Geometry and statistical patterns in zeros and critical points of random analytic functions

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Based on joint work (partly in progress) with O. Zeitouni and B.Shiffman

Themes of talk

- Use geometry (volume forms, hermitian metrics) to define inner products on spaces of polynomials or 'sections of a line bundle' over a Kähler manifold (M, ω) of dimension m.

- Each inner product induces a Gaussian measure on such a space.

- Study statistics of zeros and critical points of random polynomials (or holomorphic sections) as the degree $N \to \infty$. How do they depend on the underlying geometry? Which metrics have 'minimal complexity', i.e. cause the minimal number of critical points on average.

Principal results

The main results concern large degree asymptotics:

1. Zeros and critical points concentrate asymptotically as the degree $N \to \infty$ in regions of high curvature;

2. The metrics which minimize the expected number of critical points of a Gaussian random 'section' of large degree N are those which minimize the Calabi functional (known as canonical metrics in complex geometry; e.g. Calabi-Yau metrics in the case of the 'canonical bundle').

Complex Kac-Hammersley polynomials

Consider the random holomorphic polynomial of one complex variable,

$$f(z) = \sum_{j=1}^{N} c_j z^j$$

where the coefficients c_j are independent complex Gaussian random variables of mean zero and variance one. Complex Gaussian:

$$\mathbf{E}\,(c_j) = 0 = E(c_j c_k), \quad E(c_j \bar{c}_k) = \delta_{jk}.$$

This defines a Gaussian measure γ_{KAC} on $\mathcal{P}_N^{(1)}$:

$$d\gamma_{KAC}(f) = e^{-|c|^2/2} dc.$$

Expected distribution of zeros

The distribution of zeros of a polynomial of degree N is the probability measure on $\mathbb{C}$ defined by

$$Z_f = \frac{1}{N} \sum_{z: f(z)=0} \delta_z,$$

where δ_z is the Dirac delta-function at z.

Definition: The expected distribution of zeros of random polynomials of degree N with measure P is the probability measure $\mathbf{E}_P Z_f$ on $\mathbb{C}$ defined by

$$\langle \mathbf{E}_P Z_f, \varphi \rangle = \int_{\mathcal{P}_N^{(1)}} \{ \frac{1}{N} \sum_{z: f(z)=0} \varphi(z) \} dP(f),$$

for $\varphi \in C_c(\mathbb{C})$.

How are zeros of complex Kac polynomials distributed?

Complex zeros concentrate in small annuli around the unit circle S^1. In the limit as the degree $N \to \infty$, the zeros asymptotically concentrate exactly on S^1:

Theorem **1 (Kac-Hammersley-Shepp-Vanderbei)** *The expected distribution of zeros of polynomials of degree N in the Kac ensemble has the asymptotics:*

$$\mathbf{E}^N_{KAC}(Z^N_f) \to \delta_{S^1} \quad \textit{as} \quad N \to \infty \, ,$$

$$\textit{where} \qquad (\delta_{S^1}, \varphi) := \frac{1}{2\pi} \int_{S^1} \varphi(e^{i\theta}) \, d\theta.$$

The real zeros concentrate at the intersection points of $S^1 \cap \mathbb{R} = \{\pm 1\}$.

Gaussian measure and inner product

It was the (implicit) choice of inner product that produced this concentration of zeros on S^1.

The inner product underlying the Kac Gaussian measure on $\mathcal{P}_N^{(1)}$ is defined by the basis $\{z^j\}$ being orthonormal. Thus, they were orthonormalized on S^1. An inner product induces an orthonormal basis $\{S_j\}$ and associated associated Gaussian measure dγ:

$$S = \sum_{j=1}^{d} c_j S_j,$$

where $\{c_j\}$ are independent complex normal random variables.

Orthonormalizing on S^1 made zeros concentrate on S^1.

Gaussian random polynomials adapted to domains and weights

We now orthonormalize polynomials on the interior Ω or boundary $\partial\Omega$ of any simply connected, bounded domain $\Omega \subset \mathbb{C}$. Introduce a weight $e^{-N\varphi}$ and a probability measure $d\nu$ on Ω and define

$$\langle f, \bar{g} \rangle_{\Omega,\varphi} := \int_\Omega f(z)\overline{g(z)}\, e^{-N\varphi(z)} d\nu \ .$$

Let $\gamma^N_{\Omega,\varphi}$ = the Gaussian measure induced by $\langle f, \bar{g} \rangle_{\Omega,\varphi}$ on $\mathcal{P}^{(1)}_N$.

How do zeros of random polynomials adapted to Ω concentrate?

Equilibrium distribution of zeros

Denote the expectation relative to the ensemble $(\mathcal{P}_N, \gamma^N_{\partial\Omega})$ by $\mathbf{E}^N_{\partial\Omega}$.

Theorem **2**

$$\mathbf{E}^N_{\partial\Omega}(Z^N_f) = \nu_\Omega + O\left(1/N\right) \ ,$$

where ν_Ω is the equilibrium measure of $\bar{\Omega}$ with respect to φ.

The equilibrium measure of a compact set K is the unique probability measure $d\nu_K$ which minimizes the energy

$$E(\mu) = -\int_K \int_K \log|z-w|\, d\mu(z)\, d\mu(w) + \int_K \varphi d\mu.$$

Thus, zeros behave like electric charges in the potential φ.

Warm-Up for line bundles: $SU(2)$ polynomials

There exists an inner product in which the expected distribution of zeros is 'uniform' on $\mathbb{CP}^1$ w.r.t. to the usual Fubini-Study area form ω_{FS}.

We define an inner product on $\mathcal{P}_N^{(1)}$ which depends on N:

$$\langle z^j, z^k \rangle_N = \frac{1}{\binom{N}{j}} \delta_{jk}.$$

Thus, a random $SU(2)$ polynomial has the form

$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \sqrt{\binom{N}{\alpha}}\, z^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \overline{\lambda}_\beta) = \delta_{\alpha\beta}.$$

Proposition **3** *In the $SU(2)$ ensemble, $\mathbf{E}(Z_f) = \omega_{FS}$, the Fubini-Study area form on $\mathbb{CP}^1$.*

$SU(2)$ and holomorphic line bundles

The $SU(2)$ inner products may be written in the form

$$\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-N\,\mathrm{log}(1+|z|^2)}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}.$$

The factor $e^{-N\,\mathrm{log}(1+|z|^2)}$ defines a Hermitian metric on $\mathcal{O}(N)$, and its curvature form is $\omega = \frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$.

This gives a geometric interpretation of the inner product

$$\langle f,\bar{g}\rangle_{\Omega,\varphi} := \int_{\Omega} f(z)\overline{g(z)}\, e^{-N\varphi(z)}d\nu \ \ .$$

We should regard f, g as sections of the Nth power of a line bundle with Hermitian metric $e^{-N\varphi}$.

Gaussian random holomorphic sections of line bundles

We now consider more general Hermitian metrics $h = e^{-\varphi}$ on $\mathcal{O}(1) \to \mathbb{CP}^1$ and area forms on $\mathbb{CP}^1$. In fact, everything we do generalizes to any Riemann surface M of any genus.

The Hermitian metric h on $\mathcal{O}(1)$ induces Hermitian metrics $h^N = e^{-N\varphi}$ on the powers $\mathcal{O}(N)$, a volume form dV, and an inner product

$$\langle s_1, s_2 \rangle_N = \int_M s_1(z)\overline{s_2(z)} e^{-N\varphi} dV(z).$$

We let $\{S_j\}$ denote an orthonormal basis of the space $H^0(M, L^N)$ of holomorphic sections of L^N.

Then define the Gaussian measure γ_{h^N} on $s \in H^0(M, L^N)$ by

$$s = \sum_j c_j S_j, \quad \langle S_j, S_k \rangle = \delta_{jk}$$

with $\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k)$, $\mathbf{E}(c_j \overline{c_k}) = \delta_{jk}$.

Statistics of critical points

From now on we focus on critical points

$$\nabla s(z) = 0,$$

where ∇ is a metric connection.

Critical points of Gaussian random functions come up in may areas of physics–

- as peak points of signals (S.O. Rice, 1945);
- as vacua in compactifications of string/M theory on Calabi-Yau manifolds with flux (Giddings-Kachru-Polchinski, Gukov-Vafa-Witten);
- as extremal black holes (Strominger, Ferrara-Gibbons-Kallosh) , peak points of galaxy distributions (Szalay et al, Zeldovich), etc.

Critical points with respect to a metric connection

Definition: *Let* $(L, h) \to M$ *be a Hermitian holomorphic line bundle over a complex manifold* M*, and let* $\nabla = \nabla_h$ *be its Chern connection.*

A critical point of a holomorphic section $s \in H^0(M, L)$ *is defined to be a point* $z \in M$ *where* $\nabla s(z) = 0$*.*

In a local frame e critical point equation for $s = fe$ reads:

$$\partial f(w) + f(w)\partial\varphi(w) = 0,$$

where $||e(z)||_h = e^{-\varphi}$.

The critical point equation is only C^∞ and not holomorphic since φ is not holomorphic.

Statistics of critical points

The distribution of critical points of $s \in H^0(M, L)$ with respect to h (or ∇_h) is the measure on M

$$(1) \qquad C_s^h := \sum_{z:\ \nabla_h s(z)=0} \delta_z.$$

Definition: The (expected) distribution $\mathbf{E}_\gamma C_s^h$ *of critical points of* $s \in H^0(M, L)$ *w.r.t.* ∇_h *and* γ_h *is the measure on* M *defined by*

$$\langle \mathbf{E}_\gamma C_s^h, \varphi \rangle := \int_{H^0(M,L)} \left[\sum_{z:\nabla_h s(z)=0} \varphi(z) \right] d\gamma(s).$$

The expected number of critical points is defined by

$$\mathcal{N}^{crit}(h, \gamma) \;=\; \int_{\mathcal{S}} \# Crit(s, h) d\gamma(s).$$

Problems of interest

1. Calculate $\mathbf{E}_{\gamma} C_s^h$. How are critical points distributed? How are they correlated. As the degree $N \to \infty$, how is $C_s^{h^N}$ concentrated around the equilibrium measure?

2. How large is $\mathcal{N}^{\text{crit}}(h, \gamma)$? How does the expected number of critical points depend on the metric?

3. The 'best' metrics are the ones which minimize this quantity. Which are they?

How the curvature affects the expected number of critical points

Let us consider the simplest case:

Theorem **4** *The expected number of critical points of a random section* $s_N \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$ *(with respect to the Gaussian measure* γ_{FS} *on* $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ *induced from the Fubini-Study metrics on* $\mathcal{O}(N)$ *and* $\mathbb{CP}^1$*) is*

$$\frac{5N^2 - 8N + 4}{3N - 2} = \frac{5}{3}N - \frac{14}{9} + \frac{8}{27}N^{-1} \cdots .$$

Of course, relative to the flat connection d/dz the number is $N - 1$. Thus, the positive curvature of the Fubini-Study hermitian metric and connection causes sections to oscillate much more than the flat connection. There are $\frac{N}{3}$ new local maxima and $\frac{N}{3}$ new saddles.

Asymptotic expansion for the expected number of critical points as $N \to \infty$

Theorem **5** *(Douglas, Shiffman,Zelditch) Let $(L, h) \to M$ be a positive hermitian line bundle over any Kähler manifold of any dimension. Let $\mathcal{N}^{\text{crit}}(h^N)$ denote the expected number of critical points of random $s \in H^0(M, L^N)$ with respect to the Hermitian Gaussian measure. Then $\exists \Gamma_m^{\text{crit}} > 0$ s.th.*

$$\mathcal{N}(h^N) = \left(\frac{\pi^m}{m!}\Gamma_m^{\text{crit}}\ c_1(L)^m\right) N^m$$

$$+\left(\int_M \rho dV_\omega\right) N^{m-1}$$

$$+\left[C_m \int_M \rho^2 dV_\Omega + \mathit{top}\right] N^{m-2} + O(N^{m-3})\,.$$

Here, ρ is the scalar curvature of ω_h, the curvature of h.

$\Gamma_m^{\text{crit}}\, c_1(L)^m$ is larger than for a flat connection.

Is the expected number of critical points a topological invariant?

The first two terms are topological invariants of a positive line bundle, i.e. independent of the metric! (Both are Chern numbers of L). But the non-topological part of the third term

$$C_m \int_M \rho^2 dV_\Omega N^{m-2}$$

is a non-topological invariant, as long as $C_m \neq 0$. It is a multiple of the Calabi functional. It was proved by Douglas-Shiffman-Zelditch in $m = \dim M \leq 5$ that $C_m \neq 0$ and by B. Baugher in all dimensions (2008 PhD thesis).

(These calculations are based on the Tian-Yau-Zelditch (and Catlin) expansion of the Szegö kernel and on Zhiqin Lu's calculation of the coefficients in that expansion.)

Calabi extremal metrics are asymptotic minimizers

A Calabi extremal metric is a minimizer of the functional $\int_M \rho^2 dV$ where $\rho =$ scalar curvature of the Kähler form ω.

Theorem **6** *Douglas-Shiffman-Zelditch (2006)-Baugher (2008) Calabi extremal metrics (asymptotically minimize) the metric invariant given by $\mathcal{N}^{\text{crit}}(\mathcal{H}^N)$, the expected number of critical points of a random $s \in H^0(M, L^N)$.*

For instance, Fubini-Study metrics are extremal metrics on $\mathbb{CP}^m$, so on average holomorphic sections have fewer critical points with respect to FS metrics than any other metric on $\mathbb{CP}^m$.

Hints at methods

The proofs are based on:

- A general Kac-Rice formula for the expected number of critical points.

- An Itzykson-Zuber type re-working of this formula;

- For large N, an asymptotic analysis of the two point function $\Pi_{h^N}(z,w) = \mathbf{E}_{\gamma_{h^N}}(s(w)\overline{s(w)})$, i.e. the Bergman kernel for the Hermitian line bundle L^N.

General formula for density critical points

We denote by Sym$(m,\mathbb{C})$ the space of complex $m\times m$ symmetric matrices. In well-chosen local coordinates $z=(z_1,\dots,z_m)$, in a local frame e, we have:

Theorem **1** *There exist positive-definite Hermitian matrices*

$A(z):\mathbb{C}^m\to\mathbb{C}^m$,

$\Lambda(z):\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}\to \mathit{Sym}(m,\mathbb{C})\oplus\mathbb{C}$, *s.th.*

$$\mathcal{K}^{\mathrm{crit}}(z)=\frac{1}{\det A(z)\det\Lambda(z)}\times\int_{\mathbb{C}}\int_{\mathrm{Sym}(m,\mathbb{C})}$$

$$|\det\begin{pmatrix}H' & x\,\Theta(z)\\ \bar{x}\,\bar{\Theta}(z) & \bar{H}'\end{pmatrix}|\;e^{-\langle\Lambda(z)^{-1}(H'\oplus x),\,H'\oplus x\rangle}\,dH'\,dx\,.$$

Formulae for $A(z)$ and $\Lambda(z)$

$A(z)$ and $\Lambda(z)$ depend only on ∇ and on the Szegö kernel, i.e. orthogonal projection

$$\Pi_{\mathcal{S}} : L^2(M,L) \to \mathcal{S} \subset H^0(M,L),$$

for $\mathcal{S}$ and for the inner product. Let $F_{\mathcal{S}}(z,w)$ be the local expression for $\Pi_{\mathcal{S}}(z,w)$ in the frame e_L. Then $\Lambda = C - B^* A^{-1} B$, where

$$\begin{array}{l} A = \left(\frac{\partial^2}{\partial z_j \partial \bar{w}_{j'}} F_{\mathcal{S}}(z,w)|_{z=w} \right), \\ B = \left[\left(\frac{\partial^3}{\partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}}) F_{\mathcal{S}}|_{z=w} \right) \quad \left((\frac{\partial}{\partial z_j} F_{\mathcal{S}}|_{z=w} \right) \right], \\ C = \left[\begin{array}{cc} \left(\frac{\partial^4}{\partial z_q \partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}}|_{z=w} \right) & \left(\frac{\partial^2}{\partial z_j \partial z_q} F_{\mathcal{S}} \right) \\ \left(\frac{\partial^2}{\partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}} \right) |_{z=w} & F_{\mathcal{S}}(z,z) \end{array} \right], \\ 1 \le j \le m\,, 1 \le j \le q \le m\,, 1 \le j' \le q' \le m\,. \end{array}$$

In the above, A, B, C are $m \times m$, $m \times n$, $n \times n$ matrices, respectively, where $n = \frac{1}{2}(m^2 + m + 2)$.