integral over $[0,\infty)$.) What made that homework problem a bit challenging was in integrating over the line segment forming the top of the contour, where the parametrization $z=te^{i\pi/4}$ was used. With this $\frac{1}{1+z^2}$ becomes $\frac{1}{1+it^2}$, and then multiplying numerator and denominator by the conjugate of the denominator is what led to $\frac{t^2}{1+t^4}$.

This contour was necessary to use back then since at that time we only had Cauchy's theorem available, so we needed a contour that enclosed no singularities of $\frac{1}{1+z^2}$. But now that we can work with more general residues, we have a simpler approach to this same integral, namely by integrating $\frac{z^2}{1+z^4}$ over the top half disk contour C_R



Restricting $\frac{z^2}{1+z^4}$ to the real axis immediately gives the improper integral we care about, so as long as we can find the value of the integral over the entire contour using residues and what happens to the integral over the circular piece, we are good to go. (In the previous method, we also needed to know $\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$ since this is what we get when restricting $\frac{1}{1+z^2}$ to the part of the previous contour on the real axis, which is also something that makes this previous method more challenging. All of this is avoided by using a different function and contour and making use of residues.)

The roots of $1+z^4$ are the fourth roots of -1, and only two of these lie within C_R, at least for $R > 1$. Thus we have

$$\oint_{C_R} \frac{z^2}{1+z^4}\,dz = 2\pi i(\text{residue at } e^{i\pi/4} + \text{residue at } e^{3\pi i/4}).$$

Both singularities are simple zeroes of the denominator, so we can use

$$\operatorname{Res}_{z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$$

to find the residues. We have

$$\operatorname{Res}_{z_0} \frac{z^2}{1+z^4} = \frac{z_0^2}{4z_0^3} = \frac{1}{4z_0},$$

so

$$\begin{aligned}\oint_{C_R} \frac{z^2}{1+z^4}\,dz &= 2\pi i(\text{residue at } e^{i\pi/4} + \text{residue at } e^{3\pi i/4}) \\ &= 2\pi i(\tfrac{1}{4}e^{-i\pi/4} + \tfrac{1}{4}e^{-3\pi i/4}) \\ &= \tfrac{\pi i}{2}(\cos(\tfrac{\pi}{4}) - i\sin(\tfrac{\pi}{4}) + \cos(\tfrac{3\pi}{4}) - i\sin(\tfrac{3\pi}{4})) \\ &= \tfrac{\pi i}{2}(-2i\tfrac{1}{\sqrt{2}}) = \tfrac{\pi}{\sqrt{2}}.\end{aligned}$$

As $R \to \infty$, this value will remain the same.

On the other hand, the integral over the bottom portion of C_R is

$$\int_{C_R \text{ bottom}} \frac{z^2}{1+z^4}\,dz = \int_{-R}^{R} \frac{x^2}{1+x^4}\,dx$$

which gives the desired improper integral after taking $R \to \infty$. For the integral over the top circle arc of C_R, we bound:

$$\left|\int_{|z|=R, y\geq 0} \frac{z^2}{1+z^4}\,dz\right| \leq \int_{|z|=R, y\geq 0} \frac{|z|^2}{|1+z^4|}\,|dz| \leq \int_{|z|=R, y\geq 0} \frac{R^2}{R^4-1}\,|dz| = \frac{\pi R^3}{R^4-1},$$

where we use the reverse triangle inequality $|1+z^4| \geq |z^4| - 1 = R^4 - 1$ in the third step. This final expression goes to 0 as $R \to \infty$, so we are left with

$$\frac{\pi}{\sqrt{2}} = \lim_{R\to\infty} \oint_{C_R} \frac{z^2}{1+z^4}\,dz$$

$$
\begin{aligned}
&= \lim_{R\to\infty}\left(\int_{C_R \text{ bottom}} \frac{z^2}{1+z^4}\,dz + \int_{|z|=R, y\geq 0} \frac{z^2}{1+z^4}\,dx\right)\\
&= \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx + 0\\
&= \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx
\end{aligned}
$$

which agrees with the value found on the homework.

Lecture 25: More Integrals

Warm-Up 1. We write

$$\int_0^{2\pi} \frac{1}{1+\cos^2\theta}\,d\theta$$

as a contour integral over the unit circle $|z| = 1$ and determine the points at which we need the residues in order to find the value. (We will not actually evaluate the integral itself, only see the ingredients that go into doing so. You will compute a similar integral in full on the final optional homework.) We start with the substitution $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, so that

$$\frac{1}{1+\cos^2\theta} = \frac{1}{1+[\frac{1}{2}(e^{i\theta}+e^{-i\theta})]^2} = \frac{4}{4+(e^{i\theta}+e^{-i\theta})^2}.$$

(In the final step we multiplied by $\frac{4}{4}$ to clear fractions.) Looking ahead to where we will eventually integrate over the circle $|z| = 1$, with the parametrization $z = e^{i\theta}$ we have

$$\frac{1}{1+\cos^2\theta} = \frac{4}{4+(e^{i\theta}+e^{-i\theta})^2} = \frac{4}{4+(z+\frac{1}{z})^2} = \frac{4}{4+(z^2+2+\frac{1}{z^2})} = \frac{4z^2}{z^4+6z^2+1},$$

where in the last step we multiplied by $\frac{z^2}{z^2}$ in order to clear the $\frac{1}{z^2}$ term.

The given integral should thus be obtainable by integrating this final expression above over the unit circle, except that we have not yet taken into account the tangent vector $z' = ie^{i\theta}$ term. To account for this we can we multiply by $\frac{ie^{i\theta}}{ie^{i\theta}}$ to get

$$\frac{1}{1+\cos^2\theta} = \frac{4z^2}{z^4+6z^2+1}\left(\frac{ie^{i\theta}}{ie^{i\theta}}\right) = \frac{4z}{i(z^4+6z^2+1)}\,dz,$$

where $z = e^{i\theta}$ in the denominator cancels with one z in the numerator $4z^2$. Thus

$$\int_0^{2\pi} \frac{1}{1+\cos^2\theta}\,d\theta = \frac{1}{i}\oint_{|z|=1} \frac{4z}{z^4+6z^2+1}\,dz$$

is our desired contour integral.

To compute this we need the poles of the integrand, which come from the roots of the denominator. If we think of the roots as coming from the equation

$$(z^2)^2 + 6(z^2) + 1 = 0,$$

by treating z^2 as the "variable" to solve for the quadratic formula gives

$$z^2 = \frac{-6\pm\sqrt{36-4}}{2} = -3\pm 2\sqrt{2}, \text{ so } z = \pm\sqrt{-3\pm 2\sqrt{2}}.$$

So, there are four roots, but we only need the roots that fall within $|z|=1$. Since $-3-2\sqrt{2}$ falls outside $|z|=1$, so do the square roots of $-3-2\sqrt{2}$, while since $-3+2\sqrt{2}$ falls within $|z|=1$, so do $\pm\sqrt{-3+2\sqrt{2}}$. Thus in order to finish computing this integral we would need to determine the residues of $\frac{4z}{z^4+6z^2+1}$ at $\sqrt{-3+2\sqrt{2}}$ and $-\sqrt{-3+2\sqrt{2}}$. (These residues are not challenging to compute, just a bit of work because of the numbers involved. As stated at the outset, we will not finish the computation here but you will work out a similar problem on the homework.)

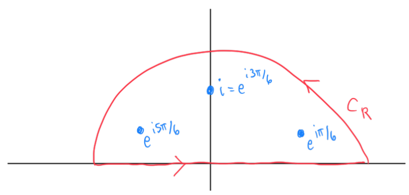
Warm-Up 2. We compute

$$\int_0^\infty \frac{1}{1+x^6}\,dx.$$

First we note that since $\frac{1}{1+x^6}$ is even with respect to x (replacing x by $-x$ gives the same function), we have that

$$\int_0^\infty \frac{1}{1+x^6}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{1+x^6}\,dx.$$

To compute the latter, we integrate $\frac{1}{1+z^6}$ over the boundary C_R of the top half of the disk $|z|\le R$ and take $R\to\infty$:



On the one hand, $\frac{1}{1+z^6}$ has poles at the sixth roots of -1, and three of these, namely

$$e^{i\pi/6},\ i,\ \text{and } e^{i5\pi/6},$$

fall within the top half of the disk. Thus

$$\oint_{C_R} \frac{1}{1+z^6}\,dz = 2\pi i(\text{sum of three residues}).$$

Each singularity is a simple zero of the denominator $1+z^6$, so all of the residues are given by

$$\operatorname{Res}_{z_0}\frac{1}{1+z^6} = \frac{1}{\frac{d}{dz}(1+z^6)}\bigg|_{z_0} = \frac{1}{6z_0^5}.$$

Hence we get

$$\begin{aligned}\oint_{C_R} \frac{1}{1+z^6}\,dz &= 2\pi i\left(\frac{1}{6e^{i5\pi/6}}+\frac{1}{6i^5}+\frac{1}{6e^{i25\pi/6}}\right)\\ &= \frac{2\pi i}{6}(e^{-i5\pi/6}-i+e^{-i25\pi/6})\\ &= \frac{\pi i}{3}(e^{-i5\pi/6}-i+e^{-i\pi/6})\\ &= \frac{\pi i}{3}(2i\sin(-\pi/6)-i)\\ &= \frac{\pi i}{3}(-2i)\end{aligned}$$

$$= \frac{2\pi}{3}.$$

(Note that $e^{-i5\pi/6} = e^{-i\pi}e^{i\pi/6}$ is the negative of the conjugate of $e^{-i\pi/6}$, so $e^{-i5\pi/6} + e^{-i\pi/6}$ is $2i$ times the imaginary part of $e^{-i\pi/6}$.)

Now, the integral of $\frac{1}{1+z^6}$ over the portion of C_R that lies on the real axis where $z = x$ is

$$\int_{-R}^{R} \frac{1}{1+x^6}\,dx.$$

The integral over the circular arc of C_R is bounded by

$$\left|\int_{|z|=R, y\geq 0} \frac{1}{1+z^6}\,dz\right| \leq \int_{|z|=R, y\geq 0} \frac{1}{|1+z^6|}\,|dz \;\leq \int_{|z|=R, y\geq 0} \frac{1}{R^6-1}\,|dz| = \frac{\pi R}{R^6-1}$$

where we use $|z^6+1| \geq |z^6| - 1 = |z|^6 - 1 = R^6 - 1$. This final expression goes to 0 as $R \to \infty$, so the integral over the circular arc vanishes in the limit. Thus

$$\begin{aligned}
\frac{2\pi}{3} &= \lim_{R\to\infty} \oint_{C_R} \frac{1}{1+z^6}\,dz \\
&= \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{1}{1+x^6}\,dx + \int_{|z|=R, y\geq 0} \frac{1}{1+z^6}\,dz \right) \\
&= \int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx + 0,
\end{aligned}$$

so $\int_0^\infty \frac{1}{1+x^6}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx = \frac{\pi}{3}$.

New type of contour. We finish now by computing

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx,$$

This will be a somewhat elaborate computation since it will require with a contour we have not used before. This integral plays an important role in the study of what's called the *Fourier transform* and in the study of certain *partial differential equations* such as the *heat equation*, so its value is important to know in those contexts.

To start, we will integrate the function

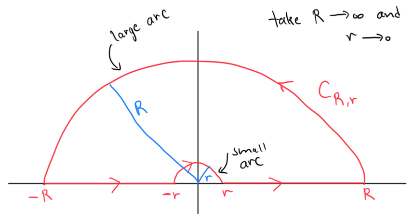
$$\frac{e^{iz}}{z}.$$

For $z = x$ real this is $\frac{e^{ix}}{x}$, whose imaginary part is $\frac{\sin x}{x}$, which is why in the end we will be able to extract the value of the integral we want. One might ask why on do not simply integrate

$$\frac{\sin z}{z}$$

instead as this too will produce $\frac{\sin x}{x}$ when $z = x$ is real, and the answer is that in order to get the bounds we will need on certain parts of our contour it will be important that we work with exponentials, which are generally simpler to bound than other functions like $\sin z$.

Since $\frac{e^{iz}}{z}$ has a singularity at 0, the contour we use cannot be one that passes through the origin, which eliminates every contour we have used so far in such improper integral computations. Instead, we consider the following contour, which is almost the boundary of the top half of a disk $|z| \leq R$ only with a small "bump" put in near the origin:



Call this contour $C_{R,r}$ with "large" radius R we will take to ∞ and "small" radius r we will take to 0. (Having to take two limits now is part of what makes this more elaborate.) The function $\frac{e^{iz}}{z}$ is perfectly holomorphic on a simply-connected domain containing $C_{R,r}$, so Cauchy's theorem gives

$$\oint_{C_{R,r}} \frac{e^{iz}}{z}\,dz = 0.$$

When integrating over the portions of $C_{R,r}$ on the real axis we get two terms

$$\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{r}^{R} \frac{e^{ix}}{x}\,dx,$$

one for each line segment making up the bottom of $C_{R,r}$. In the limit as $R \to \infty$ *and* $r \to 0$, these combine to give

$$\int_{-\infty}^{0} \frac{e^{ix}}{x}\,dx + \int_{0}^{\infty} \frac{e^{ix}}{x}\,dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx,$$

whose imaginary part is the integral we seek to compute.

Now, for the integral over the "large" circular arc of radius R making up $C_{R,r}$ (spoiler alert: this will go to zero in the limit), we need bounds. With the parametrization $z = Re^{it}$, we get

$$\int_{\text{large arc}} \frac{e^{iz}}{z}\,dz = \int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}} iRe^{it}\,dt = \int_0^{\pi} ie^{iR(\cos t + i\sin t)}\,dt,$$

so

$$\left|\int_{\text{large arc}} \frac{e^{iz}}{z}\,dz\right| \leq \int_0^{\pi} |ie^{-iR(\cos t + i\sin t)}|\,dt = \int_0^{\pi} e^{-R\sin t}\,dt = 2\int_0^{\pi/2} e^{-R\sin t}\,dt.$$

At the end here we use the fact that $\sin t$ takes on the same values for $0 \leq t \leq \frac{\pi}{2}$ as it does for $\frac{\pi}{2} \leq t \leq \pi$, so that the integral of $e^{-R\sin t}$ over $[0, \frac{\pi}{2}]$ is the same as its integral over $[\frac{\pi}{2}, \pi]$, which is why we are able to double-up in the expression above. This now allows us to use the inequality $\sin t \geq \frac{2t}{\pi}$, which is valid on $[0, \frac{\pi}{2}]$ (but not on $[\frac{\pi}{2}, \pi]$ in fact, which is why we needed to "double-up"), to get

$$2\int_0^{\pi/2} e^{-R\sin t}\,dt \leq 2\int_0^{\pi/2} e^{-R(2t/\pi)}\,dt = -\frac{\pi}{R}e^{-R(2t/\pi)}\Big|_0^{\pi/2} = \frac{\pi}{R}(1 - e^{-R}).$$

(No need to know the inequality $\sin t \geq \frac{2t}{\pi}$ used here by heart.) This final expression indeed vanishes as $R \to \infty$, so we get that the integral over the large circular arc vanishes as well.

Putting it all together gives

$$\begin{aligned} 0 &= \lim_{R\to\infty, r\to 0} \oint_{C_{R,r}} \frac{e^{iz}}{z}\,dz \\ &= \lim_{R\to\infty, r\to 0} \left(\int_{\text{bottom segments}} \frac{e^{iz}}{z}\,dz + \int_{\text{large arc}} \frac{e^{iz}}{z}\,dz + \int_{\text{small arc}} \frac{e^{iz}}{z}\,dz \right) \end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx + 0 + \lim_{r\to 0}\int_{\text{small arc}} \frac{e^{iz}}{z}\,dz.$$

What remains then is compute the limit of the integral over the small circular arc of radius r. (Note that this small arc does not depend on R, which is why we need only consider the limit as $r \to 0$ and not $R \to \infty$.) For this we note that

$$\frac{e^{iz}}{z} = \frac{1}{z} + \frac{e^{iz}-1}{z}.$$

The second function on the right actually has a *removable* singularity at 0 since

$$e^{iz} = 1 + iz + \frac{1}{2}(iz)^2 + \cdots \rightsquigarrow e^{iz} - 1 = iz + \frac{1}{2}(iz)^2 + \cdots,$$

so that after dividing by z we still left with a usual power series. In particular this means that $\frac{e^{iz}-1}{z}$ is bounded near 0 since there is no $\frac{1}{z}$ term to cause it to blow up, so we can pick some bound $|\frac{e^{iz}-1}{z}| \leq M$ near 0. This gives

$$\left|\int_{\text{small arc}} \frac{e^{iz}-1}{z}\,dz\right| \leq \int_{\text{small arc}} \left|\frac{e^{iz}-1}{z}\right| |dz| \leq \int_{\text{small arc}} M|dz| = M\pi r,$$

where πr is the circumference of the small circular arc. As $r \to 0$, this right side vanishes, so the integral of $\frac{e^{iz}-1}{z}$ over the small arc goes to 0 as $r \to 0$ as well.

Thus taking $r \to 0$ in

$$\int_{\text{small arc}} \frac{e^{iz}}{z}\,dz = \int_{\text{small arc}} \frac{1}{z}\,dz + \int_{\text{small arc}} \frac{e^{iz}-1}{z}\,dz$$

gives

$$\lim_{r\to 0}\int_{\text{small arc}} \frac{e^{iz}-1}{z}\,dz = \lim_{r\to 0}\int_{\text{small arc}} \frac{1}{z}\,dz.$$

This final integral can be computed directly using the parametrization $z = re^{it}, 0 \leq t \leq \pi$ (which gives the wrong orientation since the small arc moves clockwise), and we get

$$\lim_{r\to 0}\int_{\text{small arc}} \frac{e^{iz}-1}{z}\,dz = \lim_{r\to 0}\int_{\text{small arc}} \frac{1}{z}\,dz = -\pi i.$$

Hence

$$0 = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx + 0 + \lim_{r\to 0}\int_{\text{small arc}} \frac{e^{iz}}{z}\,dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - \pi i,$$

so

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i, \text{ and taking imaginary parts gives } \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi.$$

Huzzah! Moreover, since $\frac{\sin x}{x}$ is even with respect to x, we also get $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Lecture 26: Argument Principle

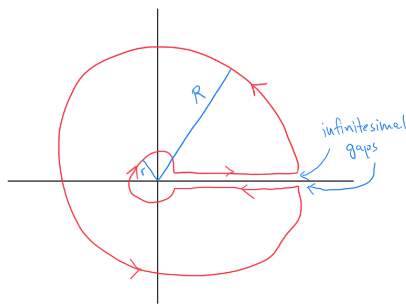
Warm-Up. We compute

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx.$$

This is the final improper integral we will consider and illustrates a final subtlety in such computations. The complex function we will use is

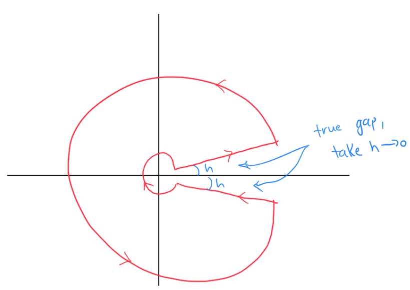
$$\frac{z^{1/3}}{1+z^2}$$

since this (appears to) restricts to the given integrand when $z = x$ is real. However, we must be careful now in working with $z^{1/3}$ since doing so requires picking a particular branch! We make a branch cut along $[0,\infty)$, meaning we pick argument values $0 < \arg z < 2\pi$. The issue now is that the contour we use, then, cannot pass through this branch cut, so we have to be clever about what we do. We integrate over the contour,



where we imagine that pieces close to positive real axis on the right are in fact "infinitesimally close". Thus, we move along the top "infinitesimally close" segment, then along a large circle $|z| = R$, then along the bottom "infinitesimally close" segment (going towards the left!), and finally along a small (clockwise) circle $|z| = r$.

To be clear, we are being a bit informal here since "infinitesimally close" does not have a precise meaning. We will literally think of these segments as if they were *on* the real axis without being the "same" as segments on the real-axis. To be more formal, we should really consider a contour like that above only where the "infinitesimal gaps" were true gaps of some size h measured by an angle, and then taking a limit as the gap $h \to 0$ shrinks in the end:



but we will not expect this level of formality here. The informal approach does give the correct value, albeit with some details swept under the rug. The segments being "infinitesimally close" to the real axis means that we can treat the large and small circles as if they were complete circles.

So, with these caveats, we proceed as follows. Denote the contour above by $C_{R,r}$. The function $\frac{z^{1/3}}{1+z^2}$ has poles at $\pm i$ (at least for large enough R and small enough r), so the residue theorem gives

$$\oint_{C_{R,r}} \frac{z^{1/3}}{1+z^2}\,dz = 2\pi i(\text{residue at } i \ + \ \text{residue at } -i).$$

Both of these poles are simple zeroes of $1+z^2$, so the residues are

$$\operatorname{Res}_{\pm i} \frac{z^{1/3}}{1+z^2} = \left.\frac{z^{1/3}}{2z}\right|_{z=\pm i} = \frac{(\pm i)^{1/3}}{\pm 2i}.$$

For the chosen branch of $z^{1/3}$, we have

$$i = e^{i\pi/2} \rightsquigarrow i^{1/3} = e^{i\pi/6} \quad \text{and} \quad -i = e^{3\pi i/2} \rightsquigarrow (-i)^{1/3} = e^{3\pi i/6},$$

so

$$\oint_{C_{R,r}} \frac{z^{1/3}}{1+z^2}\,dz = 2\pi i\left(\frac{i^{1/3}}{2i} + \frac{(-1)^{1/3}}{-2i}\right) = \pi(e^{i\pi/6} - e^{i3\pi/6}).$$

(We will simplify this further at the very end in order to get a nice numerical result.)

Now, for the integral over the large circle $|z| = R$, we use bounds:

$$\left|\oint_{|z|=R} \frac{z^{1/3}}{z^2+1}, dz\right| \leq \oint_{|z|=R} \frac{|z|^{1/3}}{|z|^2-1}\,|dz| = \frac{R^{1/3}}{R^2-1}(2\pi R).$$

(Note again that we can treat the circles as if they were complete circles due to the "infinitesimal gaps" we use, which is why the circumference is $2\pi R$.) Overall we get $R^{4/3}$ in the numerator but R^2 in the denominator, so the limit as $R \to \infty$ is zero. For the integral over the smaller circle $|z| = r$, in fact the same exact bounds apply only with a different radius:

$$\left|\oint_{|z|=r} \frac{z^{1/3}}{z^2+1}, dz\right| \leq \oint_{|z|=r} \frac{|z|^{1/3}}{|z|^2-1}\,|dz| = \frac{r^{1/3}}{r^2-1}(2\pi r).$$

This still goes to 0 as $r \to 0$, now because the numerator goes to 0 and the denominator to -1. (The small circle is oriented clockwise, so the integral changes sign, but the negative of 0 is 0 so the orientation does not matter in this case.) So, in the end neither of these circles will contribute to the contour integral value after we take limits $R \to \infty$ and $r \to 0$.

For the "top" segment infinitesimally close to $[r, R]$ on the positive real axis, we have $z = xe^{i0}$ for the argument values chosen "infinitesimally above" the real axis. This gives that

$$z^{1/3} = x^{1/3}e^{i0/3} = \sqrt[3]{x}$$

is the usual real cube root function, so as $R \to \infty, r \to 0$ the integral over this segment becomes

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx.$$

However, for the "bottom" segment infinitesimally close to $[r, R]$, in order to get the correct cube root of $z = x$ we must use $x = xe^{2\pi i}$ to describe these real values since the argument approaches 2π along this bottom direction for our chosen branch, which gives

$$z^{1/3} = (xe^{2\pi i})^{1/3} = \sqrt[3]{x}e^{2\pi i/3}$$

along this segment. Thus, in the limit, the integral over this "bottom" segment becomes

$$-\int_0^\infty \frac{\sqrt[3]{x}e^{2\pi i/3}}{1+x^2}\,dx = -e^{2\pi i/3}\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx$$

where the negative corrects for the leftward orientation along the bottom segment.

Altogether then, we get

$$\pi(e^{i\pi/6}-e^{i3\pi/6}) = \lim_{\substack{R\to\infty\\ r\to 0}} \oint_{C_{R,r}} \frac{z^{1/3}}{1+z^2}\,dz = \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx - e^{2\pi i/3}\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx,$$

which after solving for $\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx$ gives

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx = \frac{\pi(e^{i\pi/6}-e^{i3\pi/6})}{1-e^{2\pi i/3}}.$$

To put this into a nicer form, we can do some algebraic manipulation. Factoring $e^{2\pi i/6}$ out of numerator and $e^{\pi i/3} = e^{2\pi i/6}$ out of the denominator gives

$$\frac{\pi(e^{i\pi/6}-e^{i3\pi/6})}{1-e^{2\pi i/3}} = \frac{\pi e^{2\pi i/6}(e^{-i\pi/6}-e^{i\pi/6})}{e^{\pi i/3}(e^{-i\pi/3}-e^{\pi i/3})} = \frac{\pi(e^{-i\pi/6}-e^{i\pi/6})}{e^{-i\pi/3}-e^{\pi i/3}}.$$

The remaining differences of exponentials are both of the form $w-\overline{w}$, which gives $2i$ times the imaginary part of w:

$$\frac{\pi(e^{-i\pi/6}-e^{i\pi/6})}{e^{-i\pi/3}-e^{\pi i/3}} = \frac{\pi 2i\sin(-\pi/6)}{2i\sin(-\pi/3)} = \frac{\pi(-1/2)}{(-\sqrt{3}/2)} = \frac{\pi}{\sqrt{3}}.$$

Thus we conclude that

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx = \frac{\pi(e^{i\pi/6}-e^{i3\pi/6})}{1-e^{2\pi i/3}} = \frac{\pi}{\sqrt{3}}$$

is our desired value. (At least informally. As mentioned before, to be precise we would need to include an actual gap to replace the "infinitesimal gap" we used and take a limit as the gap shrinks. Our informal approach gives the correct value nonetheless because the integral over the top segment with an actual gap approaches the integral over our segment with infinitesimal gap in the limit as the gap shrinks, and similarly for the integral over the bottom segment with an actual gap.)

Back to winding numbers. Recall that the winding number of a closed contour around a point detects the numbers of times the contour "wraps" or "winds" around that point, with counter-clockwise windings contributing $+1$ and clockwise windings contributing -1. We will now give a more formal approach to winding numbers, which leads to the fact that we can use them to detect zeros and poles in a manner summed up by what's called the *argument principle*.

As motivation, let us consider the example of

$$\frac{1}{2\pi i}\oint_{|z|=1} \frac{3z^2}{z^3}\,dz.$$

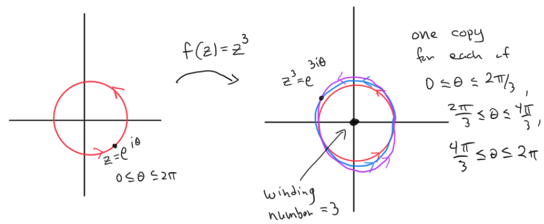
We can compute this integral directly by simplifying the integrand:

$$\frac{1}{2\pi i}\oint_{|z|=1} \frac{3z^2}{z^3}\,dz = \frac{1}{2\pi i}\oint_{|z|=1} \frac{3}{z}\,dz = \frac{3}{2\pi i}\oint_{|z|=1} \frac{1}{z}\,dz = \frac{3}{2\pi i}(2\pi i) = 3.$$

Now, consider the effect the transformation $f(z) = z^3$ in the denominator of our integrand has on the contour of integration $|z| = 1$. For $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$ on this contour we get

$$f(z) = z^3 = e^{3i\theta},$$

which as θ varies from 0 to 2π traces out the same unit circle but *three* times overall:



The upshot is that the value of the integral $\frac{1}{2\pi i} \oint_{|z|=1} \frac{3z^2}{z^3}\, dz = 3$ above is precisely the number of times the image circle is traced out under the denominator $f(z) = z^3$, so it is indeed the winding number of the image circle around 0. The numerator $3z^2$ of our integrand is precisely the derivative of $f(z) = z^3$, so this example hints at a relation between

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{f'(z)}{f(z)}\, dz$$

and the winding number of the image of $|z| = 1$ around 0. The argument principle states in part that this relation is true for more general $f(z)$, as we will soon see.

Moreover, we can interpret this winding number as encoding a *change* in argument values. After all, the reason why the circle above is traced out three times is because for $0 \leq \theta \leq \frac{2\pi}{3}$ we have that the argument of $e^{3i\theta}$ increases from 0 to 2π to give one copy of the circle, then as θ runs from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$ we pick up another argument change in $e^{3i\theta}$ of 2π from 2π to 4π giving a second copy of the circle, and finally for $\frac{4\pi}{3} \leq \theta \leq 2\pi$ we pick up one final argument change of 2π from 4π to 6π for a third copy. We say that 6π is thus the change in argument values for $f(z)$ along $|z| = 1$, so that dividing this argument change of 2π gives precisely the winding number of the image:

$$\frac{1}{2\pi} \Delta_C \arg f(z) = \text{winding number of image contour around 0.}$$

(On the left we denote the argument change by $\Delta_C \arg f(z)$. The argument change in general will be an integer multiple of 2π, which is why dividing by 2π will give an integer. For clockwise windings the argument change is *negative* since arguments decrease in clockwise directions, which matches what we also expect of winding numbers.) The upshot is that the relation between the integral above and winding numbers seems to suggest that

$$\frac{1}{2\pi} \Delta_C \arg f(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f'(z)}{f(z)}\, dz$$

is true, at least in this one example, but the argument principle will say this holds in general.

We make one more observation about the value

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{3z^2}{z^3}\, dz = 3.$$

The denominator $f(z) = z^3$ has one zero in $|z| = 1$, but in fact this is a zero of order 3, or, as we also say, of *multiplicity* 3. The value 3 we got for this integral matches this counting of zeroes with multiplicity, which suggests a relation between this number of zeroes and the winding number or change in argument in general.

Argument principle. The argument principle makes this all precise, even allowing for the presence of poles in addition to zeroes! Suppose that f is *meromorphic*—a term we briefly introduced a while back to mean holomorphic except for possible poles—on and interior to a simple closed contour C on which f is nonzero nor has poles. The argument principles states that the change in argument of f along C, the winding number of the image around 0, the integral of $\frac{f'}{f}$, and the number of zeroes and poles all encode the same information via:

$$\Delta_C \arg f(z) = \frac{1}{i} \oint_C \frac{f'(z)}{f(z)}\, dz = 2\pi(Z - P),$$

where Z denotes the number of zeroes of $f(z)$ within C counted with multiplicity and P the number of poles of f within C counted with multiplicity. (The multiplicity of a pole is just its order, just as the multiplicity of a zero is also its order.) Dividing by 2π gives the version

$$\frac{1}{2\pi}\Delta_C \arg f(z) = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = Z - P$$

where the left side is now the winding number of the image contour around 0. The name "argument princple" comes from the change in argument interpretation.

The takeaway is that we can detect argument changes in terms of the number of zeroes and poles since both of these are detectable by the integral in the middle. We will exploit this next time to give our final result of the quarter about counting numbers of zeroes of holomorphic functions. We will justify the left side of the argument principle above shortly, and will leave the right side (an application of the residue theorem) to next time.

Examples. Consider

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{z^{n-1}}{3z^n - 1}\, dz.$$

To put this into the form required of the argument principle the numerator of the integrand should be the derivative of the denominator, so we multiply and divide by $3n$:

$$\frac{1}{2\pi i(3n)} \oint_{|z|=1} \frac{3nz^{n-1}}{3z^n - 1}\, dz.$$

The argument principle says that this integral value is $\frac{1}{3n}$ times the number of zeroes minus the number of poles of $f(z) = 3z^n - 1$ inside $|z| = 1$. There are no poles and the zeroes are the n-th roots of $\frac{1}{3}$, all n of which (counted with multiplicity) lie within the unit circle since $\frac{1}{3}$ is within the unit circle. Thus

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{3nz^{n-1}}{3z^n - 1}\, dz = Z - P = n,$$

so

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{z^{n-1}}{3z^n - 1}\, dz = \frac{1}{3n}\left(\frac{1}{2\pi i} \oint_{|z|=1} \frac{3nz^{n-1}}{3z^n - 1}\, dz\right) = \frac{1}{3n}(n) = \frac{1}{3}.$$

Moreover, $Z - P = n$ is the winding number of the image of $|z| = 1$ under $3z^n - 1$ (or $\frac{1}{2\pi}$ times the change in argument of $f(z) = 3z^n - 1$ around $|z| = 1$), which makes sense intuitively because an n-th power will cause things to wind around n times.

For

$$\frac{1}{i}\oint_{|z|=2} \frac{\frac{d}{dz}[z/(z-1)^2]}{z/(z-1)^2}\,dz,$$

where the numerator is just literally the derivative of the denominator (I just wanted to avoid computing this and putting a complicated thing in the numerator), the argument principle gives the value as

$$\frac{1}{i}\oint_{|z|=2} \frac{\frac{d}{dz}[z/(z-1)^2]}{z/(z-1)^2}\,dz = 2\pi(Z-P) = 2\pi(1-2) = -2\pi.$$

Indeed, $f(z) = \frac{z}{(z-1)^2}$ has one (simple) zero in $|z| = 2$ at $z = 0$ and one pole in $|z| = 2$ at $z = 1$, but the pole is of order 2, so $Z - P = 1 - 2$ when counted with multiplicity. This implies that the winding number of the image of $|z| = 2$ under the transformation $f(z) = \frac{z}{(z-1)^2}$ is

$$\frac{1}{2\pi}\Delta_C \arg f(z) = Z - P = -1,$$

even without having to know what this image contour actually looks like!

Argument side of argument principle. We finish for now by justifying the

$$\Delta_C \arg f(z) = \frac{1}{i}\oint_C \frac{f'(z)}{f(z)}\,dz$$

side of the argument principle. We give two approaches, one highlighting the winding number interpretation and the other the change in argument interpretation. (Both of which are different than the justification the book gives!) First, we use the fact that change of variables (or substitutions) in complex integrals works the same way as they do for real integrals. Namely, in

$$\frac{1}{i}\oint_C \frac{f'(z)}{f(z)}\,dz$$

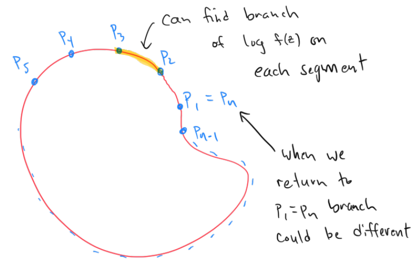
we make the substitution $w = f(z)$, so that $dw = f'(z)\,dz$ and hence

$$\frac{1}{i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{i}\oint_{f(C)} \frac{1}{w}\,dw.$$

But we know that integrating $\frac{1}{w}$ over a closed contour gives $2\pi i$ for each time the contour wraps around the origin, so this final integral is $\frac{1}{i}$ times $2\pi i$ times the winding number of the image $f(C)$ around 0, which is $\frac{1}{2\pi}\Delta \arg f(z)$:

$$\frac{1}{i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{i}\oint_{f(C)} \frac{1}{w}\,dw = \frac{1}{i}(2\pi i \cdot \text{winding}) = \frac{1}{i}(2\pi i[\tfrac{1}{2\pi}\Delta_C \arg f(z)]) = \Delta_C \arg f(z).$$

Alternatively we note that $\frac{f'(z)}{f(z)}$ is the thing you would get by differentiating $\log(f(z))$, assuming that we had a branch of log for which $\log(f(z))$ made sense as a holomorphic function. We cannot guarantee that such a branch exists on the entirety of the given contour C, but by breaking C up into pieces we can in fact use logarithms to compute the integral. Namely, consider



By picking these sample points p_i close enough to one another, we can ensure that we can find branches of $\log(f(z))$ that at least makes sense on each segment of C between such pairs of points. (We will not justify this, but it is a similar idea to what we used way back when to compute $\oint_{|z|=1} \frac{1}{z}\,dz$ using branches of log.) The branch we need might change as we shift from one segment to the next, but that is OK; moreover, the different branches used give the same value at the endpoint where one segment overlaps the next.

For the segment between p_1 and p_2 for example, we thus get that the chosen branch of $\log(f(z))$ is a valid anti-derivative of $\frac{f'(z)}{f(z)}$, so the fundamental theorem of calculus gives

$$\int_{\text{piece from } p_1 \text{ to } p_2} \frac{f'(z)}{f(z)}\,dz = \log(f(p_2)) - \log(f(p_1)).$$

For the next piece between p_2 and p_3 we get

$$\int_{\text{piece from } p_2 \text{ to } p_3} \frac{f'(z)}{f(z)}\,dz = \log(f(p_3)) - \log(f(p_2)).$$

Adding these gives the integral over the piece from p_1 all the way to p_3, and the point is that the two $\log(f(p_2))$ terms cancel out. (Recall that even with different branches the values at the point where one segment switches to the next are the same.) Continuing in this manner by moving to p_4, then p_5, and so on eventually gets us back to $p_n = p_1$, but where the branch of $\log(f(z))$ at the end as we moved all the way around is not necessarily the same as the one with which we began. The integral over all of C is thus

$$\oint_C \frac{f'(z)}{f(z)}\,dz = \log(f(p_n)) - \log(f(p_1)),$$

where again to be clear p_n is the same point at p_1 but the two $\log(f(z))$ terms on the right could denote different branches. But either way $\ln|f(p_n)|$ and $\ln|f(p_1)|$ are the same, so

$$\begin{aligned}\log(f(p_n)) - \log(f(p_1)) &= [\ln|f(p_n)| + i\arg f(p_n)] - [\ln|f(p_1)| + i\arg f(p_1)] \\ &= i(\arg f(p_n) - \arg f(p_1)),\end{aligned}$$

which is precisely $i\Delta_C \arg f(z)$. Thus $\oint_C \frac{f'(z)}{f(z)}\,dz = i\Delta_C \arg f(z)$ as the argument side of the argument principle states.

Lecture 27: Rouché's Theorem

Warm-Up. We justify the zeros and poles sides of the argument principle, namely

$$\oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i(Z - P),$$

which is an application of the residue theorem. The singularities of $\frac{f'(z)}{f(z)}$ come from either zeroes of the denominator f or poles of the denominator, so

$$\oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i\left(\sum_{\text{zeros within } C} \text{residues } + \sum_{\text{poles within } C} \text{residues}\right).$$

All we need to do is compute the necessary residues.

Suppose first that z_0 is a zero f of order m inside C. Then, as we have seen before, we can "factor out" $(z-z_0)^m$ and write f as

$$f(z) = (z-z_0)^m h(z)$$

for some holomorphic h with $h(z_0) \neq 0$. With this we get

$$f'(z) = m(z-z_0)^{m-1}h(z) + (z-z_0)^m h'(z)$$

from the product rule, so

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}h(z) + (z-z_0)^m h'(z)}{(z-z_0)^m h(z)} = \frac{m}{z-z_0} + \frac{h'(z)}{h(z)}.$$

The quotient $\frac{h'(z)}{h(z)}$ is differentiable at z_0 (because the denominator is nonzero at z_0), so this is expressible as a power series on a disk around z_0, which means that adding $\frac{m}{z-z_0}$ to this power series gives the Laurent expansion of $\frac{f'(z)}{f(z)}$ around z_0:

$$\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + (\text{some power series}).$$

This means that z_0 is a simple pole of $\frac{f'}{f}$ with residue m, so the sum of residues over the zeros in the integral above is

$$\sum_{\text{zeros within } C} \text{residues} = \sum_{\text{zeros within } C} \text{order/multiplicity} = Z.$$

If instead z_0 is a pole of f of order m inside C, in fact the same reasoning applies only with m replaced by $-m$. Indeed, for a pole of order m we have

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots,$$

so that we can factor out $(z-z_0)^{-m} = \frac{1}{(z-z_0)^m}$ to get

$$f(z) = (z-z_0)^{-m}[a_{-m} + a_{-m+1}(z-z_0) + \cdots] = (z-z_0)^{-m}[\text{some power series nonzero at } z_0].$$

Thus $f(z) = (z-z_0)^{-m}h(z)$ for some holomorphic $h(z)$ nonzero at z_0, so $\frac{f'}{f}$ can be computed in the same way as before. The difference is that we get $-m(z-z_0)^{-m-1}$ (with a negative in front) for the derivative of the first term, and this results in

$$\frac{f'(z)}{f(z)} = \frac{-m}{z-z_0} + \frac{h(z)}{h'(z)}.$$

The residue at the (simple) pole z_0 is now the negative of the order, so

$$\sum_{\text{poles within } C} \text{residues} = \sum_{\text{poles within } C} -\text{order/multiplicity} = -P.$$

Hence

$$\oint_C \frac{f'(z)}{f(z)}\, dz = 2\pi i \left(\sum_{\text{zeros within } C} \text{residues } + \sum_{\text{poles within } C} \text{residues} \right) = 2\pi i(Z - P)$$

as claimed by the zeros/poles side of the argument principle.

Rouché's theorem. For our final result we use the argument principle to give a method for determining the number of zeros a given holomorphic function has in a given region. For such a function (assumed to be holomorphic) there are no poles so the argument principle gives

$$\frac{1}{2\pi}(\text{change in argument}) = \frac{1}{i}(\text{integral of derivative divided by function}) = (\text{number of zeroes}).$$

One way to determine the number of zeroes is thus by computing the integral in the middle, and indeed this is something that people do with the aide of computers since the types of integrals you get are in general going to be too complicated to compute by hand. But here we consider a different approach, where we are able to determine the desired number of zeroes by comparing our given function to a hopefully *simpler* one for which the number of zeroes is obvious.

The precise statement of this technique is known as *Rouché's theorem.* The claim is that if f and g are holomorphic on a domain containing a simple closed contour C and its interior, and if

$$|g(z)| < |f(z)| \text{ at all points on } C,$$

then $f(z)$ and $f(z) + g(z)$ have the same number zeros (counted with multiplicity) within C. In practice, the "dominating" function f is one for which the number of zeroes is easy to determine, and $f + g$ will then be something more complicated, but if we can ensure a bound like $|g| < |f|$ on C then we will know that $f + g$ has as many zeroes as the simpler function f inside C. (Note that bound is only required to hold *on* the contour, but we derive information about the behavior *within* the contour.)

To justify this we appeal to the

$$(\text{winding number}) = \frac{1}{2\pi}(\text{change in argument}) = (\text{number of zeroes})$$

identity in the argument principle. Consider

$$\frac{1}{2\pi}\Delta_C \arg(f(z) + g(z)) = \tfrac{1}{2\pi}\Delta_C \arg(f(z)[1 + \tfrac{g(z)}{f(z)}]),$$

where on the right we factored $f(z)$ out of $f(z)+g(z)$. Note that on C, since $|f(z)| > |g(z)|$ we have that $f(z) \neq 0$ since $|g(z)|$ is always at least zero, and hence dividing by $f(z)$ as above is possible. Moreover, the reverse triangle inequality gives

$$|f(z) + g(z)| \geq |f(z)| - |g(z)| > 0 \text{ on } C$$

by the assumption that $|g(z)| < |f(z)|$, which is good because $f+g$ being nonzero on C is necessary to make the argument principle applicable.

Taking arguments of products corresponds to adding arguments, so

$$\tfrac{1}{2\pi}\Delta_C \arg(f(z)[1+\tfrac{g(z)}{f(z)}] = \tfrac{1}{2\pi}\Delta_C \arg(f(z)) + \tfrac{1}{2\pi}\Delta_C \arg[1+\tfrac{g(z)}{f(z)}]$$

and thus

$$\frac{1}{2\pi}\Delta_C \arg(f(z)+g(z)) = \tfrac{1}{2\pi}\Delta_C \arg(f(z)) + \tfrac{1}{2\pi}\Delta_C \arg[1+\tfrac{g(z)}{f(z)}].$$

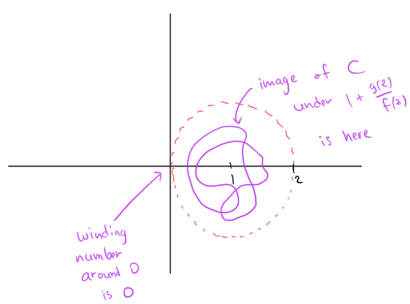
The left side is the number of zeros (with multiplicities) of $f+g$ within C by the argument princple, and the first term on the right is the number of zeroes (with multiplicities) of f within C, so Rouché's theorem will be justified once we know that the remaining term on the right must be zero:

$$\tfrac{1}{2\pi}\Delta_C \arg[1+\tfrac{g(z)}{f(z)}] = 0.$$

Here's the magic: the assumption that $|g(z)| < |f(z)|$ on C gives that

$$\left|\left(1+\frac{g(z)}{f(z)}\right) - 1\right| = \frac{|g(z)|}{|f(z)|} < 1 \text{ on } C,$$

so the complex number $w = 1+\frac{g(z)}{f(z)}$ always satisfies $|w-1| < 1$. Hence $w = 1+\frac{g(z)}{f(z)}$ lies in the open disk of radius 1 centered at 1, so the image of C under $1+\frac{g(z)}{f(z)}$ fully lies in this disk:



But $\frac{1}{2\pi}\Delta_C \arg[1+\frac{g(z)}{f(z)}]$ is the winding number of this image around the origin, which is thus zero (!!!) since the image does not wind around the origin at all. Therefore

$$\frac{1}{2\pi}\Delta_C \arg(f(z)+g(z)) = \tfrac{1}{2\pi}\Delta_C \arg(f(z)) + \underbrace{\frac{1}{2\pi}\Delta_C \arg[1+\frac{g(z)}{f(z)}]}_{0} = \tfrac{1}{2\pi}\Delta_C \arg(f(z)),$$

so the number of zeros of $f+g$ (the left side) within C equals the number of zeroes of f (the right side) within C. Boom!

Example. Let us use Rouché's theorem to determine the number of zeroes (I do not believe I have been consistent throughout these notes as to whether to write the plural of zero as "zeros" or "zeroes", c'est la vie!) of the polynomial

$$z^7 + 4z^4 + z^3 + 1$$

that lie within the circle $|z| = 2$. This polynomial is the function $f(z) + g(z)$ in the statement of Rouché's theorem, so we need to come up with a simpler function $f(z)$ dominating the remaining $g(z)$ for which the number of zeroes is easy to determine. We use

$$f(z) = z^7, \text{ so that } g(z) = \text{everything else} = 4z^4 + z^3 + 1.$$

Certainly the number of zeroes of $f(z) = z^7$ within $|z| = 2$ is easy to find: there is only one root at the origin, but itis counted with multiplicity 7, so we get 7 zeroes.

Thus if we know that $|f(z)| > |g(z)|$ on $|z| = 2$, we will be able to conclude that $f(z) + g(z) = z^7 + 4z^4 + z^3 + 1$ has 7 zeroes within $|z| = 2$ as well. But for $|z| = 2$, we have

$$|f(z)| = |z^7| = |z|^7 = 2^7 = 128$$

and

$$|g(z)| = |4z^4 + z^3 + 1| \leq 4|z|^4 + |z|^3 + 1 = 4(2^4) + 2^3 + 1 = 73,$$

(where we used the triangle inequality $|a + b| \leq |a| + |b|$ to bound $|4z^4 + z^3 + 1|$ by taking a sum of individual moduli), so we do have $|f(z)| > |g(z)|$ on $|z| = 2$, and thus everything works out.

Interlude: fundamental theorem of algebra. If we instead had

$$z^7 + 9z^4 + z^3 + 1,$$

the inequality $|9z^4 + z^3 + 1| < |z^7|$ no longer holds for $|z| = 2$ since using the triangle inequality on the left gives a largest value of $9(2^4) + 2^3 + 1 = 153$, which is larger than 2^7. So Rouché's theorem does not apply to this polynomial over $|z| = 2$, but it *does* apply for a larger radius like $|z| = 100$, so we would get that $z^7 + 9z^4 + z^3 + 1$ has 7 zeroes (counted with multiplicities) inside $|z| = 100$. A similar idea works if we have even larger coefficients, as long as we make our circle large enough.

In general, if $a_n z^n + \cdots + a_1 z + z_0$ is any nonconstant polynomial (and with leading coefficient $a_n \neq 0$ so that our polynomial is of degree n), since $|z|^n$ will increase more rapidly than any smaller power as $|z|$ increases, we can get a bound like

$$|a_n z^n| > |a_{n-1} z^{n-1} + \cdots + a_0| \text{ for large enough } |z|.$$

Rouché's theorem then says that

$$a_n z^n + \cdots + a_1 z + z_0$$

will have the same number of zeroes as does $a_n z^n$ in some large disk. This thus gives $n > 0$ roots, which is hence another proof of the fundamental theorem of algebra!

Back to example. We know that $z^7 + 4z^4 + z^3 + 1$ has 7 zeroes within $|z| = 2$, and now we want to determine how many of these lie within the annulus $1 \leq |z| < 2$. To do so we need only determine now many zeroes lie within $|z| = 1$ since the number within the annulus will be 7 minus the number within $|z| = 1$. For $|z| = 1$ we cannot hope to use the same "dominating" function z^7 as before since

$$|4z^4 + z^3 + 1| = 5 \text{ at } z = 1 \text{ on } |z| = 1 \text{ for example,}$$

so $|4z^4 + z^3 + 1| < |z^7|$ is not true on $|z| = 1$. We will instead use $f(z) = 4z^4$ as the dominating function (for which the number of zeroes is still easy to determine), and $g(z) = z^7 + z^3 + 1$ as "everything else". We have

$$|z^7 + z^3 + 1| \leq |z|^7 + |z|^3 + 1 = 3 < 4 = 4|z^4| = |4z^4| \text{ for } |z| = 1,$$

so $|g(z)| < |f(z)|$ holds for these functions on $|z| = 1$. Thus Rouché's theorem implies that $4z^4$ and $z^7 + 4z^4 + z^3 + 1$ have the same number of zeroes within $|z| = 1$, which is thus 4 since $4z^4$ has one root of order 4 in this disk. Therefore, there are $7 - 4 = 3$ zeroes in the annulus $1 \leq |z| < 2$. (Actually, as we saw in the proof of Rouché's theorem, the condition $|f| > |g|$ on $|z| = 1$ implies that $f + g$ is not zero on $|z| = 1$, so these 3 zeroes actually lie in the fully open annulus $1 < |z| < 2$.)

Another example. Now we determine how many zeroes

$$4z^{100} + z^{50} - e^z$$

has in the annulus $\frac{1}{2} < |z| < 1$. The strategy is the same as before: count the number of zeroes within $|z| = 1$ and subtract from this number within $|z| = \frac{1}{2}$. Since

$$|e^z| = e^x \text{ where } x \text{ is the real part of } z,$$

we have that $|e^z| \leq e^1 = e$ since the largest real part of a point on the circle $|z| = 1$ is $x = 1$. Thus we have

$$|z^{50} - e^z| \leq |z|^{50} + |e^z| \leq 1 + e < 4 = |4z^{100}| \text{ on } |z| = 1,$$

so $4z^{100} + z^{50} - e^z$ has 100 zeroes (counted with multiplicity) within $|z| = 1$ since $4z^{100}$ does.

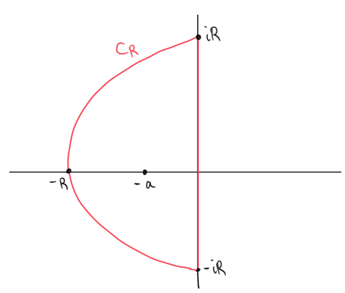
On the smaller circle $|z| = \frac{1}{2}$, note that $|z|^{100} = \frac{1}{2^{100}}$ and $|z|^{50} = \frac{1}{2^{50}}$ are incredibly small so we cannot expect that these can serve as good "dominating" functions in Rouché's theorem. Instead we thus use $-e^z$ as the dominating function, which is OK because we still know how many zeroes this function has... none! The smallest possible real part a point on $|z| = \frac{1}{2}$ can have is $-\frac{1}{2}$, so

$$|4z^{100} + z^{50}| \leq 4|z|^{100} + |z|^{50} = \frac{4}{2^{100}} + \frac{1}{2^{50}} < e^{-1/2} \leq |e^z| \text{ on } |z| = \frac{1}{2}.$$

(You can verify the inequality in the middle using a calculator: $\frac{4}{2^{100}} + \frac{1}{2^{50}}$ is something like 0.0000... with a good number of zeroes in front, while $e^{-1/2} = \frac{1}{\sqrt{e}}$ is something like 0.6....) Thus $4z^{100}+z^{50}-e^z$ has the same number of zeroes within $|z| = \frac{1}{2}$ as does $-e^z$, which is zero. The inequality that holds on $|z| = \frac{1}{2}$ also prevents there being any zeroes on this circle, so all 100 zeroes of $4z^{100} + z^{50} - e^z$ inside $|z| = 1$ actually lie within the annulus $\frac{1}{2} < |z| < 1$.

Final example. Finally, fix some real $a > 1$. We show that $z + a - e^z$ has exactly one zero on the left half plane $\operatorname{Re} z < 0$. This is not a region enclosed by a contour (such regions are always bounded), so we have to be more careful about how we apply Rouché's theorem.

Let us take some radius value $R > 0$ (to be specified later) and consider the contour C_R consisting of the left half of the circle $|z| = R$ together with the line segment from $-iR$ to iR on the imaginary axis:



If we want to determine how many zeroes $z + a - e^z$ has in the half-disk enclosed by this contour, we need to single-out a piece of this function for which zeroes are easy to find. Let us thus take the function $z + a$, which has one zero at $-a$ within C_R as long as R is large enough, say for $R > 2a$. (We will see why we use $2a$ instead of just a shortly.) With such a choice of R, we thus have

$$|z + a| \geq |z| - |a| = R - a > 2a - a = a > 1 \text{ for } |z| = R.$$

The modulus of e^z is e^x, which is at most 1 on the left half-plane, so in particular we get

$$|e^z| \leq 1 < |z+a| \text{ for } z \text{ on the circular arc part of } C_R.$$

(This is why we wanted $R > 2a$ instead of $R > a$; $R > a$ still makes C_R enclose $-a$, but we do not get that $z + a$ dominates e^z in this case.) For $z = it$ (with $-R \leq t \leq R$) on the part of C_R on the imaginary axis we have

$$|z+a| = |it+a| = \sqrt{t^2+a^2} \geq a > 1 = |e^{it}|,$$

so $z+a$ dominates e^z on this segment as well, so $z+a$ dominates $z+a$ on all of C_R. Thus Rouché's theorem implies that $z + a - e^z$ has one zero within C_R since $z + a$ has one zero within C_R.

But this is true for *any* $R > 2a$. As we increase R we still only get one zero within C_R, so by taking $R \to \infty$ we get exactly one zero in the entire left half plane as claimed: a second zero in the left-half plane would have to occur within some large enough C_R that would thus contain two zeroes, which is not possible. Thus, by taking contours that increase in size we can use Rouché's theorem to detect zeroes even over unbounded regions.

Thanks for reading!