CRITICAL POINTS AND SUPERSYMMETRIC VACUA, II: ASYMPTOTICS AND EXTREMAL METRICS

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ABSTRACT. Motivated by the vacuum selection problem of string/M theory, we study a new geometric invariant of a positive hermitian line bundle $(L,h) \to M$ over a compact Kähler manifold: the expected distribution $\mathcal{K}_h^{\rm crit}(z)$ of critical points $d \log |s(z)|_h = 0$ of a Gaussian random holomorphic section $s \in H^0(M,L)$ with respect to h. It is a measure on M whose total mass is the average number $\mathcal{N}_h^{\rm crit}$ of critical points of a random holomorphic section. We are interested in the metric dependence of $\mathcal{N}_h^{\rm crit}$, especially metrics h which minimize $\mathcal{N}_h^{\rm crit}$. We concentrate on the asymptotic minimization problem for the sequence of tensor powers $(L^N, h^N) \to M$ of the line bundle and their critical point densities $\mathcal{K}_{h^N}^{\rm crit}(z)$. We prove that $\mathcal{K}_{h^N}^{\rm crit}(z)$ has a complete asymptotic expansion in N whose coefficients are curvature invariants of h. The first two terms in the expansion of $\mathcal{N}_{h^N}^{\rm crit}$ are topological invariants of (L,M). The third term is a topological invariant plus a constant β_2^m (depending only on the dimension m of M) times the Calabi functional $\int_M \rho^2 dVol_h$, where ρ is the scalar curvature of the curvature form of h. We give an integral formula for β_2^m and show, by a computer assisted calculation, that $\beta_2^m > 0$ for $m \leq 3$, hence that $\mathcal{N}_{h^N}^{\rm crit}$ is asymptotically minimized by the Calabi extremal metric (when one exists). We conjecture that $\beta_2^m > 0$ in all dimensions, i.e. that the Calabi extremal metric is always the asymptotic minimizer.

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1. Introduction

This paper is the second in a series of articles on the statistics of vacua in string/M theory and associated effective supergravity theories. Mathematically, vacua are critical points of a holomorphic section $s \in H^0(M, L)$ of a line bundle $L \to M$ over a complex manifold relative to a connection ∇, which we always choose to be the Chern connection ∇_h of a hermitian metric h on L. In several papers [AD, D], M. R. Douglas has proposed a program of studying the statistics of critical points of a random holomorphic section with respect to a physically natural Gaussian measure γ on the space $H^0(M, L)$ of holomorphic sections, or on certain distinguished subspaces $\mathcal{S} \subset H^0(M, L)$. The basic idea is that the (supersymmetric) vacua of string/M theory are critical points of a holomorphic section (known as a superpotential) over the moduli space of complex structures on a Calabi-Yau manifold. But there exists at this time no reasonable selection principle to decide which superpotential nor which of its critical points gives the vacuum state which correctly describes our universe in string/M theory. So it makes sense to study the statistics of vauca of random superpotentials.

In this article, we study a purely geometric simplification of the physical problem where (L, h) is a positive Hermitian line bundle over a (usually compact) manifold M and where the Gaussian measure on $H^0(M, L)$ is derived from the inner product induced by h. Our aim is to understand the metric dependence of the statistics of the random critical point set

$$Crit(s, h) = \{z : \nabla_h(s) = 0\} = \{z : d\, |s(z)|_h^2 = 0, \;\; s(z) \neq 0\}. \tag{1}$$

of a Gaussian random section of $H^0(M, L)$ relative to the Chern connection ∇_h of h.

From the probabalistic viewpoint, the critical points of random holomorphic sections relative to the Chern connection ∇_h of a fixed hermitian metric on L define a *point process* on M, that is, a measure on the configuration space $Conf(M)$ of finite subsets of M. Each holomorphic section gives rise to the almost surely discrete set $Crit(s, h)$ of its critical points. The critical point process is the measure on $Conf(M)$ which gives the probability distribution of $X \subset M$ being the critical point set of a holomorphic section. It is determined by its n-point correlations $\mathbf{K}^{\mathrm{crit}}_{\nabla,\gamma,n}(z_1, \dots, z_n)$ which give the probabilities of critical points occurring at the points $z_1, \dots, z_n \in M$. They determine whether critical points tend to cluster or to repel each other. Since both ∇, γ are determined by h in this article, we simplify the notation to $\mathbf{K}^{\mathrm{crit}}_h$.

In this paper, we focus on the simplest (1-point) correlation function, namely the expected distribution of critical points

$$\mathbf{K}^{\mathrm{crit}}_h(z) = \mathbf{E}_{\gamma} \left[\sum_{z \in Crit(s,h)} \delta_z \right] \tag{2}$$

where δ_z is the Dirac point mass at z, and where $\mathbf{E}_\gamma$ denotes expectation relative to the Gaussian measure γ on $H^0(M,L)$ (cf. Definition 2). We shall see that $\mathbf{K}_h^{\rm crit}$ is a smooth measure on M. In particular, we are interested in the expected (average) number of critical points

$$\mathcal{N}_h^{\rm crit} = \int_M \mathbf{K}_h^{\rm crit}(z) \tag{3}$$

of a random section. This is a purely geometric invariant of (M,L). If we view $Conf(M) = \bigcup_{n=1}^\infty Sym^n(M)$ as the union of symmetric products of M, a model problem is to find the values of n where the critical point process is concentrated.

It is important to realize that the number $\#Crit_\nabla(s)$ is a (non-constant) random variable on $H^0(M,L)$, unlike the number of zeros of m independent sections which is a topological invariant of L. As indicated in (1), connection critical points $\nabla_h s(z) = 0\}$ are the same as critical points of $|s(z)|_h^2$ for which $s(z) \neq 0$, or equivalently as critical points of $\log|s(z)|_h$ (see [DSZ] for the simple proof). Hence, there are critical points of each Morse index $\geq m$ (see [B, DSZ]), and only the alternating sum of the number of critical points of each index is a topological invariant. Another way to understand the metric dependence of the number of critical points is to write the covariant derivative in a local frame e_L as

$$\nabla_{z_j} s = \left(\frac{\partial f}{\partial z_j} - f\frac{\partial K}{\partial z_j}\right) e_L = e^K \frac{\partial}{\partial z_j}\left(e^{-K} f\right) e_L\ , \qquad \nabla_{\bar z_j} s = \frac{\partial f}{\partial \bar z_j}\, e_L\ , \tag{4}$$

where we locally express a section as $s = fe_L$. Hence, the critical point equation

$$\left(\frac{\partial f}{\partial z_j} - f\frac{\partial K}{\partial z_j}\right) = 0 \tag{5}$$

in the local frame fails to be holomorphic when the connection form is only smooth.

Although $Crit(s,h)$ and $\#Crit(s,h)$ depend on h, it is not clear at the outset whether $\mathcal{N}_h^{\rm crit}$ is a topological invariant or whether it truly depends non-trivially on the metric h. To investigate the metric dependence of $\mathbf{K}_h^{\rm crit}$ and $\mathcal{N}_h^{\rm crit}$ we consider their asymptotic behavior as we take powers L^N of L. As in [SZ, BSZ1], it is natural to expect that the density and number of critical points will have simple asymptotic expansions which reveal their metric dependence. The study of such asymptotics does not have a physical interpretation at present, but is undertaken to gain insight into the nature of $\mathcal{N}_h^{\rm crit}$ as an invariant.

We therefore let $\mathbf{K}_{N,h}^{\rm crit}(z)$ denote the expected distribution of critical points of random holomorphic sections $s \in H^0(M,L^N)$ with respect to the Chern connection and Hermitian Gaussian measure induced by h^N, as given by (13)–(14) in §2. We also let

$$\mathcal{N}_{N,h}^{\rm crit} = \int_M \mathbf{K}_{N,h}^{\rm crit}(z) \tag{6}$$

denote the expected number of critical points. The covariant derivative associated to h^N has the semi-classical form

$$\nabla_{z_j} s_N = \left(\frac{\partial f}{\partial z_j} - Nf\frac{\partial K}{\partial z_j}\right) e_L^{\otimes N} = e^{NK} \frac{\partial}{\partial z_j}\left(e^{-NK} f\right) e_L^{\otimes N}\ , \qquad \nabla_{\bar z_j} s_N = \frac{\partial f}{\partial \bar z_j}\, e_L^{\otimes N}\ . \tag{7}$$

Our first result is a complete asymptotic expansion for the distribution of critical points and for the distribution of critical points for powers $L^N \to M$ in terms of curvature invariants of (L,h,M).

THEOREM **1.1.** *For any positive Hermitian line bundle* $(L,h)\to(M,\omega)$ *over any compact Kähler manifold with* $\omega=\frac{i}{2}\Theta_h := i2\bar\partial\partial K$*, the expected critical point distribution of sections of* L^N *relative to the Hermitian Gaussian measure induced by* h *has an asymptotic expansion of the form*

$$N^{-m}\,\mathbf{K}^{\mathrm{crit}}_{N,h}(z)\sim\{b_0+b_1(z)N^{-1}+b_2(z)N^{-2}+\cdots\}\frac{\omega^m}{m!}\,,$$

where the coefficients $b_j=b_j(m)$ *are curvature invariants of order* j *of* ω*. In particular,* b_0 *is the universal constant*

$$b_0=\pi^{-\binom{m+2}{2}}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}}\left|\det(2HH^*-|x|^2I)\right|\,e^{-\langle(H,x),(H,x)\rangle}\,dH\,dx$$

$b_1=\beta_1\rho$*, where* ρ *is the scalar curvature of* ω *and* β_1 *is a universal constant, and* b_2 *is a quadratic curvature polynomial. The values of the constant* b_0 *for low dimensions are:*

$$b_0(1)=\frac{5}{3}\,,\quad b_0(2)=\frac{59}{3^3}\,,\quad b_0(3)=\frac{637}{3^5}\,,\quad b_0(4)=\frac{6571}{3^7}\,.$$

Here, $\mathrm{Sym}(m,\mathbb{C})$ is the space of $m\times m$ complex symmetric matrices. It follows that the density of critical points is asymptotically uniform relative to the curvature volume form with a universal asymptotic density.

With only minor changes in the proofs, our methods give refinements of the asymptotic results which take the Morse indices of the critical points into account. By the Morse index q of a critical point, we mean its Morse index as a critical point of $\log\|s\|_{h^N}$; it is well known that $m\le q\le 2m$ for positive line bundles. Thus we let $\mathbf{K}^{\mathrm{crit}}_{N,q,h}(z)=\mathbf{K}^{\mathrm{crit}}_{N,q}(z)$ denote the expected distribution of critical points of $\log\|s\|_{h^N}$ of Morse index q, and we let $\mathcal{N}^{\mathrm{crit}}_{N,q,h}$ denote the expected number of these critical points. Thus we have

$$\mathbf{K}^{\mathrm{crit}}_{N,h}(z)=\sum_{q=m}^{2m}\mathbf{K}^{\mathrm{crit}}_{N,q,h}(z)\,,\qquad \mathcal{N}^{\mathrm{crit}}_{N,h}=\sum_{q=m}^{2m}\mathcal{N}^{\mathrm{crit}}_{N,q,h}\,. \tag{8}$$

Then we have:

THEOREM **1.2.** *Let* $M,L,h,\omega,\mathbf{K}^{\mathrm{crit}}_{N,q,h}$ *be as above. Then*

$$N^{-m}\,\mathbf{K}^{\mathrm{crit}}_{N,q,h}(z)\sim\{b_{0q}+b_{1q}(z)N^{-1}+b_{2q}(z)N^{-2}+\cdots\}\frac{\omega^m}{m!}\,,\qquad 0\le q\le m\,,$$

where the $b_{jq}=b_{jq}(m)$ *are curvature invariants of order* j *of* ω*. In particular,* b_{0q} *is given by the same formula as* b_0 *except that the domain of integration* $\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}$ *is replaced by*

$$\mathbf{S}_{m,k}:=\{(H,x)\in\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}:\mathrm{index}(HH^*-|x|^2I)=k\}\,, \tag{9}$$

with $k=q-m$.

Since each b_{0q} is strictly positive and their sum equals b_0 we have:

COROLLARY **1.3.** $b_0>\sum_{q=0}^m(-1)^{m+q}\,b_{0q}=\chi(L\otimes T^{*1,0})$.

It follows that positive curvature causes sections to have substantially more critical points on average than required by the topology. The integrals are very complicated to evaluate except in dimension one, where we obtain a very precise formula:

THEOREM **1.4.** *Let (L,h) be a positive line bundle on a compact complex curve C of genus g. Then*

$$\mathcal{N}^{\rm crit}_{N,h} = \frac{5}{3}\, c_1(L)\, N + \frac{7}{9}\,(2g-2) + \left(\frac{2}{27\pi}\int_C \rho^2\omega_h\right) N^{-1} + O(N^{-2})\ ,$$

where $\omega_h = \frac{i}{2}\Theta_h$ and ρ is the Gaussian curvature of the metric ω_h.

Thus we gain a quantitative sense of how many additional critical points there are in the metric sense by comparison with the classical sense of $\frac{\partial f}{\partial z} = 0$. In the case of $\mathcal{O}(N) \to \mathbb{CP}^1$, whose sections are polynomials of degree N, we may view this classical critical point equation as a connection critical point equation by viewing the derivative $\frac{\partial}{\partial z}$ as a flat meromorphic connection with pole at ∞. Alternately, it is the Chern connection of a singular hermitian metric. The critical point equation being purely holomorphic, the number of critical points of a generic section is a constant $N-1$. All critical points relative to this connection are saddle points. By comparison, critical points of $s \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$ relative to a smooth Chern connection have an additional $\sim \frac{N}{3}$ local maxima and $\sim \frac{N}{3}$ additional saddles. The study of critical points relative to meromorphic connections (known as Minkowski vacua) is simpler than that relative to Chern connections and will be explored further in a subsequent work.

Next we observe that, as a corollary of Theorem 1.2, the asymptotics of the expected number of critical points is universal to two orders.

COROLLARY **1.5.** *Let $(L,h) \to (M,\omega)$ be a positive holomorphic line bundle on a compact Kähler manifold, with $\omega = \frac{i}{2}\Theta_h$. Then the expected total number of critical points of Morse index q $(m \le q \le 2m)$ on M is of the form*

$$\begin{aligned}\mathcal{N}^{\rm crit}_{N,q,h} \;\sim\;& \left[\frac{\pi^m b_{0q}}{m!}\, c_1(L)^m\right] N^m + \left[\frac{\pi^m \beta_{1q}}{(m-1)!}\, c_1(M)\cdot c_1(L)^{m-1}\right] N^{m-1}\\ &+ \left[\beta_{2q}\int_M \rho^2 d{\rm Vol}_h + \beta'_{2q}\, c_1(M)^2\cdot c_1(L)^{m-2} + \beta''_{2q}\, c_2(M)\cdot c_1(L)^{m-2}\right] N^{m-2} + \cdots\ ,\end{aligned}$$

where $b_{0q}, \beta_{1q}, \beta_{2q}, \beta'_{2q}, \beta''_{2q}$ are universal constants depending only on the dimension m.

In §4, we obtain exact formulas for $\mathcal{N}^{\rm crit}_{N,q,h}$ for the case where M is projective space of dimension ≤ 3 and (L,h) is the hyperplane section bundle with the Fubini-Study metric.

This renews the question raised above whether $\mathcal{N}^{\rm crit}_{q,h}$ is a topological invariant or at least whether the asymptotic expansion of $\mathcal{N}^{\rm crit}_{N,q,h}$ is a topological invariant. We see from Corollary 1.5 that the expansion is not topological provided that the constant $\beta_{2q} = \beta_{2q}(m)$ does not vanish. Indeed from our computations in dimensions ≤ 3, we see that it is positive for these cases, and we expect that $\beta_{2q}(m) > 0$ for all m.

We pause to explain that the sign of β_{2q} has a natural interpretation in terms of extremal metrics [T, Don]. This interpretation is based on a notion of asymptotic minimality of $\mathcal{N}^{\rm crit}_{N,h}$, and we introduce it by revisiting the original problem of determining how $\mathcal{N}^{\rm crit}_h$ varies as h varies over hermitian metrics on L. One could consider all hermitian metrics on L, but we focus on the smaller class of metrics of concern in this article,

$$P(M,L) = \{h : \frac{i}{2}\Theta_h \ \text{ is a positive } \ (1,1)\text{-form }\}.$$

If we fix one such metric $h_0 = e^{-K_0}$, the others may be expressed as $h_\varphi := e^\varphi h_0$ with $\varphi \in C^\infty(M)$. It is plausible (though we do not have a proof) that $\mathcal{N}^{\rm crit}_{h_\varphi}$ is unbounded as h_φ varies over $P(M,L)$. In view of the equation (5), the number of critical points of a section should be 'large' if the 'degree' of the connection form $-\partial K$ is 'large'. Here, $K = -\log h = K_0 - \varphi$ in a local frame. The connection form is constrained by the positivity condition that $h_\varphi \in P(M,L)$, but $e^{\varepsilon\varphi}h_0 \in P(M,L)$ for any φ if ε is small enough, so it is plausible that this constraint does not suffice to bound $\mathcal{N}^{\rm crit}_{h_\varphi}$ from above. On the other hand, $\mathcal{N}^{\rm crit}_h$ is bounded below by the Euler number of $L \otimes T^{*1,0}$, and that suggests it has a smooth minimum. It would be interesting to determine this minimal metric, which would be a least entropy metric for the physical problem in the sense of the uncertainty as to which critical point is the correct vacuum. To study this minimization problem, one could study the variation $\delta\mathcal{N}^{\rm crit}_h$ of $\mathcal{N}^{\rm crit}_h$ with respect to h. However, the equation $\delta\mathcal{N}^{\rm crit}_h = 0$ has so far resisted analysis.

The asymptotic problem is simpler and brings this question into contact with Calabi extremal metrics. Since the first two leading coefficients in the expansion of $\mathcal{N}^{\rm crit}_{N,h}$ are topological, the issue is to find the metrics for which the first non-topological term is critical. The first non-topological term is the Calabi functional

$$\int_M \rho_h^2 \, d\mathrm{Vol}_h \ ,$$

where ρ_h is the scalar curvature of the Kähler metric $\omega_h = -\frac{i}{2}\partial\bar\partial \log h$, and $d\mathrm{Vol}_h = \frac{1}{m!}\omega_h^m$. Thus the problem of finding metrics which are critical for the metric invariant $\mathcal{N}^{\rm crit}_{N,h}$ is very closely related to the problem of find critical points (necessarily minima) of Calabi's functional.

Existence of critical metrics is one of the fundamental problems in complex geometry, and we refer to [T, Don] for background. It is believed that such a canonical metric exists if and only if L is stable in a suitable sense. One class of canonical metrics are hermitian metrics h for which Θ_h is a Kähler metric of constant scalar curvature, i.e. for which $\rho = C$ in our notation. By a theorem due to S. Donaldson ([Don], Corollary 5), there exists at most one Kähler metric of constant scalar curvature in the cohomology class of $2\pi c_1(L)$. Hence if there exists such a metric of constant scalar curvature, there exists a unique hermitian metric minimizing Calabi's functional.

This leads us to make the following definition:

Definition: Let $L \to M$ be an ample holomorphic line bundle over a compact Kähler manifold. We say that $h \in P(M,L)$ *has asymptotically minimal critical numbers* if for all $h_1 \neq h$ in $P(M,L)$, there exists $N_0 = N_0(h_1)$ such that

$$\mathcal{N}^{\rm crit}_{N,q,h} < \mathcal{N}^{\rm crit}_{N,q,h_1} \qquad \text{for} \quad N \geq N_0\,,\ m \leq q \leq 2m\,. \tag{10}$$

Assuming (M,L) has a hermitian metric h minimizing Calabi's functional, we see from Corollary 1.5 that h has asymptotically minimal critical numbers as long as $\beta_{2q}(m) > 0$. Since we believe this to be the case for all dimensions, we state the following conjecture.

Conjecture **1.6.** *A metric $h \in P(M,L)$ has asymptotically minimal critical numbers if and only if it minimizes Calabi's functional.*

In Lemma 6.1, we show that

$$\beta_{2q}(m) = \frac{1}{4\,\pi^{\binom{m+2}{2}}} \int_{\mathbf{S}_{m,q-m}} \gamma(H)\ \left|\det(2HH^* - |x|^2 I)\right|\ e^{-\langle (H,x),(H,x)\rangle}\, dH\, dx\ , \tag{11}$$

where

$$\mathbf{S}'_{m,k} = \{(H,x) \in \mathrm{Sym}(m,\mathbb{C}) \times \mathbb{C} : \mathrm{index}(2HH^* - |x|^2 I) = k\}\ ,$$

and

$$\gamma(H) = \frac{1}{2}|H_{11}|^4 - 2|H_{11}|^2 + 1.$$

It is unfortunately not clear from this formula which sign β_{2q} has. In §6, we give a final integral formula for β_{2q} (Lemma 6.2) which we evaluate using Maple to obtain:

Theorem **1.7.** *The constants $\beta_{2q}(m)$ are positive for $m \leq 3$, and hence Conjecture 1.6 is true for* $\dim M \leq 3$.

In particular, we have:

Corollary **1.8.** *Suppose that* $\dim M \leq 3$ *and that L possesses a metric h for which the scalar curvature of Θ_h is constant. Then h is the unique metric on L with asymptotically minimal critical numbers.*

Thus, for instance, the Fubini-Study metric h on the hyperplane section bundle $\mathcal{O} \to \mathbb{CP}^m$ is the unique metric with asymptotically minimal critical numbers on this bundle, at least for $m \leq 3$.

We close the introduction with some comments on the organization of the paper. The proof of Theorems 1.1 and 1.2 are based on the Tian-Yau-Zelditch asymptotic expansion of the Szegö kernel $\Pi_N(z,w)$ [Ze, Lu] and on formulas from [DSZ] for the density of critical points. We then need to evaluate the coefficients explicitly to obtain concrete results linking geometry to numbers of critical points. Once we know the leading coefficient is universal, we may calculate it for $\mathcal{O}(N) \to \mathbb{CP}^m$ and this is done in §4. We further obtain an exact formula for $\mathcal{N}^{\mathrm{crit}}_{h_{FS},N}$ for the Fubini-Study metric on $\mathcal{O}(N) \to \mathbb{CP}^m$ in any dimension. Unfortunately, Fubini-Study is not useful for finding the sign of β_2 since it is impossible to separate out the topologically invariant terms from the Calabi functional in an example. Hence in §6, we analyze this term in the case of $M = \mathbb{CP}^1 \times E^{m-1}$ for E an elliptic curve, where the topological terms vanish. This leads to an explicit integral which we analyze by a variant of the Itzykson-Zuber formula in random matrix integrals.

2. Background

Let $(L,h) \to M$ be a Hermitian holomorphic line bundle over a complex manifold M, and let $\nabla = \nabla_h$ be its Chern connection, i.e. the unique connection of type $(1,0)$ on compatible with both the metric and complex structure of L. Thus, it satisfies $\nabla'' s = 0$ for any holomorphic section s where $\nabla = \nabla' + \nabla''$ is the splitting of the connection into its $L \otimes T^{*1,0}$, resp. $L \otimes T^{*0,1}$ parts. It follows that

$$Crit(s,h) = \{z : \nabla'_h s(z) = 0\}. \tag{12}$$

We denote by $\Theta_h = d\partial \log h = -\partial\bar\partial \log h$ the curvature of h and $\omega_h = \frac{i}{2}\Theta_h$.

We now introduce the Gaussian measures γ_h, called *Hermitian Gaussian measures* in [DSZ] which we use exclusively in this paper. They are determined by the inner product

$$\langle s_1, s_2 \rangle = \int_M h(s_1(z), s_2(z)) dV_h(z) \tag{13}$$

on $H^0(M,L)$, where $dV_h = \frac{1}{m!}\omega_h^m$. By definition,

$$d\gamma_h(s) = \frac{1}{\pi^d} e^{-\|c\|^2} dc \,, \qquad s = \sum_{j=1}^d c_j e_j, \tag{14}$$

where dc is Lebesgue measure and $\{e_j\}$ is an orthonormal basis basis for $\mathcal{S}$ relative to $\langle , \rangle$. We also denote the expected value of a random variable X on with respect to γ_h by $\mathbf{E}_{\gamma_h}$ or simply by $\mathbf{E}$.

Definition: The expected distribution of critical points of $s \in \mathcal{S} \subset H^0(M,L)$ with respect to γ_h is defined by

$$\mathbf{K}_h^{\rm crit}(z) = \mathbf{E}_{\gamma_h} \left[\sum_{z \in Crit(s,h)} \delta_z \right] , \tag{15}$$

where δ_z is the Dirac point mass at z; i.e.,

$$\int_M \varphi(z) \mathbf{K}_h^{\rm crit}(z) \; = \int_{H^0(M,L)} \left[\sum_{z: \nabla_h s(z) = 0} \varphi(z) \right] d\gamma_h(s). \tag{16}$$

The density of $\mathbf{K}_h^{\rm crit}$ with respect to dV_h is denoted $\mathcal{K}_h^{\rm crit}(z)$; i.e.,

$$\mathbf{K}_h^{\rm crit} = \mathcal{K}_h^{\rm crit}(z)\, dV_h \;.$$

2.1. **Formulas for the expected distribution of critical points.** Let $(L,h) \to (M,\omega)$ be a Hermitian holomorphic line bundle on an m-dimensional Kähler manifold. We say that $H^0(M,L)$ has the 2-jet spanning property if all possible values and derivatives of order ≤ 2 are attained by the global sections $s \in H^0(M,L)$ at every point of M In [DSZ] we obtained an integral formula for $\mathbf{K}_h^{\rm crit}(z_0)$ in terms of the Szegö kernel $\Pi(z,\bar{w})$ of $H^0(M,L)$ with respect to h. To describe this formula, we choose normal coordinates about $z_0 \in M$ and define the following matrices:

$$A(z_0) \;=\; \left(\nabla_{z_j} \nabla_{\bar{w}_{j'}} \Pi\right), \tag{17}$$

$$B(z_0) \;=\; \left[\left(\tau_{j'q'} \nabla_{z_j} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi\right) \quad \left(\nabla_{z_j} \Pi\right) \right], \tag{18}$$

$$C(z_0) \;=\; \begin{bmatrix} \left(\tau_{jq}\tau_{j'q'} \nabla_{z_q} \nabla_{z_j} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi\right) & \left(\tau_{jq} \nabla_{z_q} \nabla_{z_j} \Pi\right) \\ \left(\tau_{j'q'} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi\right) & \Pi \end{bmatrix}, \tag{19}$$

$$\tau_{jq} = \sqrt{2} \quad \text{if} \;\; j < q \,, \quad \tau_{jj} = 1 \,, \qquad 1 \leq j \leq m \,,\; 1 \leq j \leq q \leq m \,,\; 1 \leq j' \leq q' \leq m \,, \tag{20}$$

where Π and its derivatives are evaluated at the point $(z_0, 0; z_0, 0)$. In the above matrices, j, q index the rows, and j', q' index the columns. Note that A, B, C are $m \times m$, $m \times d_m$, $d_m \times d_m$ matrices, respectively, where

$$d_m = \dim_{\mathbb{C}}(\mathrm{Sym}(m, \mathbb{C}) \times \mathbb{C}) = \frac{m^2 + m + 2}{2} \, .$$

We then let

$$\Lambda(z_0) = C(z_0) - B(z_0)^* A(z_0)^{-1} B(z_0) \, . \tag{21}$$

The matrices A, B, C give the second moments of the joint probability distribution of the random variables $\nabla s(z_0)$ and $\nabla^2 s(z_0)$ on $\mathcal{S}$.

THEOREM **2.1.** [DSZ] *Let $(L, h) \to M$ denote a positive holomorphic line bundle with the 2-jet spanning property. Give M the volume form $dV = \frac{1}{m!} \left(\frac{i}{2}\Theta_h\right)^m$ induced from the curvature of L. Then the expected density of critical points relative to dV is given by*

$$\mathcal{K}_h^{\mathrm{crit}}(z) = \frac{\pi^{-\binom{m+2}{2}}}{\det A(z) \det \Lambda(z)} \int_{\mathrm{Sym}(m,\mathbb{C}) \times \mathbb{C}} \left|\det(HH^* - |x|^2 I)\right| e^{-\langle \Lambda(z)^{-1}(H,x),(H,x)\rangle} \, dH \, dx \, .$$

Here, $H \in \mathrm{Sym}(m, \mathbb{C})$ is a complex symmetric matrix, dH and dx denote Lebesgue measure, and Λ^{-1} is the Hermitian operator on the complex vector space $\mathrm{Sym}(m, \mathbb{C}) \times \mathbb{C}$ described as follows:

Let S^{jq}, $1 \le j \le q \le m$, be the basis for $\mathrm{Sym}(m, \mathbb{C})$ given by

$$(S^{jq})_{j'q'} = \begin{cases} \frac{1}{\sqrt{2}}(\delta_{jj'}\delta_{qq'} + \delta_{qj'}\delta_{jq'}) & \text{for } j < q \\ \delta_{jj'}\delta_{qq'} & \text{for } j = q. \end{cases}$$

I.e., for $j < q$, S^{jq} is the matrix with $\frac{1}{\sqrt{2}}$ in the jq and qj places and 0 elsewhere, while S^{jj} is the matrix with 1 in the jj place and 0 elsewhere. We note that $\{S^{jq}\}$ is an orthonormal basis (over $\mathbb{C}$) for $\mathrm{Sym}(m, \mathbb{C})$ with respect to the Hilbert-Schmidt Hermitian inner product

$$\langle S, T \rangle_{\mathrm{HS}} = \mathrm{Tr}(ST^*). \tag{22}$$

For $H = (H_{jq}) \in \mathrm{Sym}(m, \mathbb{C})$, we have

$$H = \sum_{1 \le j \le q \le m} \widehat{H}_{jq} \, E^{jq} \, , \qquad \widehat{H}_{jq} = \tau_{jq} H_{jq} \, , \tag{23}$$

where τ_{jq} is given by (20). Lebesgue mesure dH (with respect to the Hilbert-Schmidt norm) is given by

$$dH = \prod_{j \le q} d\mathrm{Re}\widehat{H}_{jq} \wedge d\mathrm{Im}\widehat{H}_{jq}$$

Writing

$$\Lambda = \begin{bmatrix} \left(\Lambda_{jq}^{j'q'}\right) & \left(\Lambda_{jq}^{0}\right) \\ \left(\Lambda_{0}^{j'q'}\right) & \Lambda_0^0 \end{bmatrix} ,$$

we then define

$$\langle \Lambda(z)^{-1}(H, x), (H, x) \rangle = \sum (\Lambda^{-1})_{jq}^{j'q'} \widehat{H}_{jq} \overline{\widehat{H}_{j'q'}} + 2\mathrm{Re} \sum (\Lambda^{-1})_{jq}^{0} \widehat{H}_{jq} \bar{x} + (\Lambda^{-1})_0^0 |x|^2. \tag{24}$$

To study the asymptotics, we consider powers L^N and we use the following result.

Corollary **2.2.** *With the same notation and assumptions as above, the density of the expected distribution $\mathbf{K}^{\rm crit}_{N,h}$ of critical points of random sections $s_N \in H^0(M, L^N)$ relative to dV_h is given by*

$$\mathcal{K}^{\rm crit}_{N,h}(z) = \frac{\pi^{-\binom{m+2}{2}}}{\det A_N(z) \det \Lambda_N(z)} \int_{{\rm Sym}(m,\mathbb{C})\times\mathbb{C}} \left|\det(HH^* - |x|^2 I)\right| e^{-\langle \Lambda_N(z)^{-1}(H,x),(H,x)\rangle}\, dH\, dx\,.$$

where

$$\Lambda_N(z_0) = C_N(z_0) - B_N(z_0)^* A_N(z_0)^{-1} B_N(z_0)\,, \tag{25}$$

$$A_N(z_0) = \left[\left(\nabla_{z_j}\nabla_{\bar w_{j'}}\Pi_N\right)\right], \tag{26}$$

$$B_N(z_0) = \left[\left(\tau_{j'q'}\,\nabla_{z_j}\nabla_{\bar w_{q'}}\nabla_{\bar w_{j'}}\Pi_N\right) \quad \left(N\,\nabla_{z_j}\Pi_N\right)\right], \tag{27}$$

$$C_N(z_0) = \left[\begin{array}{cc} \left(\tau_{jq}\tau_{j'q'}\,\nabla_{z_q}\nabla_{z_j}\nabla_{\bar w_{q'}}\nabla_{\bar w_{j'}}\Pi_N\right) & \left(\tau_{jq}N\,\nabla_{z_q}\nabla_{z_j}\Pi_N\right) \\ \left(\tau_{j'q'}N\,\nabla_{\bar w_{q'}}\nabla_{\bar w_{j'}}\Pi_N\right) & N^2\,\Pi_N \end{array}\right], \tag{28}$$

$$\tau_{jq} = \sqrt{2} \quad \textit{if} \;\; j < q\,, \quad \tau_{jj} = 1\,,$$
$$1 \le j \le m\,,\; 1 \le j \le q \le m\,,\; 1 \le j' \le q' \le m\,,$$

where Π_N and its derivatives are evaluated at the point $(z_0, 0; z_0, 0)$.

Proof. Rescale $z_j = \widetilde z_j/\sqrt{N}$. Then the curvature of L^N is given by

$$\Theta_{h^N} = N\Theta_h = \frac{1}{2}\sum d\widetilde z_j \wedge d\overline{\widetilde z_j},$$

so that the $\widetilde z_j$ are normal coordinates (at a point z_0) for the curvature of L^N. Apply Theorem 2.1, using the coordinates $\{\widetilde z_j\}$ to obtain A, B, C, Λ . Since $d\widetilde V = N^m\, dV$ and the transformation $(A, \Lambda) \mapsto (NA, N^2\Lambda)$ introduces a factor N^{-m}, we let $A_N = NA,\ B_N = N^{3/2}B,\ C_N = N^2 C$ to obtain the desired formula. □

We also have a formula for the density of critical points of specific Morse indices:

Theorem **2.3.** *Under the above assumptions, the density relative to dV_h of the expected distribution $\mathbf{K}^{\rm crit}_{N,q,h}$ of critical points of Morse index q of $\log\|s_N\|_h$ for random sections $s_N \in H^0(M, L^N)$ is given by*

$$\mathcal{K}^{\rm crit}_{N,q,h}(z) = \frac{\pi^{-\binom{m+2}{2}}}{\det A_N(z) \det \Lambda_N(z)} \int_{\mathbf{S}_{m,q-m}} \left|\det(HH^* - |x|^2 I)\right| e^{-\langle \Lambda_N(z)^{-1}(H,x),(H,x)\rangle}\, dH\, dx\,.$$

where

$$\mathbf{S}_{m,k} = \{(H,x) \in {\rm Sym}(m,\mathbb{C}) \times \mathbb{C} : {\rm index}(HH^* - |x|^2 I) = k\}\,.$$

Proof. The case $N = 1$ is given as Theorem 6 in [DSZ]. The general case follows immediately by rescaling as in the proof of Corollary 2.2. □

Recall that the index of a nonsingular Hermitian matrix is the number of its negative eigenvalues, and the Morse index of a nondegenerate critical point of a real-valued function is the index of its (real) Hessian.

2.2. **The Szegö kernel.** As in our previous work, it is useful to lift the analysis on positive line bundles $L \to M$ to the associated principal S^1 bundle $X \to M$. Sections then become scalar functions and it is simpler to formulate various asymptotic properties for powers L^N [BSZ1, BSZ2]. The same analysis is also useful for general line bundles although the asymptotic results no longer hold.

Given a holomorphic line bundle L and a Hermitian metric h on L, we obtain a Hermitian metric h^* on the dual line bundle L^* and we define the associated circle bundle by $X = \{\lambda \in L^* : \|\lambda\|_{h^*} = 1\}$. Thus, X is the boundary of the disc bundle $D = \{\lambda \in L^* : \rho(\lambda) > 0\}$, where $\rho(\lambda) = 1 - \|\lambda\|^2_{h^*}$. When (L, h) is a positive line bundle, the disc bundle D is strictly pseudoconvex in L^*, hence X inherits the structure of a strictly pseudoconvex CR manifold. When L is negative, as is the case for the line bundles relevant to string theory, X is pseudoconcave. We endow X with the contact form $\alpha = -i\partial\rho|_X = i\bar{\partial}\rho|_X$ and the associated volume form

$$dV_X = \frac{1}{m!}\alpha \wedge (d\alpha)^m = \alpha \wedge \pi^* dV_M \,. \tag{29}$$

We define the Hardy space $\mathcal{H}^2(X) \subset \mathcal{L}^2(X)$ of square-integrable CR functions on X, i.e., functions that are annihilated by the Cauchy-Riemann operator $\bar{\partial}_b$ and are $\mathcal{L}^2$ with respect to the inner product

$$\langle F_1, F_2\rangle = \frac{1}{2\pi}\int_X F_1\overline{F_2}dV_X\,, \quad F_1, F_2 \in \mathcal{L}^2(X)\,. \tag{30}$$

We let $r_\theta x = e^{i\theta}x$ ($x \in X$) denote the S^1 action on X and denote its infinitesimal generator by $\frac{\partial}{\partial\theta}$. The S^1 action on X commutes with $\bar{\partial}_b$; hence $\mathcal{H}^2(X) = \bigoplus_{N=0}^\infty \mathcal{H}^2_N(X)$ where $\mathcal{H}^2_N(X) = \{F \in \mathcal{H}^2(X) : F(r_\theta x) = e^{iN\theta}F(x)\}$. A section s_N of L^N determines an equivariant function $\hat{s}_N$ on L^* by the rule

$$\hat{s}_N(\lambda) = \left(\lambda^{\otimes N}, s_N(z)\right)\,, \quad \lambda \in L^*_z\,,\ z \in M\,,$$

where $\lambda^{\otimes N} = \lambda \otimes \cdots \otimes \lambda$. We henceforth restrict $\hat{s}$ to X and then the equivariance property takes the form $\hat{s}_N(r_\theta x) = e^{iN\theta}\hat{s}_N(x)$. The map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^N)$ and $\mathcal{H}^2_N(X)$.

We let e_L be a nonvanishing local section, or local frame, of L. As above, we write

$$\|e_L(z)\|^2_h = e^{-K(z,\bar{z})}\,. \tag{31}$$

Thus, a positive line bundle L induces the Kähler form $\omega = \frac{i}{2}\partial\bar{\partial}K$ with Kähler potential K.

The Szegö kernel $\Pi_N(x, y)$ is the kernel of the orthogonal projection $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}^2_N(X)$; it is defined by

$$\Pi_N F(x) = \int_X \Pi_N(x, y)F(y)dV_X(y)\,, \quad F \in \mathcal{L}^2(X)\,. \tag{32}$$

Let $\{s^N_j = f_j e_L^{\otimes N} : j = 1, \dots, d_N\}$ be an orthonormal basis for $H^0(M, L^N)$. Then $\{\hat{s}^N_j\}$ is an orthonormal basis of $\mathcal{H}^2(X)$, and the Szegö kernel can be written in the form

$$\Pi_N(x, y) = \sum_{j=1}^{d_N} \hat{s}^N_j(x)\overline{\hat{s}^N_j(y)}\,. \tag{33}$$

It is the lift of the section

$$\widetilde{\Pi}_N(z,\bar{w}) := F_N(z,\bar{w})\, e_L^{\otimes N}(z) \otimes \overline{e_L^{\otimes N}(w)}\,, \tag{34}$$

where

$$F_N(z,\bar{w}) = \sum_{j=1}^{d_N} f_j(z)\overline{f_j(w)}\ . \tag{35}$$

We let (z,θ) denote the coordinates of the point $x = e^{i\theta}\|e_L(z)\|_h e_L^*(z) \in X$. The equivariant lift of a section $s = f e_L^{\otimes N} \in H^0(M, L^N)$ is given explicitly by

$$\hat{s}(z,\theta) = e^{iN\theta}\|e_L^{\otimes N}\|_h f(z) = e^{N\left[-\frac{1}{2}K(z,\bar{z})+i\theta\right]} f(z)\ . \tag{36}$$

The Szegö kernel is then given by

$$\Pi_N(z,\theta;w,\varphi) = e^{N\left[-\frac{1}{2}K(z,\bar{z})-\frac{1}{2}K(w,\bar{w})+i(\theta-\varphi)\right]} F_N(z,\bar{w})\ . \tag{37}$$

2.2.1. *The connection.* We denote by $H = \ker\alpha$ and obtain a splitting $T_X = H \oplus \mathbb{C}\frac{\partial}{\partial\theta}$ into horizontal and vertical spaces. The Chern connection ∇ on L^N then lifts to X as the horizontal derivative d^H, i.e.

$$(\nabla s_N)\widehat{\ } = d^H \hat{s}_N\ . \tag{38}$$

To describe the connection explicitly, we choose local holomorphic coordinates $\{z_1,\dots,z_m\}$ in M, and we write

$$\nabla = \nabla' + \nabla'', \qquad \nabla' s_N = \sum dz_j \otimes \nabla_{z_j} s_N, \qquad \nabla'' s_N = \sum d\bar{z}_j \otimes \nabla_{\bar{z}_j} s_N\ .$$

In particular, $(\nabla'' s_N)\widehat{\ } = \bar\partial_b \hat{s}_N$, which vanishes when the section s_N is holomorphic, or equivalently, when $\hat{s}_N \in \mathcal{H}^2_N(X)$.

For a section $s_N = f e_L^{\otimes N}$ of L^N, we have

$$\nabla_{z_j} s_N = \left(\frac{\partial f}{\partial z_j} - N f \frac{\partial K}{\partial z_j}\right) e_L^{\otimes N} = e^{NK}\frac{\partial}{\partial z_j}\left(e^{-NK} f\right) e_L^{\otimes N}\ , \qquad \nabla_{\bar{z}_j} s_N = \frac{\partial f}{\partial \bar{z}_j}\, e_L^{\otimes N}\ . \tag{39}$$

We also write

$$d^H \Pi_N(z,\theta;w,\varphi) = \sum dz_j \otimes \nabla_{z_j}\Pi_N + \sum d\bar{w}_j \otimes \nabla_{\bar{w}_j}\Pi_N\ , \tag{40}$$

where d^H is the horizontal derivative on $X \times X$. (We used the fact that the horizontal derivatives of Π_N with respect to the $\bar{z}_j$ and w_j variables vanish.) By (37)–(39), we have

$$\nabla_{z_j}\Pi_N \;=\; e^{N\left[-\frac{1}{2}K(z,\bar{z})-\frac{1}{2}K(w,\bar{w})+i(\theta-\varphi)\right]}\left(\frac{\partial}{\partial z_j} - N\frac{\partial K}{\partial z_j}(z,\bar{z})\right) F_N(z,\bar{w})\ , \tag{41}$$

$$\nabla_{\bar{w}_j}\Pi_N \;=\; e^{N\left[-\frac{1}{2}K(z,\bar{z})-\frac{1}{2}K(w,\bar{w})+i(\theta-\varphi)\right]}\left(\frac{\partial}{\partial \bar{w}_j} - N\frac{\partial K}{\partial \bar{w}_j}(w,\bar{w})\right) F_N(z,\bar{w})\ . \tag{42}$$

3. Alternate formulas for the density of critical points

The integrals in Theorems 2.1–2.3 are difficult to evaluate because of the absolute value sign, which prevents a direct application of Wick methods. To compute the densities, we shall replace our integral by another one which can be evaluated by residue calculus in certain cases. This new integral is given by the following lemma:

Lemma 3.1. *Let* Λ *be a positive definite Hermitian operator on* $\mathrm{Sym}(m,\mathbb{C}) \times \mathbb{C}$*. Then*

$$\frac{1}{\pi^{d_m}\,\det\Lambda}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \left|\det(HH^* - |x|^2 I)\right| e^{-\langle\Lambda^{-1}(H,x),(H,x)\rangle}\, dH\, dx$$
$$= \frac{(-i)^{m(m-1)/2}}{(2\pi)^m\prod_{j=1}^m j!}\lim_{\varepsilon'\to 0^+}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathrm{U}(m)} \frac{\Delta(\xi)\,\Delta(\lambda)\,|\prod_j\lambda_j|\, e^{i\langle\xi,\lambda\rangle}e^{-\varepsilon|\xi|^2-\varepsilon'|\lambda|^2}}{\det\left[i\widehat{D}(\xi)\rho(g)\Lambda\rho(g)^*+I\right]}\, dg\, d\xi\, d\lambda,$$

where

- $\Delta(\lambda) = \Pi_{i<j}(\lambda_i - \lambda_j)$,
- dg *is unit mass Haar measure on* $\mathrm{U}(m)$,
- $\widehat{D}(\xi)$ *is the Hermitian operator on* $\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}$ *given by*

$$\widehat{D}(\xi)\big((H_{jk}),x\big) = \left(\left(\frac{\xi_j+\xi_k}{2}\,H_{jk}\right),\ -\left(\textstyle\sum_{q=1}^m \xi_q\right)x\right)\ ,$$

- ρ *is the representation of* $\mathrm{U}(m)$ *on* $\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}$ *given by*

$$\rho(g)(H,x) = (gHg^t, x)\ .$$

The integrand is analytic in ξ, g but rather complicated. Its principal features are:

- $\Delta(\xi), \Delta(\lambda)$ are homogeneous polynomials of degree $m(m-1)/2$, and $|\prod_j \lambda_j|$ is homogeneous of degree m;
- $P_{g,z}(\xi) = \det\left[i\widehat{D}(\xi)\rho(g)\Lambda(z)\rho(g)^* + I\right]$ is a (family of) polynomial(s) in ξ of degree $m(m+1)/2+1$ with no real zeros $\xi\in\mathbb{R}^m$. But the polynomials are not elliptic (or even hypo-elliptic), that is, do not satisfy $|P(\xi)| \geq C|\xi|^{m(m+1)/2+1}$ (or any other power $|\xi|^\mu$). Indeed, for large $|\xi|$ we may drop the second term I and find that the growth at infinity is that of $\det\left[i\widehat{D}(\xi)\right]$. Since $\widehat{D}(\xi)$ is a diagonal matrix as described in Theorem 3.2, its determinant is a product of linear polynomials in ξ, and hence vanishes along a union of real hyperplanes.
- The ratio $p_g(\xi) = \frac{\Delta(\xi)}{\det\left[i\widehat{D}(\xi)\rho(g)\Lambda(z)\rho(g)^*+I\right]}$ is thus a rational function in ξ which is a 'symbol' of order $-m-1$, i.e. each ξ-derivative decays to one extra order. Repeated partial integrations in $d\xi$ using $\frac{1}{1+|\lambda|^2}[I-\Delta_\xi]e^{i\langle\lambda,\xi\rangle} = e^{i\langle\lambda,\xi\rangle}$ simultaneously lowers the order in both ξ and λ by two and renders the $d\lambda$ integral absolutely convergent without the Gaussian factor.

The proof of Lemma 3.1 is given in §3.1 below.

As a consequence, we have the following alternative formula for the expected critical point density:

THEOREM **3.2.** *Under the hypotheses of Theorem 2.1 and notation of Lemma 3.1, the density of the expected distribution of critical points of sections of $H^0(M, L^N)$ is also given by:*

$$\mathcal{K}^{\mathrm{crit}}_{N,h}(z) = \frac{c_m}{\det A_N} \lim_{\varepsilon'\to 0^+} \int_{\mathbb{R}^m} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^m} \int_{\mathrm{U}(m)} \frac{\Delta(\xi)\,\Delta(\lambda)\,|\prod_j \lambda_j|\, e^{i\langle\xi,\lambda\rangle} e^{-\varepsilon|\xi|^2-\varepsilon'|\lambda|^2}}{\det\left[i\widehat{D}(\xi)\rho(g)\Lambda_N(z)\rho(g)^* + I\right]}\, dg\, d\xi\, d\lambda,$$

where

$$c_m = \frac{(-i)^{m(m-1)/2}}{2^m\,\pi^{2m}\,\prod_{j=1}^m j!}\ .$$

Proof. Corollary 2.2 and Lemma 3.1. □

In §4 we shall use Theorem 3.2 to calculate the density of critical points for random sections $s_N \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ of the N-th power of the hyperplane bundle. In this case the $\mathrm{U}(m)$ integral drops out, and the integral can be evaluated as an iterated integral without the Gaussian factor $e^{-\varepsilon|\xi|^2-\varepsilon'|\lambda|^2}$.

We also have an alternative formula for the Morse index densities, which follows by a similar argument (given in §3.2):

THEOREM **3.3.** *Under the above assumptions, the density of the expected distribution of critical points of Morse index q of $\log\|s_N\|_h$ is also given by:*

$$\mathcal{K}^{\mathrm{crit}}_{N,q,h}(z) = \frac{m!\,c_m}{\det A_N} \lim_{\varepsilon'\to 0^+} \int_{Y_{2m-q}} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^m} \int_{\mathrm{U}(m)} \frac{\Delta(\xi)\,\Delta(\lambda)\,|\prod_j \lambda_j|\, e^{i\langle\xi,\lambda\rangle} e^{-\varepsilon|\xi|^2-\varepsilon'|\lambda|^2}}{\det\left[i\widehat{D}(\xi)\rho(g)\Lambda_N(z)\rho(g)^* + I\right]}\, dg\, d\xi\, d\lambda,$$

where

$$Y_p = \{\lambda \in \mathbb{R}^m : \lambda_1 > \cdots > \lambda_p > 0 > \lambda_{p+1} > \cdots > \lambda_m\}\ .$$

3.1. **Proof of Lemma 3.1.** We write

$$\mathcal{I}(z_0) = \frac{1}{\pi^{d_m}\det\Lambda(z_0)} \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \left|\det(HH^* - |x|^2 I)\right| \exp\left(-\langle\Lambda(z_0)^{-1}(H,x),(H,x)\rangle\right) dH dx\ . \tag{43}$$

Here, H (previously denoted by H') is a complex $m\times m$ symmetric matrix, so $H^* = \overline{H}$. The proof is basically to rewrite (43) using the Itzykson-Zuber integral and Gaussian integration.

We first observe that

$$\mathcal{I}(z_0) = \lim_{\varepsilon'\to 0} \lim_{\varepsilon\to 0} \mathcal{I}_{\varepsilon,\varepsilon'}(z_0)\ ,$$

where $\mathcal{I}_{\varepsilon,\varepsilon'}(z_0)$ is the absolutely convergent integral,

$$\begin{aligned}\mathcal{I}_{\varepsilon,\varepsilon'}(z_0) \;&=\; \frac{1}{(2\pi)^{m^2}\pi^{d_m}\det\Lambda} \int_{\mathcal{H}_m}\int_{\mathcal{H}_m}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} |\det P|\, e^{-\varepsilon Tr\Xi^*\Xi - \varepsilon' Tr P^*P} e^{i\langle\Xi, P-HH^*+|x|^2 I\rangle} \\ &\quad\times\ \exp\left(-\langle\Lambda^{-1}(H,x),(H,x)\rangle\right)\, dH\, dx\, dP\, d\Xi.\end{aligned} \tag{44}$$

Absolute convergence is guaranteed by the Gaussian factors in each variable (H, x, Ξ, P). If the $d\Xi$ integral is done first, we obtain a dual Gaussian which converges (in the sense of tempered distributions) to the delta function $\delta_{HH^*-\frac{1}{2}|x|^2}(P)$ as $\varepsilon\to 0$. Then, as $\varepsilon'\to 0$, the dP integral then evaluates the integrand at $P = HH^* - |x|^2 I$ and we retrieve the original integral $\mathcal{I}(z_0)$.

We next conjugate P in (44) to a diagonal matrix $D(\lambda)$ with $\lambda = (\lambda_1, \dots, \lambda_m)$ by an element $h \in \mathrm{U}(m)$. Recalling that

$$\int_{\mathcal{H}_m} \varphi(P)\, dP = c'_m \int_{\mathbb{R}^m} \int_{\mathrm{U}(m)} \varphi(hD(\lambda)h^*) \Delta(\lambda)^2 \, dh \, d\lambda \,, \qquad c'_m = \frac{(2\pi)^{\binom{m}{2}}}{\prod_{j=1}^m j!} \tag{45}$$

(see for example [ZZ, (1.9)]), we then obtain

$$\begin{aligned} \mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{c'_m}{(2\pi)^{m^2}\, \pi^{d_m} \det \Lambda} \int_{\mathrm{U}(m)} \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \int_{\mathcal{H}_m} \int_{\mathbb{R}^m} |\det(D(\lambda))| \\ &\quad \times\, e^{-\varepsilon(TrD(\lambda)^*D(\lambda) + Tr\Xi^*\Xi)} e^{i\langle \Xi, hD(\lambda)h^* + |x|^2 I - H^*H\rangle} \Delta(\lambda)^2 \\ &\quad \times\, \exp\left(-\langle \Lambda^{-1}(H,x),(H,x)\rangle\right) \, d\lambda \, d\Xi \, dH \, dx \, dh \end{aligned}$$

Since the factor $\int_{\mathrm{U}(m)} e^{i\langle \Xi, hD(\lambda)h^*\rangle} dh$ is invariant under the conjugation $\Xi \to g^*\Xi g$ with $g \in \mathrm{U}(m)$, we apply the same identity (45) in the Ξ variable. We write $\Xi = g^{-1}D(\xi)g$ where $D(\xi)$ is diagonal. This replaces $d\Xi$ by $\Delta(\xi)^2 d\xi$. The inner product is bi-invariant so we may transfer the conjugation to HH^*. We thus obtain:

$$\begin{aligned} \mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{(c'_m)^2}{(2\pi)^{m^2}\, \pi^{d_m} \det \Lambda} \int_{\mathrm{U}(m)} \int_{\mathrm{U}(m)} \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |\det(D(\lambda))| \, e^{-\varepsilon(|\xi|^2 + |\lambda|^2)} \\ &\quad \times\, \exp\left[i\langle D(\xi), hD(\lambda)h^* + |x|^2 I - gHH^*g^*\rangle - \langle \Lambda^{-1}(H,x),(H,x)\rangle \right] \\ &\quad \times\, \Delta(\lambda)^2 \Delta(\xi)^2 \, d\xi \, d\lambda \, dH \, dx \, dg \, dh \,. \end{aligned} \tag{46}$$

Next we recognize the integral $\int_{\mathrm{U}(m)} e^{i\langle D(\xi), hD(\lambda)h^*\rangle} dh$ as the well-known Itzykson-Zuber-Harish-Chandra integral [Ha] (cf., [ZZ]):

$$J(D(\lambda), D(\xi)) = (-i)^{m(m-1)/2} \left(\textstyle\prod_{j=1}^{m-1} j! \right) \frac{\det[e^{i\lambda_j \xi_k}]_{j,k}}{\Delta(\lambda)\Delta(\xi)} \,. \tag{47}$$

We note that both numerator and denominator are anti-symmetric in ξ_j and λ_j under permutation, so that the ratio is well-defined.

We substitute (47) into (46) and expand

$$\det[e^{i\xi_j\lambda_k}]_{jk} = \sum_{\sigma \in S_m} (-1)^\sigma \; e^{i\langle \xi, \sigma(\lambda)\rangle},$$

obtaining a sum of $m!$ integrals. However, by making the change of variables $\lambda' = \sigma(\lambda)$ and noting that $\Delta(\sigma(\lambda)) = (-1)^\sigma \Delta(\lambda)$, we see that these integrals are equal, and (46) then becomes

$$\begin{aligned} \mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{(-i)^{m(m-1)/2}}{(2\pi)^m (\prod_{j=1}^m j!) \pi^{d_m} \det \Lambda} \int_{\mathrm{U}(m)} \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \Delta(\lambda)\Delta(\xi) \; |\det(D(\lambda))| \\ &\times \;\; e^{i\langle \lambda, \xi\rangle} e^{-\varepsilon(|\xi|^2 + |\lambda|^2)} e^{i\langle D(\xi), |x|^2 I - gHH^*g^*\rangle - \langle \Lambda^{-1}(H,x),(H,x)\rangle} \, d\xi \, d\lambda \, dH \, dx \, dg \,. \end{aligned} \tag{48}$$

Further we observe that the $dHdx$ integral is a Gaussian integral. We simplify the phase by noting that

$$\langle D(\xi), gHH^*g^* - |x|^2 I\rangle = Tr(D(\xi) gHg^t \bar{g} H^* g^*) - TrD(\xi)\, |x|^2 = \left\langle \widehat{D}(\xi)\rho(g)(H,x), \rho(g)(H,x) \right\rangle$$

where $\widehat{D}(\xi)$ and $\rho(g)$ are as in the statement of the theorem. Thus,

$$\begin{aligned}\frac{1}{\pi^{d_m}\det\Lambda}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} &\exp\left[i\langle D(\xi),|x|^2I-gHH^*g^*\rangle-\langle\Lambda^{-1}(H,x),(H,x)\rangle\right]dH\,dx\\ &= \frac{1}{\det\Lambda\,\det\left[i\rho(g)^*\widehat{D}(\xi)\rho(g)+\Lambda^{-1}\right]}\\ &= \frac{1}{\det\left[i\rho(g)^*\widehat{D}(\xi)\rho(g)\Lambda+I\right]}\\ &= \frac{1}{\det\left[i\widehat{D}(\xi)\rho(g)\Lambda\rho(g)^*+I\right]}\,. \qquad (49)\end{aligned}$$

Substituting (49) into (48), we obtain the desirred formula. ☐

3.2. **Proof of Theorem 3.3.** By the proof of Lemma 3.1, we also see that

$$\begin{aligned}&\frac{1}{\det\Lambda}\int_{\mathbf{S}_{m,k}}\left|\det(HH^*-|x|^2I)\right|e^{-\langle\Lambda^{-1}(H,x),(H,x)\rangle}\,dH\,dx\\ &=\frac{(-i)^{m(m-1)/2}}{(2\pi)^m\prod_{j=1}^m j!}\lim_{\varepsilon'\to0^+}\int_{Y'_{m-k}}\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^m}\int_{\mathrm{U}(m)}\frac{\Delta(\xi)\,\Delta(\lambda)\,|\prod_j\lambda_j|\,e^{i\langle\xi,\lambda\rangle}e^{-\varepsilon|\xi|^2-\varepsilon'|\lambda|^2}}{\det\left[i\widehat{D}(\xi)\rho(g)\Lambda\rho(g)^*+I\right]}\,dg\,d\xi\,d\lambda,\end{aligned}$$

where Y'_p denotes the set of points in $\mathbb{R}^m$ with exactly p coordinates positive. Since the integrand on the right is invariant under identical simultaneous permutations of the ξ_j and the λ_j, it follows that the integral equals $m!$ times the corresponding integral over over Y_{m-k}. The desired formula then follows from Theorem 2.3. ☐

4. Exact formula for $\mathbb{CP}^m$

To illustrate our results for fixed N, we compute the density $\mathcal{K}^{\mathrm{crit}}_{N,q}(z)$ of the expected distribution of critical points of Morse index q of $\log\|s_N\|_{h^N}$ for random sections $s_N\in H^0(\mathbb{CP}^m,\mathcal{O}(N))$, where h^N is the Fubini-Study metric on $\mathcal{O}(N)$. Here, the probability measure on $H^0(\mathbb{CP}^m,\mathcal{O}(N))$ is the Gaussian measure induced from h^N and the volume form $V=\frac{1}{m!}\omega_{\mathrm{FS}}^m$ on $\mathbb{CP}^m$. Since this Hermitian metric and Gaussian measure are invariant under the $\mathrm{SU}(m+1)$ action on $\mathbb{CP}^m$, the density is independent of the point $z\in\mathbb{CP}^m$, and hence the expected number of critical points of Morse index q is given by

$$\mathcal{N}^{\mathrm{crit}}_{N,q}(\mathbb{CP}^m)=\frac{\pi^m}{m!}\mathcal{K}^{\mathrm{crit}}_{N,q}(z)\;.$$

These numbers turn out to be rational functions of N, which we state explicitly in §4.2 for $m\le4$. The following lemma is the starting point for our computation.

Lemma **4.1.** *We have:*

$$\begin{aligned}\mathcal{K}^{\mathrm{crit}}_{N,q}(z) &= i^{m+1}\frac{m!\,|c_m|}{N^m}\lim_{\varepsilon'\to0^+}\int_{Y_{2m-q}}d\lambda\,\left|\prod_j\lambda_j\right|\,\Delta(\lambda)\,e^{-\varepsilon'|\lambda|^2}\\ &\quad\times\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^m}\frac{\Delta(\xi)\,e^{i\langle\lambda,\xi\rangle}e^{-\varepsilon|\xi|^2}\,d\xi}{(N^2\sum\xi_j+i)\prod_{1\le j\le k\le m}\{i-N(N-1)(\xi_j+\xi_k)\}}\,,\end{aligned}$$

where c_m and Y_{2m-q} are as in Theorems 3.2 and 3.3.

Proof. Since the critical point density $\mathcal{K}^{\rm crit}_{N,q}$is constant, it suffices to compute it at $z = 0 \in \mathbb{C}^m \subset \mathbb{CP}^m$, using the local frame e_L corresponding to the homogeneous (linear) polynomial z_0. We recall that the Szegö kernel is given by

$$\Pi_{H^0(\mathbb{CP}^m,\mathcal{O}(N))}(z,w) = \frac{(N+m)!}{\pi^m N!}(1+z\cdot\bar{w})^N e_L(z)\otimes\overline{e_L(w)}\ .$$

(See, for example, [BSZ1, §1.3].) Since the formula in Theorem 2.1 is invariant when the Szegö kernel is multiplied by a constant, we can replace the above by the *normalized Szegö kernel*

$$\widetilde{\Pi}_N(z,w) := (1+z\cdot\bar{w})^N \tag{50}$$

in our computation.

We notice that

$$K(z) = -\log\|e_L(z)\|_h^2 = \log(1+\|z\|^2)\ ,$$

$$K(0) = \frac{\partial K}{\partial z}(0) = \frac{\partial^2 K}{\partial^2 z}(0) = 0\ .$$

Hence when computing the (normalized) matrices $\widetilde{A}_N,\ \widetilde{B}_N,\ \widetilde{C}_N$, we can take the usual derivatives of $\widetilde{\Pi}_N$. Indeed, we have

$$\begin{aligned}
\frac{\partial\widetilde{\Pi}_N}{\partial z_j} &= N(1+z\cdot\bar{w})^{N-1}\bar{w}_j\ ,\\
\frac{\partial^2\widetilde{\Pi}_N}{\partial z_j\partial\bar{w}_{j'}} &= \delta_{jj'}N(1+z\cdot\bar{w})^{N-1} + N(N-1)(1+z\cdot\bar{w})^{N-2}z_{j'}\bar{w}_j\ ,\\
\frac{\partial^4\widetilde{\Pi}_N}{\partial z_j\partial z_q\partial\bar{w}_{j'}\partial\bar{w}_{q'}}(0,0) &= N(N-1)(\delta_{jj'}\delta_{qq'}+\delta_{j'q}\delta_{jq'})\ .
\end{aligned}$$

It follows that

$$\widetilde{A}_N = NI\ ,\quad \widetilde{B}_N = 0\ ,\quad \widetilde{\Lambda}_N = \widetilde{C}_N = \begin{pmatrix} 2N(N-1)\hat{I} & 0\\ 0 & N^2\end{pmatrix}\ , \tag{51}$$

where $\hat{I}$ is the identity matrix of rank $\binom{m+1}{2}$.

The stated formula now follows from Theorem 3.3 by observing that $\rho(g)\widetilde{\Lambda}_N\,\rho(g)^* = \widetilde{\Lambda}_N$, and

$$\det\left[i\widehat{D}(\xi)\widetilde{\Lambda}_N + I\right] = (-i)^{\frac{m^2+m+2}{2}}\left(N^2\sum\xi_j + i\right)\prod_{1\le j\le k\le m}\{i - N(N-1)(\xi_j+\xi_k)\}\ .$$

□

4.1. **Evaluating the inner integral by residues.** We first evaluate

$$\mathcal{I}_{N,\lambda} := \lim_{\varepsilon\to 0^+}\frac{1}{N^m}\int_{\mathbb{R}^m}\frac{\Delta(\xi)\,e^{i\langle\lambda,\xi\rangle}e^{-\varepsilon|\xi|^2}\,d\xi}{(N^2\sum\xi_j + i)\prod_{1\le j\le k\le m}\{i - N(N-1)(\xi_j+\xi_k)\}}.$$

To simplify the constant factors, we make the redefinitions $\xi_j = (t_j + i)/2N(N-1)$ and $\lambda_j \to 2N(N-1)\lambda_j$, after which

$$\mathcal{I}_{N,\lambda} = (-1)^{\frac{m(m+1)}{2}}\, 2^{\frac{(m+1)(m+2)}{2}}\, \frac{(N-1)^{m+1}}{N} e^{-\sum \lambda_j}\, \mathcal{I}(\lambda; c)\ ,$$

with

$$\mathcal{I}(\lambda; c) = \lim_{\varepsilon\to 0^+} \int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; c)\, e^{-\varepsilon \sum |t_j|^2}\, dt\ ,$$

where

$$\mathcal{I}(\lambda, t; c) = \frac{\Delta(t)\, e^{i\langle\lambda, t\rangle}}{(\sum t_j + ic) \prod_{1\le j\le k\le m}(t_j + t_k)}\, dt\ , \qquad c = m + 2 - 2/N\ . \tag{52}$$

We note that $\int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; c)\, dt$ is a tempered distribution (in λ). Furthermore, the map

$$(\varepsilon_1, \dots, \varepsilon_m) \mapsto \int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; c)\, e^{-\sum \varepsilon_j |t_j|^2}\, dt$$

is a continuous map from $[0, +\infty)^m$ to the tempered distributions. Hence

$$\mathcal{I}(\lambda; c) = \int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; c)\, dt = \lim_{\varepsilon_m\to 0^+} \cdots \lim_{\varepsilon_1\to 0^+} \int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; c)\, e^{-\sum \varepsilon_j |t_j|^2}\, dt\ . \tag{53}$$

We now use (53) evaluate $\mathcal{I}(\lambda; c)$ by iterated residues. We assume that $\lambda \in Y_{2m-q}$, and we let $p = 2m - q$, so that

$$\lambda_1 > \cdots > \lambda_p > 0 > \lambda_{p+1} > \cdots > \lambda_m\ .$$

We first suppose that $p > 0$, and we start by doing the integral over t_1. Since the t_1 integral is absolutely convergent when $\varepsilon_1 = 0$, we can set $\varepsilon_1 = 0$ and do the integral by residues. If $p > 0$ we close the contour in the upper half plane, and pick up poles at $t_1 = 0$, and at $t_1 = -t_j$ for $j \ne 1$. The pole at $t_1 = -ic - \sum_{j\ne i} t_j$ is below the contour.

The residue of $\mathcal{I}(\lambda, t; c)$ at the pole $t_1 = 0$ is

$$\frac{(-1)^{m-1}}{2} \mathcal{I}(\lambda_2, \dots, \lambda_m, t_2, \dots, t_m; c)\ . \tag{54}$$

The residue at the pole $t_1 = -t_2$ is

$$\frac{\pm e^{i[(\lambda_2-\lambda_1)t_2 + \lambda_3 t_3 + \cdots \lambda_m t_m]}\, 2t_2(t_2+t_3)\cdots(t_2+t_m)\Delta(t_2, \dots, t_m)}{(t_3 + \cdots + t_m + ci)\, 2t_2(-t_2+t_3)\cdots(-t_2+t_m) \prod_{2\le j\le k\le m}(t_j+t_k)}$$

$$= \frac{\pm e^{i(\lambda_2-\lambda_1)t_2}\, e^{-\varepsilon_2 |t_2|^2}}{2t_2} \mathcal{I}(\lambda_3, \dots, \lambda_m, t_3, \dots, t_m; c)\ .$$

When we then do the t_2 integral and let $\varepsilon_2 \to 0^+$, we get zero. Indeed,

$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}-i} \frac{e^{i(\lambda_2-\lambda_1)t_2}\, e^{-\varepsilon_2 |t_2|^2}}{2t_2}\, dt_2 = 0\ ,$$

since $\lambda_2 - \lambda_1 < 0$ and the pole at $t_2 = 0$ is above the contour. Similarly, when we compute the residue of the pole $t_1 = -t_j$, $j > 2$, and then perform the t_j integration, we also get zero. Hence we can ignore the residues of the poles $t_1 = -t_j$.

Applying (54) recursively, the integral with $p > 0$ can be reduced to the case with all λ's negative:

$$\mathcal{I}(\lambda; c) = (-1)^{(m-1)+(m-2)+\cdots+(m-p)}(\pi i)^p\,\mathcal{I}(\lambda_{p+1}, \ldots, \lambda_m; c)\ . \tag{55}$$

We now treat the case $p = 0$ (i.e., $0 > \lambda_1 > \cdots > \lambda_m$). This time, we do the t_m contour integral first. We close it in the lower half plane, picking up the residue at $t_m = -ic - \sum_{1\le k<m} t_k$, which is

$$\frac{\Delta(t_1, \ldots, t_{m-1}) \prod_{k<m}(ic + \sum_{l<m} t_l + t_k) e^{c\lambda_m + i\sum_j(\lambda_j - \lambda_m)t_j}}{2(-ic - \sum_{l<m} t_l) \prod_{1\le j\le k\le m-1}(t_j + t_k) \prod_{k<m}(-ic - \sum_{l<m, l\neq k} t_l)}\ . \tag{56}$$

(To simplify the discussion, we set $\varepsilon = 0$, and regard the intgrals as distributions, as above.) Next we perform the t_1 integration. Since λ_m is the most negative eigenvalue, we close the contour in the upper half plane. The terms in the denominator with ic all give poles in the lower half plane, so can be ignored. And, the poles $t_1 = -t_j$ will be ignorable, by the same type of reasoning we saw earlier. Indeed, after computing the residue at $t_1 = -t_j$ we find that t_j appears in the exponent as $e^{i(\lambda_j - \lambda_1)t_j}$ with $\lambda_j - \lambda_1 < 0$ and the only factor of the denominator with a zero below the contour is $ic + t_2 + \cdots + t_m$; but this factor also appears in the numerator and hence the t_j integral gives zero.

This leaves the residue at all $t_j = 0$ with $1 \le j \le m - 1$. The residue at $t_1 = 0$ of

$$\mathcal{R}(\lambda_1, \ldots, \lambda_{m-1}, t_1, \ldots, t_{m-1}; c) := (56)$$

is

$$\frac{(-1)^{m-1}}{2}\mathcal{R}(\lambda_2, \ldots, \lambda_{m-1}, t_2, \ldots, t_{m-1}; c)\ .$$

Continuing recursively, for the case $p = 0$, we obtain (remembering that the t_m pole below the contour contributes negatively):

$$\mathcal{I}(\lambda; c) = (-1)^{m(m-1)/2}\,(\pi i)^m\,\left(\frac{-i}{c}\right)\,e^{c\lambda_m}\ . \tag{57}$$

Combining (57) (with m replaced by $m - p$) and (55), we find

$$\mathcal{I}(\lambda; c) = \begin{cases} i^{m^2-1}\,\dfrac{\pi^m}{c}\,e^{c\lambda_m} & \text{for}\ \ p < m\ , \\ i^{m^2-1}\,\dfrac{\pi^m}{c} & \text{for}\ \ p = m\ . \end{cases} \tag{58}$$

(Note that the sign $i^{m^2-1} = -i$ or 1 if m is even or odd, respectively.)

4.2. **Exact formulas for dimensions** ≤ 4. The resulting λ integrals were computed using Maple 7.[1] For $m = 1$, we reproduce the result from [DSZ]:

$$\mathcal{N}^{\rm crit}_{N,1}(\mathbb{CP}^1) = \frac{4(N-1)^2}{3N-2}\ ,\quad \mathcal{N}^{\rm crit}_{N,2}(\mathbb{CP}^1) = \frac{N^2}{3N-2}\ ;\quad \text{hence}\ \ \mathcal{N}^{\rm crit}_{N}(\mathbb{CP}^1) = \frac{5N^2-8N+4}{3N-2}\ .$$

For $m = 2$, we obtain:

$$\mathcal{N}^{\rm crit}_{N,2}(\mathbb{CP}^2) = \tfrac{3\,(N-1)^3}{(2\,N-1)}\ ,\quad \mathcal{N}^{\rm crit}_{N,3}(\mathbb{CP}^2) = \tfrac{16\,(N-1)^3 N^2}{(3\,N-2)^3}\ ,\quad \mathcal{N}^{\rm crit}_{N,4}(\mathbb{CP}^2) = \tfrac{N^5(5\,N-4)}{(3\,N-2)^3(2\,N-1)}\ .$$

[1]The Maple programs are included in the source files of the arXiv.org posting.

Hence, the expected total number of critical points is:

$$\mathcal{N}^{\rm crit}_N(\mathbb{CP}^2) = \frac{59\,N^5 - 231\,N^4 + 375\,N^3 - 310\,N^2 + 132\,N - 24}{(3\,N-2)^3}\ .$$

To check the computation, we note that

$$\mathcal{N}^{\rm crit}_{N,2}(\mathbb{CP}^2) - \mathcal{N}^{\rm crit}_{N,3}(\mathbb{CP}^2) + \mathcal{N}^{\rm crit}_{N,4}(\mathbb{CP}^2) = N^2 - 3N + 3 = \chi(T^{*1,0}_{\mathbb{CP}^2} \otimes \mathcal{O}(N))\ .$$

Similarly, for $m = 3$, we obtain:

$$\mathcal{N}^{\rm crit}_{N,3}(\mathbb{CP}^3) = \tfrac{8\,(N-1)^4}{(5\,N-2)}\ ,\quad \mathcal{N}^{\rm crit}_{N,4}(\mathbb{CP}^3) = \tfrac{(N-1)^4 N^2 \left(63\,N^2 - 50\,N + 10\right)}{(2\,N-1)^4 (5\,N-2)}\ ,$$

$$\mathcal{N}^{\rm crit}_{N,5}(\mathbb{CP}^3) = \tfrac{256\,(N-1)^4 N^5}{(5\,N-2)(3\,N-2)^5}\ ,\quad \mathcal{N}^{\rm crit}_{N,6}(\mathbb{CP}^3) = \tfrac{N^9\left(451\,N^4 - 1248\,N^3 + 1280\,N^2 - 576\,N + 96\right)}{(2\,N-1)^4 (3\,N-2)^5 (5\,N-2)}\ .$$

The expected total number of critical points is:

$$\mathcal{N}^{\rm crit}_N(\mathbb{CP}^3) = \tfrac{637\,N^8 - 3978\,N^7 + 11022\,N^6 - 17608\,N^5 + 17736\,N^4 - 11552\,N^3 + 4768\,N^2 - 1152\,N + 128}{(3\,N-2)^5}\ .$$

To check the computation:

$$\sum_{q=3}^{6} \mathcal{N}^{\rm crit}_{N,q}(\mathbb{CP}^3) = N^3 - 4N^2 + 6N - 4 = \chi(T^{*1,0}_{\mathbb{CP}^3} \otimes \mathcal{O}(N))\ .$$

Finally, for $m = 4$, we obtain:

$$\mathcal{N}^{\rm crit}_{N,4}(\mathbb{CP}^4) = \tfrac{5\,(N-1)^5}{(3\,N-1)}\ ,\quad \mathcal{N}^{\rm crit}_{N,5}(\mathbb{CP}^4) = \tfrac{16\,(N-1)^5 N^2\left(183\,N^2 - 120\,N + 20\right)}{(5\,N-2)^5}\ ,$$

$$\mathcal{N}^{\rm crit}_{N,6}(\mathbb{CP}^4) = \tfrac{(N-1)^5 N^5\left(396227\,N^7 - 1078546\,N^6 + 1261212\,N^5 - 821326\,N^4 + 321695\,N^3 - 75780\,N^2 + 9940\,N - 560\right)}{(5\,N-2)^5 (2\,N-1)^7 (3\,N-1)}\ ,$$

$$\mathcal{N}^{\rm crit}_{N,7}(\mathbb{CP}^4) = \tfrac{4096\,(N-1)^5 N^9\left(109\,N^2 - 102\,N + 24\right)}{(5\,N-2)^5 (3\,N-2)^7}\ ,$$

$$\mathcal{N}^{\rm crit}_{N,8}(\mathbb{CP}^4) = \tfrac{\alpha}{(5\,N-2)^5 (3\,N-2)^7 (2\,N-1)^7 (3\,N-1)}\ ,$$

$$\alpha = N^{14}(14251551\,N^{10} - 86984891\,N^9 + 237134546\,N^8 - 380216704\,N^7 + 397067360\,N^6 - 282219280\,N^5 + 138269792\,N^4 - 46114432\,N^3 + 10020608\,N^2 - 1281280\,N + 73216)\ .$$

The expected total number of critical points is:

$$\mathcal{N}^{\rm crit}_N(\mathbb{CP}^4) = (6571\,N^{11} - 56373\,N^{10} + 221376\,N^9 - 524190\,N^8 + 831075\,N^7 - 926382\,N^6 + 741276\,N^5 - 426392\,N^4 + 173200\,N^3 - 47520\,N^2 + 8000\,N - 640)/(3\,N-2)^7\ .$$

Again, to check the computation:

$$\sum_{q=4}^{8} \mathcal{N}^{\rm crit}_{N,q}(\mathbb{CP}^4) = N^4 - 5N^3 + 10N^2 - 10N + 5 = \chi(T^{*1,0}_{\mathbb{CP}^4} \otimes \mathcal{O}(N))\ .$$

Remark: From these computations, we guess that

$$\mathcal{N}^{\rm crit}_{N,m}(\mathbb{CP}^m) = \frac{2(m+1)(N-1)^{m+1}}{(m+2)N - 2}\ .$$

5. Asymptotics of the expected number of critical points

In this section, we compute the asymptotics of the expected density and number of critical points of sections of powers L^N of a positive holomorphic line bundle. In particular, we prove Theorems 1.1, 1.4, and 1.2 as well as Corollary 1.5.

5.1. Proof of Theorem 1.1.

We begin with some further background on the Szegö kernel.

5.1.1. *Szegö kernel asymptotics.* We first use the asymptotic expansion of the Szegö kernel to show that $\mathcal{K}_N^{\rm crit}(z)$ has an expansion of the type given in Theorems 1.1–1.2. It is evident from Theorem 2.1 and Theorem 2.3, respectively, and from formulas (26)–(25) for A and Λ that the asymptotics of the critical-point densities $\mathcal{K}_N^{\rm crit}(z)$ and $\mathcal{K}_{N,q}^{\rm crit}(z)$, respectively, can be determined by canonical algebraic operations on the asymptotics of the following derivatives of the Szegö kernel ($1 \le j \le m\,, 1 \le j \le q \le m\,, 1 \le j' \le q' \le m\,,$)

- $\nabla_{z_j} \Pi_N(z,w)|_{z=w}$;
- $\nabla_{z_j} \nabla_{\bar w_{j'}} \Pi_N(z,w)|_{z=w}$;
- $\nabla_{z_q} \nabla_{z_j} \Pi_N(z,w)|_{z=w}$ and $\nabla_{\bar w_{q'}} \nabla_{\bar w_{j'}} \Pi_N(z,w)|_{z=w}$;
- $\nabla_{z_j} \nabla_{\bar w_{q'}} \nabla_{\bar w_{j'}} \Pi_N(z,w)|_{z=w}$;
- $\nabla_{z_q} \nabla_{z_j} \nabla_{\bar w_{q'}} \nabla_{\bar w_{j'}} \Pi_N(z,w)|_{z=w}$

(Here we write $\Pi(z,w) = \Pi(z,0;w,0)$.) We can obtain their asymptotics by differentiating the following expansion:

Theorem **5.1.** [Ze] *Let $(L,h) \to M$ be a positive Hermitian holomorphic line bundle over a compact complex manifold M of dimension m with Kähler form $\omega = \frac{i}{2}\Theta_h$. Then there is a complete asymptotic expansion:*

$$\Pi_N(z,z) \sim \frac{N^m}{\pi^m}\left[1 + a_1(z)N^{-1} + a_2(z)N^{-2} + \cdots\right]\,, \tag{59}$$

for certain smooth coefficients $a_j(z)$.

To apply (59) to the differentiated Szegö kernel, we use (41)–(42). By a change of frame in L, we can assume that K and its holomorphic derivatives up to any fixed order, as well as the anti-holomorphic derivatives, vanish at z_0. Writing $\partial_j = \partial/\partial z_j$, we then have:

$$\begin{aligned}
\nabla_{z_j}\Pi_N(z_0,z_0) &= \frac{\partial F_N}{\partial z_j}(z_0,\bar z_0) = \partial_j F_N(z,\bar z)|_{z_0} = \partial_j\left[e^{NK(z)}\Pi_N(z,z)\right]\Big|_{z_0}\,,\\
\nabla_{z_j}\nabla_{\bar w_{j'}}\Pi_N(z_0,z_0) &= \frac{\partial^2 F_N}{\partial z_j \partial \bar w_{j'}}(z_0,\bar z_0) = \partial_j \bar\partial_{j'} F_N(z,\bar z)|_{z_0} = \partial_j\bar\partial_{j'}\left[e^{NK(z)}\Pi_N(z,z)\right]\Big|_{z_0}\,,\\
&\vdots\\
\nabla_{z_j}\nabla_{z_q}\nabla_{\bar w_{j'}}\nabla_{\bar w_{q'}}\Pi_N(z_0,z_0) &= \partial_j\partial_q\bar\partial_{j'}\bar\partial_{q'}\left[e^{NK(z)}\Pi_N(z,z)\right]\Big|_{z_0}\,.
\end{aligned} \tag{60}$$

Here, we used the fact that $F_N(z,\bar w)$ is holomorphic in z and anti-holomorphic in w. (In these expressions, we have no $\nabla_{\bar z_k}$ or ∇_{w_j} derivatives of $\Pi_N(z,w)$.)

It follows by substituting (59) into (60) that the components of A and Λ have asymptotic expansions in powers of N, and hence by Theorem 2.1, resp. Theorem 2.3, that $\mathcal{K}_N^{\rm crit}(z)$, resp. $\mathcal{K}_{N,q}^{\rm crit}(z)$, does. Next we study the coefficients b_0, b_1, b_2 of the expansion of $\mathcal{K}_N^{\rm crit}(z)$.

5.1.2. *The first three terms of the expansion.* Integrating the density of critical points, we find that the expected total number of critical points has the expansion

$$N^{-m}\,\mathcal{N}^{\mathrm{crit}}_{N,h} = \frac{\pi^m}{m!}\,b_0\,c_1(L)^m + N^{-1}\int_M b_1 dV_h + N^{-2}\int_M b_2 dV_h + O(N^{-3})\,.$$

The leading order term is universal.

We will use Theorem 2.1 and the following result of Z. Lu [Lu] to calculate the coefficients in these expansions:

THEOREM **5.2.** [Lu] *With the notation as in Theorem 5.1, each coefficient $a_j(z)$ is a polynomial of the curvature and its covariant derivatives at x. In particular,*

$$\begin{cases} a_1 = \frac{1}{2}\rho \\ a_2 = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3\rho^2) \end{cases}$$

where R, Ric and ρ denotes the curvature tensor, the Ricci curvature and the scalar curvature of ω, respectively, and Δ denotes the Laplace operator of (M,ω).

We now calculate A_N and Λ_N to two orders. The key point is to calculate the mixed derivatives of Π_N on the diagonal. It is convenient to do the calculation in Kähler normal coordinates about a point z_0 in M.

It is well known that in terms of Kähler normal coordinates $\{z_j\}$, the Kähler potential K has the expansion:

$$K(z,\bar z) = \|z\|^2 - \frac{1}{4}\sum R_{j\bar k p\bar q}(z_0) z_j \bar z_{\bar k} z_p \bar z_{\bar q} + O(\|z\|^5)\,. \tag{61}$$

(In general, K contains a pluriharmonic term $f(z) + \overline{f(z)}$, but a change of frame for L eliminates that term up to fourth order.)

We further use the notation $K_j = \partial_j K,\ K_{\bar j} = \bar\partial_j K$. We first claim that

$$A = NI + a_1 I + N^{-1}\left\{a_2 I + \left(\partial_j \bar\partial_{j'} a_1\right)\right\} + \cdots\,. \tag{62}$$

Indeed, by (60),

$$\begin{aligned}
\partial_j\left[e^{NK(z)}\Pi_N(z,z)\right] &= e^{NK}\left[NK_j(1 + a_1N^{-1} + a_2N^{-2}) + \partial_j a_1 N^{-1} + \partial_j a_2 N^{-2} + \cdots\right] \\
\partial_j\bar\partial_{j'}\left[e^{NK(z)}\Pi_N(z,z)\right] &= e^{NK}\left[N^2 K_j K_{\bar j'}(1 + a_1N^{-1} + a_2N^{-2}) + K_{\bar j'}\partial_j a_1 + K_{\bar j'}\partial_j a_2 N^{-1}\right. \\
&\quad + NK_{j\bar j'}(1 + a_1N^{-1} + a_2N^{-2}) + K_j\bar\partial_{j'}a_1 + K_j\bar\partial_{j'}a_2N^{-1} \\
&\quad \left. + \partial_j\bar\partial_{j'}a_1N^{-1}\cdots\right].
\end{aligned}$$

Evaluating at z_0 using (61), we then obtain (62).

We now compute the expansion of Λ. Continuing the above computation,

$$\begin{aligned}
\partial_j\bar\partial_{j'}\bar\partial_{q'}\left[e^{NK(z)}\Pi_N(z,z)\right] &= e^{NK}\left[N^2(K_{j\bar q'}K_{\bar j'} + K_{j\bar j'}K_{\bar q'})(1 + a_1N^{-1} + a_2N^{-2})\right. \\
&\quad + K_{\bar j'}\bar\partial_{q'}a_1 + K_{\bar q'}\partial_j\bar\partial_{j'}a_1 + K_{j\bar j'}\bar\partial_{q'}a_1 + K_{j\bar q'}\bar\partial_{j'}a_1 \\
&\quad \left. + NK_{j\bar j'\bar q'}(1 + a_1N^{-1})\cdots\right] + \text{unimportant terms.}
\end{aligned}$$

(The 'unimportant terms' are those which vanish at z_0 and whose holomorphic derivatives also vanish at z_0.) We have

$$\begin{aligned} B(z_0) &= \Big[\big(\nabla_{z_j}\nabla_{\bar w_{j'}}\nabla_{\bar w_{q'}}\Pi_N(z_0,z_0)\big) \quad \big(N\nabla_{z_j}\Pi_N(z_0,z_0)\big)\Big] \\ &= \Big[\big(\delta_{jj'}\bar\partial_{q'}a_1+\delta_{jq'}\bar\partial_{j'}a_1\big) \quad \big(\delta_j a_1\big)\Big] \;+\; O(N^{-1})\,. \end{aligned} \tag{63}$$

Differentiating again and evaluating at z_0 using (61), we obtain

$$\begin{aligned} \partial_j\bar\partial_{j'}\bar\partial_q\bar\partial_{q'}\left[e^{NK(z)}\Pi_N(z,z)\right]\Big|_{z_0} &= \big[N^2(\delta_{jj'}\delta_{qq'}+\delta_{jq'}\delta_{qj'})(1+a_1N^{-1}+a_2N^{-2}) \\ &\quad +\delta_{jj'}\partial_q\bar\partial_{q'}a_1+\delta_{qq'}\partial_j\bar\partial_{j'}a_1+\delta_{jq'}\partial_q\bar\partial_{j'}a_1+\delta_{qj'}\partial_q\bar\partial_{j'}a_1 \\ &\quad +NK_{j\bar j'q\bar q'}(1+a_1N^{-1})\cdots\big]\Big|_{z_0}\,. \end{aligned}$$

Noting that $K_{j\bar j'q\bar q'}|_{z_0}=-R_{j\bar j'q\bar q'}(z_0)$, and recalling that $\Lambda=C-B^*A^{-1}B$, where

$$C=\begin{bmatrix} \big(\nabla_{z_q}\nabla_{z_j}\nabla_{\bar w_{q'}}\nabla_{\bar w_{j'}}\Pi_N\big) & \big(N\nabla_{z_q}\nabla_{z_j}\Pi_N\big) \\ \big(N\nabla_{\bar w_{q'}}\nabla_{\bar w_{j'}}\Pi_N\big) & N^2\Pi_N \end{bmatrix},$$

we obtain

$$\Lambda(z_0)=N^2\Lambda_0^{\frac12}\left(I+N^{-1}\Lambda_{-1}+N^{-2}\Lambda_{-2}+\cdots\right)\Lambda_0^{\frac12}\,,$$

with

$$\Lambda_0=\begin{pmatrix}2\hat I & 0\\ 0 & 1\end{pmatrix}\,, \tag{64}$$

$$\Lambda_{-1}=\begin{pmatrix}a_1\hat I-\frac12\big(R_{j\bar j'q\bar q'}\big) & 0\\ 0 & a_1\end{pmatrix}\,, \tag{65}$$

$$\Lambda_{-2}=\begin{pmatrix}a_2\hat I+P-\frac{a_1}{2}\big(R_{j\bar j'q\bar q'}\big) & \frac{1}{\sqrt2}\big(\partial_j\partial_q a_1\big)\\ \frac{1}{\sqrt2}\big(\bar\partial_j\bar\partial_q a_1\big) & a_2\end{pmatrix}\,, \tag{66}$$

where $\hat I$ is the identity operator on $\mathrm{Sym}(m,\mathbb C)$, and

$$P=\frac12\big(\delta_{jj'}\partial_q\bar\partial_{q'}a_1+\delta_{qq'}\partial_j\bar\partial_{j'}a_1+\delta_{jq'}\partial_q\bar\partial_{j'}a_1+\delta_{qj'}\partial_q\bar\partial_{j'}a_1\big)\,.$$

We want the asymptotics of

$$\mathcal K_N^{\rm crit}(z_0)=\frac{\pi^{-\binom{m+2}{2}}N^m}{\det A_N\det\Lambda_N}\int_{\mathrm{Sym}(m,\mathbb C)\times\mathbb C}\left|\det(H'H'^*-|x|^2I)\right|e^{-\langle\Lambda_N(z_0)^{-1}(H',x),(H',x)\rangle}\,dH'\,dx\,.$$

Making the change of variables $H'\mapsto\sqrt2\,NH',\ x\mapsto Nx$, the integral is transformed to

$$\mathcal K_N^{\rm crit}(z_0)=\frac{\pi^{-\binom{m+2}{2}}N^m}{\det\widetilde A\,\det\widetilde\Lambda}\int_{\mathrm{Sym}(m,\mathbb C)\times\mathbb C}\left|\det(2H'H'^*-|x|^2I)\right|e^{-\langle\widetilde\Lambda^{-1}(H',x),(H',x)\rangle}\,dH'\,dx\,, \tag{67}$$

where

$$\widetilde A=N^{-2}A_N(z_0),\quad \widetilde\Lambda=N^{-2}\Lambda_0^{-\frac12}\Lambda_N(z_0)\Lambda_0^{-\frac12}=\left(I+N^{-1}\Lambda_{-1}+N^{-2}\Lambda_{-2}+\cdots\right)\,. \tag{68}$$

Next we observe that

$$\widetilde\Lambda^{-1}=I-\frac1N\Lambda_{-1}+\frac1{N^2}[-\Lambda_{-2}+\Lambda_{-1}^2]+\cdots$$

hence

$$\begin{aligned} e^{-\langle\widetilde{\Lambda}^{-1}H,H\rangle} &\sim e^{-\langle H,H\rangle} e^{\left\langle\left[\frac{1}{N}\Lambda_{-1}+\frac{1}{N^2}(\Lambda_{-2}-\Lambda_{-1}^2)\right]H,H\right\rangle} \\ &= e^{-\langle H,H\rangle}\{1+\tfrac{1}{N}\langle\Lambda_{-1}H,H\rangle \\ &+ \tfrac{1}{N^2}[\langle\Lambda_{-2}H,H\rangle+\tfrac{1}{2}\langle\Lambda_{-1}H,H\rangle^2-\langle\Lambda_{-1}^2H,H\rangle]\}. \end{aligned}$$

Furthermore

$$\det\widetilde{\Lambda}^{-1} = 1-(\mathrm{Tr}\Lambda_{-1})N^{-1}+\left[\frac{1}{2}\mathrm{Tr}(\Lambda_{-1}^2)+\frac{1}{2}(\mathrm{Tr}\Lambda_{-1})^2-\mathrm{Tr}\Lambda_{-2}\right]N^{-2}\cdots,$$

and similarly for $\det A^{-1}$. Altogether, we obtain:

$$\begin{aligned} \mathcal{K}_N^{\mathrm{crit}}(z) &\sim \pi^{-\binom{m+2}{2}}N^m\left\{1+\tfrac{1}{N}\left[-\mathrm{Tr}A_{-1}-\mathrm{Tr}\Lambda_{-1}\right]\right. \\ &+ \left.\tfrac{1}{N^2}\left[\tfrac{1}{2}\mathrm{Tr}(\Lambda_{-1}^2)-\mathrm{Tr}\Lambda_{-2}+\tfrac{1}{2}\mathrm{Tr}(A_{-1}^2)-\mathrm{Tr}A_{-2}+\tfrac{1}{2}(\mathrm{Tr}A_{-1}+\mathrm{Tr}\Lambda_{-1})^2\right]\right\} \\ &\times \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}}|\det(2H'H'^*-|x|^2I)|\,e^{-\langle H,H\rangle}\{1+\tfrac{1}{N}\langle\Lambda_{-1}H,H\rangle \\ &+ \tfrac{1}{N^2}[\langle\Lambda_{-2}H,H\rangle+\tfrac{1}{2}\langle\Lambda_{-1}H,H\rangle^2-\langle\Lambda_{-1}^2H,H\rangle]\}\,dH'\,dx\,. \end{aligned}$$

Expanding, we obtain

$$\begin{aligned} \mathcal{K}_N^{\mathrm{crit}}(z) &\sim b_0N^m+b_1N^{m-1}+b_2N^{m-2}+\cdots, \\ b_0 &= \int d\mu\,, \\ b_1 &= \int\left[\langle\Lambda_{-1}H,H\rangle-\mathrm{Tr}A_{-1}-\mathrm{Tr}\Lambda_{-1}\right]d\mu\,, \\ b_2 &= \int\left[\tfrac{1}{2}\mathrm{Tr}(\Lambda_{-1}^2)-\mathrm{Tr}\Lambda_{-2}+\tfrac{1}{2}\mathrm{Tr}(A_{-1}^2)-\mathrm{Tr}A_{-2}+\tfrac{1}{2}(\mathrm{Tr}A_{-1}+\mathrm{Tr}\Lambda_{-1})^2\right. \\ &\quad \left.-(\mathrm{Tr}A_{-1}+\mathrm{Tr}\Lambda_{-1})\langle\Lambda_{-1}H,H\rangle+\langle(\Lambda_{-2}-\Lambda_{-1}^2)H,H\rangle+\tfrac{1}{2}\langle\Lambda_{-1}H,H\rangle^2\right]d\mu\,, \end{aligned} \tag{69}$$

where

$$d\mu = \pi^{-\binom{m+2}{2}}\left|\det(2H'H'^*-|x|^2I)\right|\,e^{-\langle H,H\rangle}\,dH'\,dx\,. \tag{70}$$

Recalling (62) and (64)–(66), we see that b_1 is of the form

$$b_1 = \sum c_{j\bar{j}'q\bar{q}'}R_{j\bar{j}'q\bar{q}'}\,,$$

where $c_{j\bar{j}'q\bar{q}'}$ is universal. Since b_1 is also invariant under the unitary group, we must have

$$b_1 = \beta_1\rho\,, \tag{71}$$

where β_1 is a universal constant (depending only on the dimension m of M). Similarly, b_2 is of the form

$$b_2 = Q(R,R)+\gamma_0\Delta\rho\,,$$

where $Q(R,R)$ is a universal quadratic form in the curvature tensor R. But b_2 is also U(m)-invariant and hence is a curvature invariant (of order 2). Thus,

$$b_2 = \gamma_0\Delta\rho+\gamma_1\rho^2+\gamma_2|R|^2+\gamma_3|Ric|^2\,, \tag{72}$$

where the γ_k are universal constants depending only on m. This, together with the exact formulas of §4.2, completes the proof of Theorem 1.1.

5.2. **Asymptotic expansions on Riemann surfaces: Proof of Theorem 1.4.** We wish to prove that on a positive line bundle (L, h) over a compact complex curve C of genus g,

$$\mathcal{N}^{\rm crit}_{N,h} = \frac{5}{3}\, c_1(L)\, N + \frac{7}{9}\,(2g-2) + \left(\frac{2}{27\pi}\int_C \rho^2\omega_h\right) N^{-1} + O(N^{-2})\,,$$

where $\omega_h = \frac{i}{2}\Theta_h$ and ρ is the Gaussian curvature of the metric ω_h.

On a Riemann surface, $\mathcal{K}^{\rm crit}_N$ has a universal expansion of the form

$$\mathcal{K}^{\rm crit}_N \sim b_0\, N + \beta_1\rho + (\beta_2\rho^2 + \gamma_0\Delta\rho)\, N^{-1} + \cdots\,.$$

There are several ways to compute the constants. A quick way to find b_0, β_1, β_2, is to consider the case of $\mathbb{CP}^1$ with the Fubini-Study metric on $L = \mathcal{O}(1)$. By an elementary computation in [DSZ] (or by §4), we showed that for this case

$$\mathcal{N}^{\rm crit}_{N,h} = \frac{5N^2 - 8N + 4}{3N-2} = \frac{5}{3}\, N - \frac{14}{9} + \frac{8}{27}\, N^{-1} \cdots\,. \tag{73}$$

Note that $\int_{\mathbb{CP}^1}\omega = \pi c_1(L) = \pi$, where ω is the Fubini-Study Kähler form on $\mathbb{CP}^1$. Therefore,

$$\mathcal{K}^{\rm crit}_N(z) \equiv \frac{1}{\pi}\mathcal{N}^{\rm crit}_{N,h}\,.$$

Furthermore, since $c_1(\mathbb{CP}^1) = \frac{1}{\pi}\int \rho\,\omega_{\rm FS} = 2$, we have $\rho \equiv 2$. (This can be checked directly as follows: the Kähler potential $K = \log(1+|z|^2) = |z|^2 - \frac{1}{2}|z|^4 + \cdots$, where z is the affine coordinate, and hence by (61), $\rho(0) = R_{1\bar{1}1\bar{1}}(0) = 2$.) Hence, for a Riemann surface, we have

$$b_0 = \frac{5}{3\pi}\,, \qquad \beta_1 = -\frac{7}{9\pi}\,, \qquad \beta_2 = \frac{2}{27\pi}\,. \tag{74}$$

5.3. **Number of critical points: Proof of Corollary 1.5.** In particular, the N^{-1} term in the expansion of $\mathcal{N}^{\rm crit}_{N,h}$ is a topological invariant, hence independent of the metric h. Furthermore, it is well known (see, e.g., [Ko, pp. 112–113]) that for any Kähler metric ω on M, we have

$$(\rho^2 - |Ric|^2)\Omega = c_1(M,\omega)^2\wedge\omega^{m-2}\,, \quad (|Ric|^2 - |R|^2)\Omega = [c_1(M,\omega)^2 - 2c_2(M,\omega)]\wedge\omega^{m-2}\,, \tag{75}$$

where $\Omega = \frac{1}{4\pi^2 m(m-1)}\,\omega^m$. Therefore

$$b_2 = \gamma_0\Delta\rho + (\gamma_1+\gamma_2+\gamma_3)\rho^2 + \text{const.}\ \frac{c_1(h)^2\wedge\omega_h^{m-2}}{\omega_h^m} + \text{const.}\ \frac{c_2(h)\wedge\omega_h^{m-2}}{\omega_h^m}\,, \tag{76}$$

where we now write $c_j(h) = c_1(M,\omega_h)$ for the j-th Chern form of the Kähler metric $\omega_h = -\frac{i}{2}\partial\bar\partial\log h$.

Integrating (71) and (76), we see that

$$\begin{aligned}\mathcal{N}^{\rm crit}_{N,h} \ &\sim\ \left[b_0\,\frac{\pi^m}{m!}\,c_1(L)^m\right] N^m + \left[\beta_1\,\frac{\pi^m}{(m-1)!}\,c_1(M)\cdot c_1(L)^{m-1}\right] N^{m-1} \\ &+\ \left[\beta_2\int_M \rho^2 d\mathrm{Vol}_h + \beta_2'\,c_1(M)^2\cdot c_1(L)^{m-2} + \beta_2''\,c_2(M)\cdot c_1(L)^{m-2}\right] N^{m-2} + \cdots\,,\end{aligned} \tag{77}$$

where $\beta_2 = \gamma_1+\gamma_2+\gamma_3$, β_2', β_2'' are constants depending only on m.

5.4. **Morse index density asymptotics: Proof of Theorem 1.2.** The computation of the expansion of $\mathcal{K}^{\rm crit}_{N,q}$ is exactly as above, except we integrate over $\mathbf{S}_{m,q}$ instead of $\mathrm{Sym}(\mathbb{C},m)\times\mathbb{C}$.

As a consequence, the expected number of critical points of Morse index q has an asymptotic expansion of the form:

$$\begin{aligned}\mathcal{N}^{\rm crit}_{N,q,h} \;\sim\; & \left[b_{0q}\,\frac{\pi^m}{m!}\,c_1(L)^m\right] N^m + \left[\beta_{1q}\,\frac{\pi^m}{(m-1)!}\,c_1(M)\cdot c_1(L)^{m-1}\right] N^{m-1}\\ & + \left[\beta_{2q}\int_M \rho^2 d\mathrm{Vol}_h + \beta'_{2q}\,c_1(M)^2\cdot c_1(L)^{m-2} + \beta''_{2q}\,c_2(M)\cdot c_1(L)^{m-2}\right] N^{m-2} + \cdots\,,\end{aligned} \tag{78}$$

where the coefficients depend only on m. □

6. Proof of Theorem 1.7: evaluating the coefficient $\beta_{2q}(m)$

We have already shown that

- $\int_M b_{1q} dV_h$ is topological;
- $\int_M b_{2q} dV_h$ is the sum of a topological term plus a positive multiple of $\int_M \rho_h^2 dV_h$.

To complete the proof of Theorem 1.7 and show that the metric with asymptotically minimal $\mathcal{N}^{\rm crit}_{N,h}$ is the one for which ω_h has minimal $\mathcal{L}^2$ norm of the scalar curvature, we must show that β_{2q} is positive.

The proof consists of a sequence of Lemmas giving ever simpler expressions for β_2. We first summarize the key results. The first is:

Lemma **6.1.** *In all dimensions,*

$$\beta_{2q}(m) = \frac{1}{4\,\pi^{\binom{m+2}{2}}}\int_{\mathbf{S}'_{m,q-m}} \gamma(H)\,\left|\det(2HH^*-|x|^2I)\right|\,e^{-\langle (H,x),(H,x)\rangle}\,dH\,dx\,, \tag{79}$$

where

$$\mathbf{S}'_{m,q-m} = \{(H,x)\in \mathrm{Sym}(m,\mathbb{C})\times\mathbb{C} : \mathrm{index}(2HH^*-|x|^2I) = q-m\}\,.$$

After a sequence of manipulations as in the proof of Lemma 3.1, the integral (79) will be rewritten in the following form:

Lemma **6.2.**

$$\beta_{2q}(m) = \frac{(-i)^{m(m-1)/2}}{4\,\pi^{2m}\prod_{j=1}^{m-1} j!}\int_{Y_{2m-q}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \Delta(\lambda)\,\Delta(\xi)\prod_{j=1}^m|\lambda_j|\;e^{i\langle\lambda,\xi\rangle}\,\mathcal{I}(\lambda,\xi)\;d\xi_1\cdots d\xi_m\,d\lambda\,,$$

where

$$\mathcal{I}(\lambda,\xi) = \frac{F(D(\lambda)) + \left[\frac{4\sum_{j=1}^m\lambda_j}{m(m+1)(m+3)} - \frac{2}{m+1}\right]\frac{1}{\left(1-\frac{i}{2}\sum_j\xi_j\right)} + \frac{2}{(m+1)(m+3)\left(1-\frac{i}{2}\sum_j\xi_j\right)^2}}{\left(1-\frac{i}{2}\sum_j\xi_j\right)\prod_{j\le k}\left[1+\frac{i}{2}(\xi_j+\xi_k)\right]}\,. \tag{80}$$

Here, $D(\lambda)$ is the diagonal matrix with diagonal entries $\lambda=(\lambda_1,\dots,\lambda_m)$, $\Delta(\lambda)=\Pi_{i<j}(\lambda_i-\lambda_j)$ is the Vandermonde determinant and

$$F(P) = 1 - \frac{4\,\mathrm{Tr}\,P}{m(m+1)} + \frac{4(\mathrm{Tr}\,P)^2 + 8\,\mathrm{Tr}(P^2)}{m(m+1)(m+2)(m+3)}\,,$$

for (Hermitian) $m \times m$ matrices P. The iterated $d\xi_j$ integrals are defined in the distribution sense.

The final step is the evaluation of $\beta_{2q}(m)$ and the proof that it is positive. Having simplified the integral as far as we could, we complete the compution for the cases $m \leq 3$ using Maple 7, and find that it is positive for these cases, thus completing the proof of Theorem 1.7.

The resulting values of the constants $\beta_{2q}(m)$, $m \leq 3$, are given in §6.4.

6.1. **Proof of Lemma 6.1.** We use the case of $M = \mathbb{CP}^1 \times E^{m-1}$ where E is an elliptic curve, and L is the product of degree 1 line bundles on the factors (with the Fubini-Study metric on $\mathcal{O}_{\mathbb{CP}^1}(1)$ and the flat metric on E). (The manifold M is a homogeneous space with respect to $SU(2) \times T^{2m-2}$, so the critical point density is invariant and hence constant.)

Since $c_1(h)^2 = c_2(h) = 0$, it follows from (76) that the coefficient b_{2q} of the expansion $N^{-m}\mathcal{K}^{\rm crit}_N(z) = b_{0q} + b_{1q}N^{-1} + b_{2q}N^{-2} + O(N^{-3})$ is given by $b_{2q} = \beta_{2q}\rho^2$, and hence

$$\beta_{2q} = \frac{1}{\rho^2}\, b_{2q} = \frac{1}{4}\, b_{2q}\ . \tag{81}$$

The Szegö kernel for (M, L) is the product of the Szegö kernels on $\mathbb{CP}^1$ and E^{m-1}. Since the universal cover of E^{m-1} is $\mathbb{C}^{m-1}$, the Szegö kernel on E^{m-1} is given by the Heisenberg Szegö kernel on $\mathbb{C}^{m-1}$ (see [BSZ1, §1.3.2]) modulo an $O(N^{-\infty})$ term, and we have:

$$\Pi_{CP^1\times E^{m-1}}(z,w) = \frac{(N+1)N^{m-1}}{\pi^m}(1+z_1\bar w_1)^N e^{N(z_2\bar w_2+\cdots+z_m\bar w_m)}\, e_L(z)\otimes\overline{e_L(w)} + O(N^{-\infty})\ .$$

As in §4, we consider the normalized Szegö kernel

$$\widetilde\Pi_N(z,w) := (1+z_1\bar w_1)^N e^{Nz'\bar w'}, \qquad z' = (z_2,\dots,z_m),\ w' = (w_2,\dots,w_m)\ . \tag{82}$$

We have:

$$\begin{aligned}
\frac{\partial\widetilde\Pi_N}{\partial z_1} &= N(1+z_1\bar w_1)^{N-1}e^{Nz'\bar w'}\bar w_1\ ,\\
\frac{\partial\widetilde\Pi_N}{\partial z_\alpha} &= N(1+z_1\bar w_1)^{N}e^{Nz'\bar w'}\bar w_\alpha\ ,\\
\frac{\partial^2\widetilde\Pi_N}{\partial z_1\partial\bar w_1} &= \{N(1+z_1\bar w_1)^{N-1} + N(N-1)(1+z_1\bar w_1)^{N-2}z_1\bar w_1\}e^{Nz'\bar w'}\ ,\\
\frac{\partial^2\widetilde\Pi_N}{\partial z_\alpha\partial\bar w_{\alpha'}} &= \{N\delta_{\alpha\alpha'} + N^2 z_{\alpha'}\bar w_\alpha\}(1+z_1\bar w_1)^N e^{Nz'\bar w'}\ ,\\
\frac{\partial^2\widetilde\Pi_N}{\partial z_1\partial\bar w_\alpha} &= N^2(1+z_1\bar w_1)^{N-1}e^{Nz_\alpha\bar w_\alpha}z_\alpha\bar w_1\ ,\\
\frac{\partial^2\widetilde\Pi_N}{\partial z_\alpha\partial\bar w_1} &= \{N^2(1+z_1\bar w_1)^{N-1}z_1\bar w_\alpha\}e^{Nz'\bar w'}\ ,\\
&\qquad\qquad\qquad\qquad\qquad\qquad 2\leq\alpha,\alpha'\leq m\ .
\end{aligned}$$

It suffices to compute the density at 0. From the above, we have:

$$\frac{\partial^4\widetilde\Pi_N}{\partial z_j\partial z_q\partial\bar w_{j'}\partial\bar w_{q'}}(0,0) = \begin{cases} 2N(N-1), & j=q=j'=q'=1\\ 2N^2, & j=q=j'=q'>1\\ N^2, & j=j'\neq q'=q\end{cases}\ . \tag{83}$$

Recalling (26)–(28), we then have:

$$\widetilde{A}_N(0) = \left(\frac{\partial^2\widetilde{\Pi}_N}{\partial z_j\partial\bar{w}_{j'}}(0,0)\right) = NI \tag{84}$$

$$\widetilde{B}_N(0) = \left[\left(\tau_{jq}\,\frac{\partial^3\widetilde{\Pi}_N}{\partial z_j\partial\bar{w}_{q'}\partial\bar{w}_{j'}}(0,0)\right) \quad \left(N\,\frac{\partial\widetilde{\Pi}_N}{\partial z_j}(0,0)\right)\right] = 0\,, \tag{85}$$

$$\widetilde{C}_N(0) = \left[\begin{array}{cc} \left(\tau_{jq}\tau_{j'q'}\,\frac{\partial^4\widetilde{\Pi}_N}{\partial z_q\partial z_j\partial\bar{w}_{q'}\partial\bar{w}_{j'}}(0,0)\right) & \left(\tau_{jq}N\,\frac{\partial^2\widetilde{\Pi}_N}{\partial z_j\partial z_q}(0,0)\right) \\ \left(\tau_{j'q'}N\,\frac{\partial^2\widetilde{\Pi}_N}{\partial\bar{w}_{q'}\partial\bar{w}_{j'}}(0,0)\right) & N^2\,\widetilde{\Pi}_N(0,0) \end{array}\right], \tag{86}$$

$$1\le j\le m\,, 1\le j\le q\le m\,, 1\le j'\le q'\le m\,.$$

It follows that

$$\widetilde{\Lambda}_N(0) = \widetilde{C}_N(0) = D(2N(N-1),\overbrace{2N^2,\dots,2N^2}^{(m-1)(m+2)/2},N^2)\ , \tag{87}$$

i.e. the diagonal matrix with diagonal entries $2N(N-1)$, $2N^2$ repeated $(m-1)(m+2)/2$ times, N^2.

We want to compute

$$\mathcal{K}^{\rm crit}_{N,q}(0) = \frac{\pi^{-\binom{m+2}{2}}}{\det\widetilde{A}_N(0)\det\widetilde{\Lambda}_N(0)}\int_{\mathbf{S}_{m,q-m}}\left|\det(HH^*-|x|^2I)\right| e^{-\langle\widetilde{\Lambda}_N(0)^{-1}(H,x),(H,x)\rangle}\,dH\,dx\,.$$

Making the change of variables $H'\mapsto\sqrt{2}\,NH',\ x\mapsto Nx$, the integral is transformed to

$$\mathcal{K}^{\rm crit}_{N,q}(0) = \frac{\pi^{-\binom{m+2}{2}}N^m}{\det\widehat{\Lambda}}\int_{\mathbf{S}'_{m,q-m}}\left|\det(2HH^*-|x|^2I)\right| e^{-\langle\widehat{\Lambda}^{-1}(H,x),(H,x)\rangle}\,dH\,dx\,, \tag{88}$$

where

$$\widehat{\Lambda} = I-\frac{1}{N}E\,,\qquad E = D(1,0,\dots,0)\,.$$

Therefore

$$\begin{aligned} N^{-m}\mathcal{K}^{\rm crit}_{N,q}(0) &= \pi^{-\binom{m+2}{2}}\left(1+\frac{1}{N}+\frac{1}{N^2}+\cdots\right)\int_{\mathbf{S}'_{m,q-m}}\left|\det(2HH^*-|x|^2I)\right| \\ &\quad\times\ \exp\left(-\|H\|^2-|x|^2-\frac{1}{N}|H_{11}|^2-\frac{1}{N^2}|H_{11}|^2-\cdots\right)dH\,dx \\ &= \left(1+\frac{1}{N}+\frac{1}{N^2}+\cdots\right)\int\exp\left(-\frac{1}{N}|H_{11}|^2-\frac{1}{N^2}|H_{11}|^2-\cdots\right)d\widetilde{\mu} \\ &= \int\left[1+\frac{1}{N}(1-|H_{11}|^2)+\frac{1}{N^2}\left(1-2|H_{11}|^2+\frac{1}{2}|H_{11}|^4\right)\right]d\widetilde{\mu}+O\left(\frac{1}{N^3}\right)\ , \end{aligned}$$

where

$$d\widetilde{\mu} = \pi^{-\binom{m+2}{2}}\left|\det(2HH^*-|x|^2I)\right|\,e^{-\langle(H,x),(H,x)\rangle}\,dH\,dx\ .$$

Therefore

$$b_{2q} = \int_{\mathbf{S}'_{m,q-m}} \left(1 - 2|H_{11}|^2 + \frac{1}{2}|H_{11}|^4\right) d\widetilde{\mu} \ ,$$

and the desired formula then follows from (81). □

6.2. $\mathrm{U}(m)$ **symmetries of the integral.** As an intermediate step between Lemmas 6.1 and 6.2, we prove:

LEMMA **6.3.**

$$\beta_{2q}(m) = \frac{1}{4\,\pi^{\binom{m+2}{2}}} \int_{\mathbf{S}'_{m,q-m}} F(HH^*) \left|\det(2HH^* - |x|^2 I)\right| e^{-\langle (H,x),(H,x)\rangle}\, dH\, dx \ , \qquad (89)$$

where

$$F(P) = 1 - \frac{4\,\mathrm{Tr}\,P}{m(m+1)} + \frac{4(\mathrm{Tr}\,P)^2 + 8\,\mathrm{Tr}(P^2)}{m(m+1)(m+2)(m+3)} \ , \qquad (90)$$

for (Hermitian) $m \times m$ *matrices* P.

Proof. Since the change of variables $H \mapsto gHg^t$ ($g \in \mathrm{U}(m)$) is unitary on $\mathrm{Sym}(m, \mathbb{C})$ (with respect to the Hilbert-Schmidt inner product), we can make this change of variables in (79), and then integrate over $g \in \mathrm{U}(m)$ to obtain

$$\beta_{2q}(m) = \frac{1}{4\,\pi^{\binom{m+2}{2}}} \int_{\mathbf{S}'_{m,q-m}} \left(\int_{\mathrm{U}(m)} \gamma(gHg^t)\, dg\right) \left|\det(2HH^* - |x|^2 I)\right| e^{-\langle (H,x),(H,x)\rangle}\, dH\, dx \ . \qquad (91)$$

We now evaluate the integral $\int_{\mathrm{U}(m)} \gamma(gHg^t)\, dg$;

Claim: For $H \in \mathrm{Sym}(m, \mathbb{C})$,

$$\int_{\mathrm{U}(m)} |(gHg^t)_{11}|^2\, dg = \frac{2}{m(m+1)} \mathrm{Tr}(HH^*) \ , \qquad (92)$$

$$\int_{\mathrm{U}(m)} |(gHg^t)_{11}|^4\, dg = \frac{8\,(\mathrm{Tr}\,HH^*)^2 + 16\,\mathrm{Tr}(HH^*HH^*)}{m(m+1)(m+2)(m+3)} \ . \qquad (93)$$

To prove the claim, we write $v = (v_1, \dots, v_m) = (g_{11}, \dots, g_{1m})$ so that $(gHg^t)_{11} = vHv^t$, and we replace $\int_{\mathrm{U}(m)} dg$ with $\int_{S^{2m-1}} d\nu(v)$, where $d\nu$ is Haar probability measure on S^{2m-1}. Next we recall that if if p is a homogeneous polynomial of degree $2k$ on $\mathbb{R}^{2m}$,

$$\int_{S^{2m-1}} p(v)\, d\nu(v) = \frac{(m-1)!}{(m-1+k)!} \int_{\mathbb{R}^{2m}} p(v)\, d\gamma(v) \ , \qquad d\gamma(v) = \frac{1}{\pi^m} e^{-\|v\|^2}\, dv \ . \qquad (94)$$

We easily see using Wick's formula that

$$\begin{aligned} \int_{\mathbb{C}^m} |vHv^t|^2\, d\gamma &= \sum_{j,k,j',k'} H_{jk} \bar{H}_{j'k'} \int_{\mathbb{C}^m} v_j v_k \bar{v}_{j'} \bar{v}_{k'}\, d\gamma \\ &= \sum_j |H_{jj}|^2 \int_{\mathbb{C}^m} |v_j|^4\, d\gamma + 2 \sum_{j \neq k} |H_{jk}|^2 \int_{\mathbb{C}^m} |v_j|^2 |v_k|^2\, d\gamma \\ &= 2\,\mathrm{Tr}(HH^*) \ . \end{aligned}$$

Formula (92) then follows from (94) with $k = 2$.

Although the above approach can also be used to verify (93), we find it easier to use invariant theory, since the integral in (93) is a $\mathrm{U}(m)$-invariant function of $H \in \mathrm{Sym}(m, \mathbb{C})$, under the $\mathrm{U}(m)$ action $H \mapsto gHg^t$. Indeed, it is a $\mathrm{U}(m)$-invariant Hermitian inner product on $S^2(\mathrm{Sym}(m, \mathbb{C})) \approx S^2(S^2(\mathbb{C}^m))$.

The action of $\mathrm{U}(m)$ on symmetric complex matrices defines a representation equivalent to $S^2(\mathbb{C}^m)$ where $\mathbb{C}^m$ is the defining representation of $U(m)$. It is well known from Schur-Weyl duality that $S^2(\mathbb{C}^m)$ is irreducible. We then consider the $\mathrm{U}(m)$ representation

$$S^2(S^2(\mathbb{C}^m)) = \mathbb{C}\{H_1 \otimes H_2 + H_2 \otimes H_1, \quad H_1, H_2 \in S^2(\mathbb{C}^m)\},$$

with the diagonal action. Henceforth we put

$$H_1 \cdot H_2 := \frac{1}{2}[H_1 \otimes H_2 + H_2 \otimes H_1].$$

We then regard $F(H)$ as the value on $H \otimes H$ of the quadratic form

$$Q(H_1 \cdot H_2) = \int_{\mathrm{U}(m)} |\langle gH_1g^t \cdot gH_2g^t \, e_1 \otimes e_1, e_1 \otimes e_1\rangle|^2 dg.$$

This defines the Hermitian inner product

$$\langle\langle H_1 \cdot H_2, H_2 \cdot H_4\rangle\rangle = \int_{\mathrm{U}(m)} \langle gH_1g^t \cdot gH_2g^t \, e_1 \otimes e_1, e_1 \otimes e_1\rangle \overline{\langle gH_3g^t \cdot gH_4g^t \, e_1 \otimes e_1, e_1 \otimes e_1\rangle} dg.$$

We next recall that $S^2(S^2(\mathbb{C}^m))$ decomposes into a direct sum of two $\mathrm{U}(m)$ irreducibles, one corresponding to the Young diagram Y_1 with 1 row of four boxes and one corresponding to the diagram Y_2 with 2 rows each with two boxes. See for instance Proposition 1 of [Ho]. The Young projectors are respectively,

$$\begin{cases} P_{Y_1}(H \otimes H)_{i_1 i_2 i_3 i_4} = \sum_{\sigma \in S_4} H_{i_{\sigma(1)} i_{\sigma(2)}} H_{i_{\sigma(3)} i_{\sigma(4)}} \\ \\ P_{Y_2}(H \otimes H)_{i_1 i_2 i_3 i_4} = \sum_{\sigma \in S_2 \times S_2} (-1)^\sigma H_{i_{\sigma(1)} i_{\sigma(2)}} H_{i_{\sigma(3)} i_{\sigma(4)}}. \end{cases}$$

For Y_2 the $S_2 \times S_2$ permutes $1 \iff 3, 2 \iff 4$.

Since an irreducible $\mathrm{U}(m)$ representation has (up to scalar multiples) a unique invariant inner product, it follows that

$$\langle\langle , \rangle\rangle = c_1 \langle , \rangle_{Y_1} + c_2 \langle , \rangle_{Y_2},$$

where $\langle , \rangle_{Y_j}$ are the invariant inner products

$$\langle A, B\rangle_{Y_j} = \mathrm{Tr}\, \Pi_{Y_j}(A) B^*$$

for the irreducibles corresponding to the Young diagrams Y_j as above.

We now calculate these inner products on $H \otimes H$. We have

$$\begin{cases} ||H \otimes H||^2_{Y_1} = \sum_{\sigma \in S_4} \sum_{i_1, i_2, i_3, i_4 = 1}^m H_{i_{\sigma(1)} i_{\sigma(2)}} H_{i_{\sigma(3)} i_{\sigma(4)}} \bar{H}_{i_1 i_2} \bar{H}_{i_3 i_4} \\ \\ ||H \otimes H||^2_{Y_2} = \sum_{\sigma \in S_2 \times S_2} \sum_{i_1, i_2, i_3, i_4 = 1}^m (-1)^\sigma H_{i_{\sigma(1)} i_{\sigma(2)}} H_{i_{\sigma(3)} i_{\sigma(4)}} \bar{H}_{i_1 i_2} \bar{H}_{i_3 i_4}. \end{cases}$$

It is easy to see that each of these expressions is a linear combination of the two quadratic forms

$$H \otimes H \mapsto \mathrm{Tr}\{[H \otimes H] \circ [H^* \otimes H^*]\}, \quad H \otimes H \mapsto [Tr H \circ H^*]^2.$$

Hence

$$\int_{\mathrm{U}(m)} |(gHg^t)_{11}|^4\,dg = c_1\,(\mathrm{Tr}\,HH^*)^2 + c_2\,\mathrm{Tr}(HH^*HH^*)\ .$$

To determine the constants c_1, c_2, it suffices to consider the case where H is diagonal. Let $s_1,\dots,s_m$ denote the eigenvalues of H. Then by Wick's formula we obtain

$$\begin{aligned}\int_{\mathbb{C}^m} |vHv^t|^4\,d\gamma &= \sum_{j,k,j',k'} s_j s_k \bar s_{j'} \bar s_{k'} \int_{\mathbb{C}^m} v_j^2 v_k^2 \bar v_{j'}^2 \bar v_{k'}^2\,d\gamma\\ &= \sum_j |s_j|^4 \int_{\mathbb{C}^m} |v_j|^8\,d\gamma + 2\sum_{j\neq k} |s_j|^2|s_k|^2 \int_{\mathbb{C}^m} |v_j|^4|v_k|^4\,d\gamma\\ &= 4!\sum_j |s_j|^4 + 8\sum_{j\neq k}|s_j|^2|s_k|^2\\ &= 8\,(\mathrm{Tr}\,HH^*)^2 + 16\,\mathrm{Tr}\,(HH^*)^2\ .\end{aligned}$$

Formula (93) now follows from (94) with $k=4$.

Having proved the claim, the formula stated in Lemma 6.3 now follows from (91) and Lemma 6.2. □

6.3. **Proof of Lemma 6.2.** We proceed exactly as in the proof of Lemma 3.1. We rewrite the integral (79) as

$$\beta_{2q}(m) = \frac{1}{4\,\pi^m\,(2\pi)^{m^2}}\ \lim_{\varepsilon'\to 0}\ \lim_{\varepsilon\to 0}\mathcal{I}_{\varepsilon,\varepsilon'}\ , \tag{95}$$

where

$$\begin{aligned}\mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{1}{\pi^{d_m}}\int_{\mathcal{H}_m}\int_{\mathcal{H}_m(m-q)}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} F(P+\tfrac12|x|^2I)\ |\det(2P)|\,e^{i\langle \Xi, P-HH^*+\frac12|x|^2I\rangle}\\ &\quad\times\ \exp\left(-\mathrm{Tr}HH^*-|x|^2\right)\,\exp\left(-\varepsilon\mathrm{Tr}\,\Xi\Xi^*-\varepsilon'\mathrm{Tr}PP^*\right)\,dH\,dx\,dP\,d\Xi\ , \qquad (96)\\ \mathcal{H}_m(m-q) &= \{P\in\mathcal{H}_m : \mathrm{index}\,P = m-q\}\ .\end{aligned}$$

Recall that $d_m = \dim_{\mathbb{C}}(\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}) = \frac12(m^2+m+2)$. As in §3.1, we note that absolute convergence is guaranteed by the Gaussian factors in each variable (H,x,P,Ξ). Evaluating $\int e^{i\langle\Xi,P-HH^*+\frac12|x|^2\rangle}e^{-\varepsilon\mathrm{Tr}\,\Xi\Xi^*}d\Xi$ first, we obtain a dual Gaussian, which approximates the delta function $\delta_{HH^*-\frac12|x|^2}(P)$. As $\varepsilon\to0$, the dP integral then yields the integrand at $P = HH^* - \frac12|x|^2I$; then letting $\varepsilon'\to0$ we obtain the original integral stated in Lemma 6.3.

Continuing as in §3.1, we conjugate P to a diagonal matrix $D(\lambda)$ with $\lambda = (\lambda_1,\dots,\lambda_m)$ by an element $h\in\mathrm{U}(m)$ and we replace dP with $\Delta(\lambda)^2\,d\lambda\,dh$. Recalling (45), we obtain:

$$\begin{aligned}\mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{2^m\,c'_m}{\pi^{d_m}}\int_{\mathrm{U}(m)}\int_{\mathcal{H}_m}\int_{Y'_{2m-q}}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}}\Delta(\lambda)^2\prod_{j=1}^m|\lambda_j|\,F\left(D(\lambda)+\tfrac12|x|^2I\right)\\ &\quad\times\ e^{i\langle\Xi,\,hD(\lambda)h^*+\frac12|x|^2I-HH^*\rangle}e^{-\left[\mathrm{Tr}HH^*+|x|^2+\varepsilon\mathrm{Tr}\Xi\Xi^*+\varepsilon'\sum\lambda_j^2\right]}\,dH\,dx\,d\lambda\,d\Xi\,dh\ .\end{aligned}$$

Again using (45) applied this time to Ξ, we obtain:

$$\begin{aligned}\mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{2^m(c'_m)^2}{\pi^{d_m}}\int_{\mathrm{U}(m)}\int_{\mathrm{U}(m)}\int_{\mathbb{R}^m}\int_{Y'_{2m-q}}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}}\Delta(\lambda)^2\,\Delta(\xi)^2\prod_{j=1}^m|\lambda_j|\,F\left(D(\lambda)+\tfrac12|x|^2I\right)\\ &\quad\times\, e^{i\langle gD(\xi)g^*,\,hD(\lambda)h^*+\frac12|x|^2I-HH^*\rangle}e^{-\mathrm{Tr}HH^*-|x|^2-\sum(\varepsilon\xi_j^2+\varepsilon'\lambda_j^2)}\,dH\,dx\,d\lambda\,d\xi\,dh\,dg\;,\end{aligned}$$

We then transfer the conjugation by g to the right side of the $\langle,\rangle$ in the first exponent and make the change of variables $h\mapsto gh, H\mapsto gHg^t$ to eliminate g from the integrand:

$$\begin{aligned}\mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{2^m(c'_m)^2}{\pi^{d_m}}\int_{\mathrm{U}(m)}\int_{\mathbb{R}^m}\int_{Y'_{2m-q}}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}}\Delta(\lambda)^2\,\Delta(\xi)^2\prod_{j=1}^m|\lambda_j|\;F\left(D(\lambda)+\tfrac12|x|^2I\right)\\ &\quad\times\, e^{i\langle D(\xi),\,hD(\lambda)h^*+\frac12|x|^2I-HH^*\rangle}e^{-\mathrm{Tr}HH^*-|x|^2-\sum(\varepsilon\xi_j^2+\varepsilon'\lambda_j^2)}\,dH\,dx\,d\lambda\,d\xi\,dh\;.\end{aligned}$$

Next we substitute the Itzykson-Zuber-Harish-Chandra integral formula (47) into the above and expand

$$\det[e^{i\xi_j\lambda_k}]_{jk}=\sum_{\sigma\in S_m}(-1)^\sigma\;e^{i\langle\xi,\sigma(\lambda)\rangle},$$

obtaining a sum of $m!$ integrals. However, by making the change of variables $\lambda'=\sigma(\lambda)$ and noting that $\Delta(\lambda')=(-1)^\sigma\Delta(\lambda)$, we see as before that the integrals of these terms are equal, and so we obtain

$$\begin{aligned}\mathcal{I}_{\varepsilon,\varepsilon'} &= (-i)^{m(m-1)/2}\frac{c''_m}{\pi^{d_m}}\int_{\mathbb{R}^m}\int_{Y'_{2m-q}}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}}\Delta(\lambda)\,\Delta(\xi)\prod_{j=1}^m|\lambda_j|\;e^{i\langle\lambda,\xi\rangle}\\ &\quad\times\;F\left(D(\lambda)+\tfrac12|x|^2I\right)\exp\left(i\left\langle D(\xi),\,\tfrac12|x|^2I-HH^*\right\rangle-\mathrm{Tr}HH^*-|x|^2\right)\\ &\quad\times\;\exp\left(-\varepsilon\textstyle\sum\xi_j^2-\varepsilon'\sum\lambda_j^2\right)\,dH\,dx\,d\lambda\,d\xi\;.\end{aligned}\tag{97}$$

where

$$c''_m=\frac{2^{m^2}\,\pi^{m(m-1)}}{\prod_{j=1}^m j!}\;.$$

The phase

$$\begin{aligned}\Phi(H,x;\xi) &:= i\left\langle D(\xi),\frac12|x|^2I-HH^*\right\rangle-\mathrm{Tr}HH^*-|x|^2\\ &= -\left[\|H\|^2_{\mathrm{HS}}+i\sum_{j,k=1}^m\xi_j|H_{jk}|^2+\left(1-\frac{i}{2}\sum_j\xi_j\right)|x|^2\right]\\ &= -\left[\sum_{j\le k}\left(1+\frac{i}{2}(\xi_j+\xi_k)\right)|\widehat{H}_{jk}|^2+\left(1-\frac{i}{2}\sum_j\xi_j\right)|x|^2\right]\;.\end{aligned}\tag{98}$$

Thus,

$$\mathcal{I}_{\varepsilon,\varepsilon'}=(-i)^{m(m-1)/2}c''_m\int_{Y'_{2m-q}}\int_{\mathbb{R}^m}\Delta(\lambda)\,\Delta(\xi)\prod_{j=1}^m|\lambda_j|\;e^{i\langle\lambda,\xi\rangle}\,\mathcal{I}(\lambda,\xi)\,e^{-\varepsilon\sum\xi_j^2-\varepsilon'\sum\lambda_j^2}\,d\xi\,d\lambda\;,\tag{99}$$

where,

$$\begin{aligned}\mathcal{I}(\lambda,\xi) &= \frac{1}{\pi^{d_m}}\int_{\mathbb{C}}\int_{\mathrm{Sym}(m,\mathbb{C})} F\left(D(\lambda)+\frac{1}{2}|x|^2 I\right) e^{\Phi(H,x;\xi)}\,dH\,dx \\ &= \frac{1}{\prod_{j\le k}\left(1+\frac{i}{2}(\xi_j+\xi_k)\right)}\int_{\mathbb{C}} F\left(D(\lambda)+\frac{1}{2}|x|^2 I\right)\, e^{-\left(1-\frac{i}{2}\sum_j \xi_j\right)|x|^2}\,dx\ .\end{aligned}$$

To evaluate the dx integral, we first expand the amplitude:

$$\begin{aligned}F\left(D(\lambda)+\frac{1}{2}|x|^2 I\right) &= F(D(\lambda)) + \left[\frac{4\sum_{j=1}^m \lambda_j}{m(m+1)(m+3)} - \frac{2}{m+1}\right]|x|^2 \\ &\quad + \frac{1}{(m+1)(m+3)}|x|^4\ ,\end{aligned}$$

and then integrate to obtain (80).

To evaluate $\lim_{\varepsilon,\varepsilon'\to 0^+}\mathcal{I}_{\varepsilon,\varepsilon'}$, we first observe as in §4 that the map

$$(\varepsilon_1,\dots,\varepsilon_m)\mapsto \int_{\mathbb{R}^m}\Delta(\xi)\ e^{i\langle\lambda,\xi\rangle}\,\mathcal{I}(\lambda,\xi)\,e^{-\sum\varepsilon_j\xi_j^2}\,d\xi$$

is a continuous map from $[0,+\infty)^m$ to the tempered distributions. Hence by (95) and (99), we have:

$$\begin{aligned}\beta_{2q}(m) &= \frac{(-i)^{m(m-1)/2}}{4\,\pi^{2m}\prod_{j=1}^m j!}\ \lim_{\varepsilon'\to 0^+}\ \lim_{\varepsilon_1,\dots,\varepsilon_m\to 0^+} m!\int_{Y_{2m-q}} d\lambda \\ &\quad \times\int_{\mathbb{R}^m}\Delta(\lambda)\,\Delta(\xi)\prod_{j=1}^m|\lambda_j|\ e^{i\langle\lambda,\xi\rangle}\,\mathcal{I}(\lambda,\xi)\,e^{-\sum\varepsilon_j\xi_j^2-\varepsilon'\sum\lambda_j^2}\,d\xi\ . \qquad (100)\end{aligned}$$

Letting $\varepsilon_1\to 0,\dots,\varepsilon_m\to 0,\varepsilon'\to 0$ sequentially, we obtain the formula of Lemma 6.2. □

6.4. **Values of the constants β_{2q}.** We use the integral formula of Lemma 6.2 to compute the constants β_{2q}. The ξ_j integrals can be evaluated using residues as in §4.2; the resulting λ integrand is a polynomial function of the λ_j. The integrals were evaluated in dimensions ≤ 3 using Maple 7.[2]

In dimension 1, we reproduce the result from [DSZ]:

$$\beta_{21}(1)=\frac{1}{3^3\cdot\pi}\ ,\quad \beta_{22}(1)=\frac{1}{3^3\cdot\pi}\ .$$

In dimension 2, we have:

$$\beta_{22}(2)=\frac{1}{2^3\cdot 5\cdot\pi^2}\ ,\quad \beta_{23}(2)=\frac{2^4}{3^4\cdot 5\cdot\pi^2}\ ,\quad \beta_{24}(2)=\frac{47}{2^3\cdot 3^4\cdot 5\cdot\pi^2}\ .$$

In dimension 3, we have:

$$\beta_{23}(3)=\frac{2^2}{5^3\cdot\pi^3}\ ,\quad \beta_{24}(3)=\frac{11\cdot 23}{2^5\cdot 5^3\cdot\pi^3}\ ,\quad \beta_{25}(3)=\frac{2^9\cdot 7}{3^6\cdot 5^3\cdot\pi^3}\ ,\quad \beta_{26}(3)=\frac{23563}{2^5\cdot 3^6\cdot 5^3\cdot\pi^3}\ .$$

This completes the proof of Theorem 1.7. □

[2]The Maple programs are included in the source files of the arXiv.org posting.

Remark: The values of the β_2 coefficient for the expected total number of critical points are:

$$\beta_2(1) = \frac{2}{3^3 \cdot \pi}, \quad \beta_2(2) = \frac{32}{405\pi^2} = \frac{2^5}{3^4 \cdot 5 \cdot \pi^2}\ , \quad \beta_2(3) = \frac{104}{729\pi^3} = \frac{2^3 \cdot 13}{3^6 \cdot \pi^3}\ .$$

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