Recent developments in mathematical Quantum Chaos, I

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Harvard,
November 21, 2009

Quantum chaos of eigenfunction

Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions

$$\Delta\varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

of the Laplacian on a (mainly compact) Riemannian manifold (M, g).
Suppose: The geodesic flow $G^t : S_g^* M \to S_g^* M$ is ergodic, or Anosov or some other notion of "chaotic".

Problem How are eigenfunctions distributed in 'phase space' $S_g^* M$ (the unit cosphere bundle w.r.t. g) as $\lambda_j \to \infty$?

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Phase space distribution of eigenfunctions

It is measured by the matrix elements

$$\rho_j(A) = \langle A\varphi_j, \varphi_j \rangle$$

where A is a special bounded operator on $L^2(M)$: namely, $A \in \Psi^0(M)$, a pseudo-differential operator of order zero.

These talks are about the limits of $\rho_j(A)$ as $\lambda_j \to \infty$.

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Outline of talks

- Talk I : Explain the problem and review the basics.
- Talk I: Hassell's scarring result for stadia. Brief glance at Faure-Nonnenmacher-de Bievre's scarring result for cat maps.
- Talk II: Recent entropy lower bounds of Anantharaman, Anantharaman-Nonnenmacher and Riviere.
- Talk II (if time permits) Quantum ergodicity and zero sets of eigenfunctions.

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Classical vs quantum mechanics

We want to use the dynamics of the geodesic flow, e.g. ergodicity or hyperbolicity, to study eigenfunctions of the Laplacian.

The wave equation and geodesic flow are closely related in the 'semi-classical limit' as Plancks constant $\hbar \to 0$. Here, $\hbar = \lambda^{-1}$ is the inverse frequency (note that λ^2 is the Δ-eigenvalue or energy.) We start by recalling basic definitions.

Classical Mechanics = Geodesic flow

Classical phase space = cotangent bundle T^*M of M, equipped with its canonical symplectic form $\sum_i dx_i \wedge d\xi_i$. The metric defines the Hamiltonian $H(x,\xi) = |\xi|_g = \sqrt{\sum_{ij=1}^n g^{ij}(x)\xi_i\xi_j}$ on T^*M, where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$. Hamilton's equations:

$$\begin{cases} \frac{dx_j}{dt} = \frac{\partial H}{\partial \xi_j} \\ \\ \frac{d\xi_j}{dt} = -\frac{\partial H}{\partial x_j}. \end{cases}$$

Its flow is the 'geodesic flow'

$$g^t : S_g^*M \to S_g^*M$$

usually restricted to the energy surface $\{H = 1\} := S_g^*M$.

Quantum Hamiltonian $= \sqrt{\Delta_g}$

The quantization of the Hamiltonian H is the square root $\sqrt{\Delta}$ of the positive Laplacian,

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} g^{ij} g \frac{\partial}{\partial x_j}$$

of (M, g). Here, $g = \det[g_{ij}]$. The eigenvalue problem on a compact Riemannian manifold

$$\Delta \varphi_j = \lambda_j^2 \varphi_j, \qquad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}.$$

Quantum evolution

The quantization of the geodesic flow is the wave group

$$U^t = e^{it\sqrt{\Delta}}.$$

Eigenfunctions are stationary states:

$$U^t \varphi_k = e^{it\lambda_k} \varphi_k$$

Since $|e^{it\lambda_k}| = 1$, the probability measure

$$|\psi(t,x)|^2 dvol$$

is constant where $\psi(t,x) = U_t \psi(x)$ is the evolving state.

Expected value of an observable in an energy state

In quantum mechanics, the functional

$$\rho_j(A) = \langle A\varphi_j, \varphi_j \rangle_{L^2(M)}$$

is the 'expected value of the observable A in the energy state φ_j (energy $= \lambda_j^2$).

An observable is a bounded operator on $L^2(M)$; mathematically, we assume $A = Op(a) \in \Psi^0(M)$, i.e. A is a zeroth order ΨDO (a pseudo-differential operator.) Here $a_\hbar \sim \sum_{j=0}^{\infty} \hbar^j a_{-j}$. $a_0 = \sigma_A$ is the principal symbol. (In the homogeneous setting, a_{-j} is homogeneous of degree $-j$ for $|\xi| \geq 1$).

Quantization of functions

If $A = \mathbf{1}_E$ is multiplication by the characteristic function of a nice open set $E \subset M$, then

$$\rho_j(\mathbf{1}_E) = \int_E |\varphi_j|^2 dV_g$$

is the "mass" or the probability that the particle represented by φ_j is located in E.

It also makes sense to quantize $\mathbf{1}_E \to Op(\mathbf{1}_E)$ the characteristic function of a set $E \subset T^*M$. Then $\langle Op(\mathbf{1}_E)\varphi_j, \varphi_j\rangle$ is the "probability amplitude that the (position, momentum) of the particle is in E."

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Wigner distributions

The map

$$a \to \rho_j(Op(a)) = \langle Op(a)\varphi_j, \varphi_j \rangle$$

defines a distribution W_j on T^*M– the Wigner distribution. We can view

$$\rho_j(Op(a)) = \int_{S_g^* M} a dW_j$$

as a distribution on the unit cosphere bundle (energy surface).

The Wigner distribution is (almost) a positive measure and is truly one if $Op(a)$ is defined in a certain way.

Egorov theorem

If $Op(a) \in \Psi^0(M)$, and $U^t = e^{it\sqrt{\Delta}}$ then,

$$\alpha_t(Op(a)) := U^t Op(a) U^{t*} \in \Psi^0(M),$$

and the principal symbol of $\alpha_t(Op(a))$ is $a \circ g^t$.
[This is a precise meaning that U^t quantizes g^t.]
Quantitatively:

$$U^t Op(a) U^{t*} = Op(a \circ g^t) + R_t,$$

where R_t is of order -1. However, its norm can blow up like $e^{C|t|}$.
This is where the Heisenberg time arises.

$\hbar \to 0, \lambda \to \infty$ and Ehrenfest time.

Quantum ergodicity exploits the long time behavior of the classical limit geodesic flow (ergodicity) to prove results about the high eigenvalue limit of eigenfunctions. It is the joint asymptotics

$$t \to \infty, \quad \lambda_j \to \infty$$

that makes the analysis difficult.
But the geodesic flow is only a good approximation to the quantum dynamics when

$$|t| \leq T_H(\lambda_j) := \kappa \log \lambda_j,$$

for a certain κ.

Weak* limit problem = quantum limit problem

Problem Let $\mathcal{Q}$ denote the set of 'quantum limits', i.e. weak* limit points of the sequence $\{W_k\}$ of distributions

$$\int_X \sigma_A dW_k := \langle A\varphi_k, \varphi_k \rangle$$

where $A \in \Psi^0(M)$ and $\sigma_A \in C^\infty(S^*M)$ is the principal symbol.

Determine $\mathcal{Q}$.

Weyl law

A fundamental connection between classical and quantum mechanics:

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\}$$

$$= \frac{|B_n|}{(2\pi)^n} Vol(M, g)\lambda^n + O(\lambda^{n-1}).$$

Here, $|B_n|$ is the Euclidean volume of the unit ball. A crucial input into quantum ergodicity is the local Weyl law:

$$\sum_{\lambda_j \leq \lambda} \langle A\varphi_j, \varphi_j \rangle = \frac{1}{(2\pi)^n} \left(\int_{B^* M} \sigma_A dx d\xi \right) \lambda^n + O(\lambda^{n-1}).$$

Remainders $O(\frac{\lambda^{n-1}}{\log \lambda})$ in all examples we discuss here. Would be important (but very hard) to improved.

Liouville state

$$\omega(A) := \frac{1}{\mu(S^*M)} \int_{S^*M} \sigma_A d\mu_L = \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \le \lambda} \langle A\varphi_j, \varphi_j \rangle \tag{1}$$

Here, μ_L is the *Liouville measure* on S^*M, i.e. the surface measure $d\mu_L = \frac{dxd\xi}{dH}$ induced by the Hamiltonian $H = |\xi|_g$ and by the symplectic volume measure $dxd\xi$ on T^*M.

Invariant states

We now develop the view that $\rho_j(A) = \langle A\varphi_j, \varphi_j\rangle$ is an *invariant state* on the C^*-algebra $\Psi^0(M)$.

A *state* is a linear functional on $\Psi^0(M)$ s.th. (i) $\rho_\psi(A^*A) \geq 0; (ii)\rho_\psi(I) = 1; (ii)\rho_\psi$ is continuous in the norm topology. It is the quantum analogue of a probability measure (a state on $C^0(S^*M)$.

An *invariant state* is a state ρ so that

$$\rho(A) = \rho(U^tAU^{-t}).$$

Clearly, ρ_j is invariant.
Another example: $\omega(A) = \int_{S^*M} \sigma_A d\mu_L$ is an invariant state (the Liouville state).

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Geodesic flow invariance of quantum limits

PROPOSITION

$\mathcal{Q} \subset \mathcal{M}_I$, where $\mathcal{M}_I$ is the convex set of invariant probability measures for the geodesic flow. They are also time-reversal invariant.

Any weak * limit of of $\{\rho_k\}$ is an invariant measure for g^t, i.e. $\mu(E) = \mu(g^t E)$. This is because ρ_k is an invariant state for the automorphism:

$$\rho_k(U_t A U_t^*) = \rho_k(A). \tag{2}$$

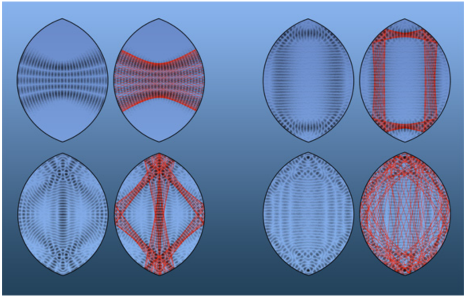
It follows by Egorov's theorem that any limit of $\rho_k(A)$ is a limit of $\rho_k(Op(\sigma_A \circ \Phi^t))$ and hence the limit measure is g^t invariant.

Possible quantum limits

There are many invariant probability measures for g^t: which occur as quantum limits?

1. Normalized Liouville measure $d\mu_L$.
2. A periodic orbit measure μ_γ defined by $\mu_\gamma(A) = \frac{1}{L_\gamma} \int_\gamma \sigma_A ds$ where L_γ is the length of γ. ("Scarring") A finite sum of periodic orbit measures.
3. A delta-function along an invariant Lagrangian manifold $\Lambda \subset S^*M$. The associated eigenfunctions are viewed as *localizing* along Λ.
4. A more general measure which is singular with respect to $d\mu_L$.

Some pictures of ‘scarring’ eigenstates along classical orbits in the case of the ellipse.



(Trajectories are in red on top of density plot of eigenfunctions; picture credit: Eric Heller)

Simplest case of finding quantum limits: S^1

S^*S^1 is the pair of circles $|\xi| = \pm 1$. The geodesic flow has two invariant sets of positive measure (the two components), but the flow is time-reversal invariant under $(x, \xi) \to (x - \xi)$ and the quotient flow is ergodic.

The only weak limit is $d\theta$: QUE!

$$\frac{1}{\pi}\int_0^{2\pi} V(x)(\sin kx)^2 dx \to \frac{1}{2\pi}\int_0^{2\pi} V(x) dx.$$

Proof: Write $\sin kx$ in terms of exponentials and use the Riemann-Lebesgue Lemma.

NOTE: Existence of the weak limits owe to the fast oscillation of the eigenfunction squares. In the limit the oscillating functions tend to their mean values in the weak sense.

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High frequency limits (= quantum limits) in the ergodic case

How does $\langle Op(\mathbf{1}_E)\varphi_j, \varphi_j\rangle$ behave as $\lambda_j \to \infty$? In the limit one should get the "classical probability" that a particle of energy λ_j^2 lies in $E \subset S_g^* M$. Ergodicity of the geodesic flow suggests that the limit should be $\frac{|E|}{|S_g^* M|}$.

Here, $|E|$ is the "Liouville measure" of E (basically, the Euclidean fiber volume times metric volume measure).

Ergodicity of geodesic flow

g^t is ergodic if the only g^t invariant sets $E \subset S^*M$ have either full Liouville measure or zero Liouville measure. Better: Louville measure is an ergodic measure for g^t.

Birkhoff: Almost every geodesic $g^t(x, \xi)$ is 'uniformly dense' in $S^*_g M$, i.e. for any set $E \subset S^*_g M$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_E(g^t(x,\xi))dt = \frac{|E|}{|S^*_g M|}.$$

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Quantum ergodic sequences

A subsequence $\{\varphi_{j_k}\}$ of eigenfunctions is called *ergodic* if the the only weak * limit of the sequence of ρ_{j_k} is $d\mu_L$ or equivalently the state ω. If $\rho_{k_j} \to \omega$ then

$$\frac{1}{Vol(M)} \int_E |\varphi_{k_j}(x)|^2 dVol \to \frac{Vol(E)}{Vol(M)}$$

for any measurable set E whose boundary has measure zero. In the interpretation of $|\varphi_{k_j}(x)|^2 dVol$ as the probability density of finding a particle of energy λ_k^2 at x, this says that the sequence of probabilities tends to uniform measure.

However, $dW_{k_j} \to \omega$ is much stronger since it says that the eigenfunctions become diffuse on the energy surface S^*M and not just on the configuration space M.

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Quantum ergodicity [Schnirelman;SZ; Colin de Verdiere; boundary case: Gerard-Leichtnam, SZ-Zworski; off-digaonal + converse SZ and Sunada]

THEOREM

Let (M, g) be a compact Riemannian manifold (possibly with boundary). Then the geodesic flow g^t is ergodic on $(S^*M, d\mu)$ $\iff$ $\forall A \in \Psi^0(M)$,

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle A\varphi_j, \varphi_j \rangle - \omega(A)|^2 = 0.$$

AND

$$(\forall \epsilon)(\exists \delta) \limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\substack{j \neq k: \lambda_j, \lambda_k \leq \lambda \\ |\lambda_j - \lambda_k| < \delta}} |\langle A\varphi_j, \varphi_k \rangle|^2 < \epsilon$$

Density one

The statement

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle A\varphi_j, \varphi_j \rangle - \omega(A)|^2 = 0$$

implies that there exists a subsequence $\{\lambda_{j_k}\}$ of counting density one for which $\langle A\varphi_{j_k}, \varphi_{j_k} \rangle \to \omega(A)$. We will call the eigenfunctions in such a sequence 'ergodic eigenfunctions'. By a diagonal argument, the sequence can be chosen independently of A.

QUE

$\sqrt{\Delta}$ is called QUE if there are no exceptional sequences, or if:

$$\langle A\varphi_j, \varphi_j\rangle \rightarrow \omega(A)$$

for the entire sequence of eigenvalues and for all $A \in \Psi^0(M)$.

Convexity proof of QE

The proof that

$$\frac{1}{N(E)} \sum_{\lambda_j \leq E} |\langle A\varphi_k, \varphi_k \rangle - \omega(A)|^2 \to 0, \qquad (3)$$

is based on two ingredients:
(i) The local Weyl law: $\frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} \rho_j \to \omega_L$: i.e. on average, eigenfunction states tend to Liouville;

(ii) Liouville is an extreme point of the compact convex set of invariant states iff g^t is ergodic.

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QE and convexity

By definition,
Ergodicity of the geodesic flow (w.r.t. Liouville measure) $\iff$ Liouville measure is an extreme point of the compact convex set $\mathcal{M}_I$.

An extreme point can not be expressed as a convex combination of other states unless they all equal the extreme point.

Yet, $\frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} \rho_j \to \omega_L$ expresses ω_L as a limit of convex combinations of ρ_j. It is only a limit, so the ρ_j do not have to equal ω_L; but almost all have to tend to it.

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Two models of chaotic eigenfunctions

1. Gaussian random functions $\varphi = \sum_{j:\lambda_j \in [\lambda, \lambda+1]} c_j \varphi_j$ with frequencies chosen from a short window. QE theorem only shows agreement to "two moments".

This is very heuristic. Gaussian random functions are much easier than eigenfunctions, but are sometimes good predictors of the latter (Berry; Heller, SZ).

2. Quantum cat map eigenfunctions: Eigenfunctions of quantizations of hyperbolic maps of the torus. "Anything that can go wrong does go wrong" : saturates eigenvalue multiplicity bounds, eigenfunctions scar on hyperbolic fixed points.... (Faure-Nonnenmacher-De Bievre).

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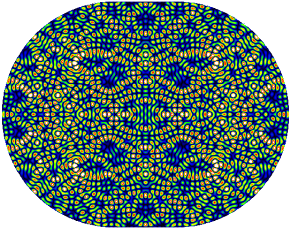
An eigenfunction in the Bunimovich stadium

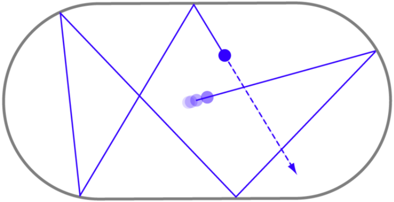


Scarring for Dirichlet (or Neumann) eigenfunctions of stadia?

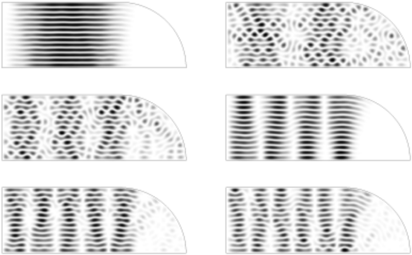
Is "density one" a defect of the proof or do exceptional sparse sequences exist?

Physicists (e.g. E. J. Heller) had numerical and heuristic results suggesting that there do exist exceptional sparse sequences of eigenfunctions in the case of the stadium.

A stadium is a domain $X_t = R_t \cup W \subset \mathbb{R}^2$ which is formed by a rectangle $R_t = [-t, t]_x \times [-1, 1]_y$ and where $W = W_- \cup W_+$ are half-discs of radius 1 attached at either end.



Scarring eigenfunctions



(Picture credit: Arnd Bäcker)

Hassell scarring theorem

Recently, Andrew Hassell proved that indeed almost all stadia do carry such sparse subsequences of eigenfunctions.

Theorem

The Laplacian on X_t is not QUE for almost every $t \in [1, 2]$.

The exceptional sequence concentrates on the Lagrangian manifold with boundary formed by bouncing ball orbits of a stadium.

Approximate eigenfunctions = quasi-modes

It is easy to construct quasi-modes for the stadium that concentrate on the unit tangent vectors to the vertical line segments in the inner rectangle:

$$\psi_n(x, y) = \chi(x) \sin n\pi y$$

where χ is a horizontal cutoff to the rectangle.

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Hassell scarring result: sketch of proof

1. Existence of scarring quasi-modes implies existence of scarring modes as long as the number of eigenvalues in small interval around the quasi-eigenvalues is uniformly bounded above (Heller-O'Connor, SZ).
2. It is all but impossible to prove such bounds for a given stadium. But Hassell showed that for a full measure family of Bunimovich stadia (as you stretch the inner rectangle), such bounds do exist.
3. Hence for a full measure of Bunimovich stadia, scarring eigenfunctions exist.

How many modes does it take to build a scarring quasi-mode?

Definition: *We say that a quasimode* $\{\psi_k\}$ *of order* 0 *with* $||\psi_k||_{L^2} = 1$ *has* $n(k)$ *essential frequencies if*

$$\psi_k = \sum_{j=1}^{n(k)} c_{kj}\varphi_j + \eta_k, \quad ||\eta_k||_{L^2} = o(1). \tag{4}$$

The frequencies λ_j of the φ_j must lie in $[\mu_k^2 - K, \mu_k^2 + K]$, where μ_k^2 are the quasi-eigenvalues of Δ for ψ_k.

Obviously, $n(k) \leq N(k, K)$, the number of eigenvalues in the interval $[\mu_k^2 - K, \mu_k^2 + K]$.

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No eigenvalue clustering + scarring qmodes → scarring modes

Proposition

(SZ '04) If there exists a quasi-mode $\{\psi_k\}$ *of order* 0 *for* Δ *with the properties:*

- *(i)* $n(k) \leq C, \ \forall \ k$*;*
- *(ii)* $\langle A\psi_k, \psi_k\rangle \to \int_{S^*M} \sigma_A d\mu$ *where* $d\mu \neq d\mu_L$*.*

Then Δ *is not QUE, i.e. there is a sequence of true modes tending to* $d\mu$*.*

How to eliminate clustering of eigenvalues near quasi-eigenvalues?

No method exists for individual stadia.

Hassell: But clustering can only occur for a measure zero set of stadia of fixed height and varying widths of the inner rectangle.

Ingredients:

- Hadamard's variational formula for the variation of Dirichlet or Neumann eigenvalues under a variation of a domain.
- Boundary QE (Hassell-SZ; Gerard-Leichtnam)

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Sketch of proof in Dirichlet case

Under the variation of X_t with infinitesimal variation vector field ρ_t, Hadamard's variational for the eigenvalues $E_j = \lambda_j^2$,

$$E_j^{-1} \frac{dE_j(t)}{dt} = E_j^{-1} \int_{\partial S_t} \rho_t (\partial_n u_j(t))^2 ds.$$

If the domain were QUE, the right side would tend to $\frac{k}{A(t)}$ where

$$k := \int_{\partial S_t} \rho_t(s) ds.$$

Hence,

$$\frac{\dot{E}_j}{E} = -kA(t)(1 + o(1)), \quad j \to \infty.$$

Hence there is a lower bound to the velocity with which eigenvalues decrease as $A(t)$ increases.

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Drifting spectrum cannot always cluster at n^2!

The quasi-eigenvalues are always $n^2 + O(1)$–the height never changes.

But eigenvalues cannot cluster in the fixed quasi-mode intervals $[n^2 - K, n^2 + K]$ for all t. Any such cluster would drift left as the domain increases. But then there are only a bounded number of eigenvalues in this interval.

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