RANDOM COMPLEX GEOMETRY AND VACUA, OR: HOW TO COUNT UNIVERSES IN STRING/M THEORY

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Abstract. This lecture is concerned with counting and equidistribution problems for critical points of random holomorphic functions and their applications to statistics of vacua of certain string/M theories. To get oriented, we begin with the distribution of complex zeros of Gaussian random holomorphic polynomials of one variable. We then consider zeros and critical points of Gaussian random holomorphic sections of line bundles over complex manifolds. The main focus is on critical points $\nabla s(z) = 0$ of a holomorphic sections relative to a smooth metric connection ∇ on a holomorphic line bundle. In string/M theory compactified on a Calabi-Yau manifold, the possible vacuum states of the universe (vacua) are critical points of a holomorphic section (the 'superpotential') of a line bundle over the moduli space of Calabi-Yau manifolds. Physicists sometimes estimate the number of possible vacua to be around 10^{500}. We describe some rigorous results from [DSZ3, DD] on the number and distribution of vacua in such string theories. Finally, we discuss some results from [DSZ1, DSZ2] on the pure geometry of critical points: how the average number of critical points is asymptotically minimized by Calabi extremal metrics and some hints on the correlations between critical points on small scales.

Contents

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1. Introduction

This is an expository article (based on the author's AMS address in Atlanta, 2005) which leads the reader from the classical mathematics of random polynomials to the contemporary physics of the vacuum selection problem of string theory. It assumes no prior background in either random polynomials or string theory. Starting with classical results of Kac and Hammersley on zeros of random polynomials, the geometric context gradually broadens to zeros and critical points of random holomorphic sections of line bundles over complex manifolds to finally encompass vacua of certain string theories compactified on Calabi-Yau manifolds. The unifying theme is the statistics of critical points of random holomorphic functions or sections. As we hope to convince the reader, the distribution of (and correlations between) critical points of random polynomials and holomorphic sections of line bundles is a subject of some intrinsic interest in complex geometry, with connections to Calabi extremal metrics (see §6). But it acquires an independent interest from its connection to string theory, in which vacua (and extremal black holes) are critical points of certain holomorphic sections known as superpotentials. There is a plethora of possible superpotentials and vacua, and in the absence of a selection principle determining a unique one, it makes sense to count the number of vacua which are consistent with known physical quantities such as the cosmological constant. After providing some background on random polynomials and sections in §2 - §4, we give a mathematical introduction in §5 to the counting and equidistribution problems in complex geometry which arise in 'statistics of vacua'. The results discussed in §5 come from joint work with M. R. Douglas and B. Shiffman [DSZ1, DSZ2, DSZ3] and from related works of Douglas and his collaborators (Ashok, Denef) [AD, DD, DD2]. They are also based on earlier joint work with P. Bleher [BSZ1]. We would like to thank B. Shiffman and M. R. Douglas for the pleasure of the collaboration, and also for many comments on and contributions to this article.

1.1. **Statistics of vacua in string/M theory.**

Let us first give an overview of what statistics of vacua in string/M theory is about (undefined terms will be discussed in detail in §5). It is a response to the so-called *vacuum selection problem* in string theory, which is easy to understand without any background in string theory: in its vacuum state, our universe according to string theory is a 10 dimensional spacetime of the form $M^{3,1} \times X$, where $M^{3,1}$ is Minkowski space and X is a small 3-complex dimensional Calabi-Yau manifold X known as the 'small' or 'extra' dimensions [CHSW, St2, Zw, P]. The vacuum selection problem is simply, "which Calabi-Yau 3-fold X?".

At this time, the set of topological types of CY 3-folds is unknown and has not been proved to be finite. We will ignore this aspect of the problem, and concentrate on the issue of which

CY metric(s) in the moduli space $\mathcal{M}_{CY}$ of CY metrics on a fixed manifold X could model the small dimensions of the universe. We focus on so-called type IIb string theories compactified on a Calabi-Yau manifold Y with flux [GKP, GVW]. When Y is d-dimensional, a 'flux' is a complex integral d-form. In the setting of this paper, $Y = X \times T^2$ where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and X is a complex 3-dimension CY manifold; the flux has the form $F \wedge dx + iH \wedge dy$ where $G = F + iH \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$.

The key point for us is that the possible vacua for a fixed choice of flux G are critical points of the scalar potential energy (cf. [WB], (21.22))

$$V_W(Z) = |\nabla W(Z)|^2 - 3|W(Z)|^2, \quad (Z = (z, \tau) \in \mathcal{M}_{CY} \times \mathcal{H}/SL(2, \mathbb{Z})) \tag{1}$$

on the 'configuration space' $\mathcal{C} = \mathcal{M}_{CY} \times \mathcal{H}/SL(2, \mathbb{Z})$ ($\mathcal{H}$ is the upper half-plane). Here, $W = W_G$ is the *flux superpotential* corresponding to the flux G in a fashion which will be defined in §5 (43)-(44). Among the critical points of V_W are the *supersymmetric vacua* (= SUSY vacua) where $\nabla W(Z) = 0$. For the sake of simplicity, we restrict our attention to these SUSY vacua (see [DD2] for developments in the non-SUSY case).

String/M theory leads to the study of critical points of holomorphic sections of line bundles with respect to Chern metric connections because:

- The superpotential W_G is a holomorphic section of a holomorphic line bundle $\mathcal{L} \to \mathcal{C}$. We denote the space of its holomorphic sections by $H^0(\mathcal{C}, \mathcal{L})$. $\mathcal{L}$ is the dual bundle of the Hodge bundle $H^{3,0}_z(X) \otimes H^{1,0}(T^2) \to \mathcal{C}$ of holomorphic volume forms;
- $\nabla = \nabla_{WP}$ is the Chern connection on $\mathcal{L}$ corresponding to the Weil-Petersson hermitian metric;
- Supersymmetric vacua are connection critical points $\nabla_{WP} W_G(Z) = 0$ (cf. Definition 4 and the local formula (18)). We denote the set of critical points of W by $Crit(W) \subset \mathcal{C}$.

Thus, to get a grasp on the number of possible (SUSY) vacua in this model, we need to count critical points of holomorphic sections satisfying constraints.

The potential V_W depends on a choice of superpotential W, and the superpotential W_G depends on a choice of flux G. How unique is the flux G? The answer is that the flux G can be any element of $H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$ satisfying a certain 'tadpole constraint' $0 \le Q[G] \le L$, where $Q[W]$ is an indefinite quadratic form derived from the intersection form on $H^3(X, \mathbb{C})$ (see §5.3). Thus, we can describe the set of relevant superpotentials as follows:

- Fluxes $G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$ give rise to flux superpotentials W_G, which form a lattice $\mathcal{F}_{\mathbb{Z}} \subset \mathcal{F}$ in a subspace $\mathcal{F} \subset H^0(\mathcal{C}, \mathcal{L})$ of dimension $2b_3 = \dim H^3(X, \mathbb{C})$. We call $\mathcal{F}$ the space of 'complex flux superpotentials'.
- The tadpole constraint $0 \le Q[G] \le L$ determines a hyperbolic shell in $\mathcal{F}$, i.e. the region between the image of the cone $Q[G] = 0$ and the hyperboloid $Q[G] = L$.

Let us summarize what this implies about the set of candidate vacua in the model. Each superpotential W_G gives rise to a number of critical points $Crit(W_G) \subset \mathcal{C}$, any one of which is a candidate for the vacuum state of the universe. The set of possible vacua is thus the union

$$\text{Vacua}_L = \bigcup_{G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}Z), \ 0 \le Q[G] \le L} \text{Crit}(W_G). \tag{2}$$

The 'tadpole' number L is fixed by X and can take on a large range of values (cf. §5.3 and especially (46)).

The seeming multitude of superpotentials and vacua is obviously a problem for string theory. Unless or until a principle emerges which selects one superpotential and one vacuum, it seems natural to investigate the whole 'discretuum' of possible vacua to answer such questions as: how many vacua are there for a fixed manifold X, as W runs over the set (2) of physically allowed superpotentials? More importantly, how many vacua are consistent with known values of the cosmological constant (the value of $V_W(Z)$ at the critical point) or masses of elementary particles? Does even one exist? How are vacua distributed in the moduli space $\mathcal{M}_{CY}$ of Calabi-Yau manifolds?

A direct enumerative approach to vacua is to pick a specific manifold X, calculate the corresponding superpotentials on $\mathcal{C}$, and solve for their critical points. Such an approach has been carried out for rigid Calabi-Yau manifolds (cf. §5.5), for the complex 3 torus $(T^2)^3$, for one or two parameter families of quintic hypersurfaces in $\mathbb{CP}^4$ or for more general hypersurfaces in weighted projective spaces (see e.g. [AD, CQ, GKT, DGKT, MN]). Enumerative problems for flux vacua in (2) may turn out to be intrinsically interesting in mathematics. Indeed, the dualities between different string models should define correspondences between their vacua. Besides the type IIb compactifications with flux on a CY 3-fold, there are other string models such as the type IIa string on a CY 3-fold, the heterotic string on a CY 3-fold or M theory on a G_2 manifold. Other dualities relate vacua with flux to vacua defined using Dirichlet branes, which are counted using Gopakumar-Vafa invariants. Thus, we may expect relations between the mathematically diverse enumeration problems which arise in vacuum counting. In addition, flux vacua are very analogous to extremal black holes, which are critical points of superpotentials of the same kind but with real fluxes [St, FGK]. The critical point equation (known as the black hole attractor equation) appears to have connections to number theory [M, MM].)

However, it is difficult and laborious to solve the critical point equations explicitly in all but the simplest models. Therefore M. R. Douglas initiated in [D] a statistical approach to count the vacua in (2) and determine the distribution of physically interesting quantities such as the cosmological constant on that set. He suggested that the discrete set of flux superpotentials could be approximated (for sufficiently large L) by a continuum ensemble resembling the Gaussian or spherical ensembles studied in [BSZ1, BSZ2]. At the same time, B. Shiffman and the author were studying critical points of random holomorphic sections relative to metric connections for their intrinsic mathematical interest (see §4). This coincidence of interests led to our joint works on statistics of vacua [DSZ1, DSZ2, DSZ3]. There is also a wealth of detail without rigorous proofs on statistics of vacua in the physics articles by Ashok-Douglas [AD], Denef-Douglas [DD, DD2] and in [GKT, DGKT, CQ].

These works aim at giving a precise and rigorous count of vacua satisfying physical constraints in a specific model. At the same time, the string theory literature contains a number of heuristic statements about numbers of vacua and the string theory landscape which do not specify any particular model. The graph of the scalar potential energy is often visualized in physics as a landscape [S] whose local minima are the candidate vacua. In the words of Bousso-Polchinski [BP], "The theory of strings predicts that the universe might occupy one random "valley" out of a virtually infinite selection of valleys in a vast landscape of possibilities." They suggest that the number of valleys might not be infinite but roughly of the order 10^{500}. In a lecture, L. Susskind also arrived at this large number of local minima by reasoning that the potential energy is a function of a large number of variables (say, $1,000$). A typical polynomial f of degree d on $\mathbb{C}^m$ may be expected to have $(d-1)^m$ critical points since critical points are solutions of the m equations $\frac{\partial f}{\partial z_j}(w) = 0$ of degree $d-1$. Thus, a figure of 10^{500} is reasonable if the potential has a degree of the order of magnitude of 10.

To connect this heuristic argument to the type IIb flux compactification model we are focussing on here, we should imagine the potential energy as defined on $\mathcal{F} \times \mathcal{C}$, i.e. the flux space $\mathcal{F}$ should be included as part of the moduli variables in the potential energy. The string theory landscape is thus pictured as the graph of a potential energy function $\mathcal{V}(W, Z)$ over $\mathcal{F} \times \mathcal{C}$, whose critical points are pairs $(G, Z) \in H^3(X, \mathbb{C}) \times \mathcal{C}$ with $G \in \mathcal{F}_{\mathbb{Z}}$ and $Z \in Crit(W_G)$; thus, Vacua_L is in $1-1$ correspondence with the critical points of $\mathcal{V}$. For a typical CY 3-fold, the third betti number b_3 is often around 300 and sometimes as high as $1,000$ (cf. [G, GHJ, Can1, Can2] for references). The space $\mathcal{F} \times \mathcal{C}$ has dimension $3b_3$ or roughly $1,000$ or so (depending on the topology of X). Thus, the potential is indeed a function of roughly $1,000$ variables.

It is not so easy to conclude from these facts how many critical points the potential should have, however. The first problem is that the critical point equation is C^∞ but not holomorphic, so vacua should be viewed as critical points of a real system of equations. Thus $\#Crit(W)$ varies with W and it is not obvious how many real critical points to expect even a polynomial of known degree to have (the answer depends on the probability measure one uses to define 'expect'). For that matter, a flux superpotential W is not a polynomial and it is not apriori clear how to assign it a 'degree' or other measure of complexity which reflects its number of critical points. Further, $\mathcal{C}$ is an incomplete Kähler manifold of rather small volume, so it is not really analogous to $\mathbb{R}^n$ in counting critical points. Such complications lie at the heart of the 'statistics of vacua'. In Theorem 5.1 of §5, we give an asymptotic formula for $\#\text{Vacua}_L$ which uses a mixture of lattice point methods and integral geometry to deal with these complications. Our rigorous count of the number of vacua of (2) lying in a given compact subset $K \subset \mathcal{C}$ shows that it is given asymptotically by

$$\#\text{Vacua}_L \cap K \sim C_{K,X} L^{b_3(X)}, \text{ as } L \to \infty, \tag{3}$$

where $C_{K,X}$ is a certain integral over K. Of course, such an asymptotic result only provides a good estimates when L is large (cf. §5 for more details). It is roughly consistent with the heuristic predictions of 10^{500} when $L \sim 1,000$, but its accuracy depends on the size of the coefficient $C_{K,X}$ and on the remainder estimate. The coefficient $C_{K,X}$ is (very roughly speaking) a measure of the number of critical points of an individual W in K. This number is often small, so it appears that the complexity of a given superpotential W is not large; it is rather the large number of possible fluxes G satisfying $0 \leq Q[G] \leq L$ which is responsible for the large value of $\#\text{Vacua}_L$. This is why we should view the landscape as rippling over $\mathcal{F} \times \mathcal{C}$.

Theorem 5.1 is only a first step in the rigorous study of statistics of vacua. Effective results would presumably be more useful than asymptotic results for counting vacua in a specific theory. Its limitations and the many issues it leaves unresolved are discussed in §5. However, at this stage in vacuum statistics, the main purpose is to provide a model for rigorous results which illuminates the complexities and subtleties underlying naive vacuum counting in string theory.

1.2. **Random polynomials and statistical algebraic geometry.** Although we are emphasizing statistics of string vacua, the study of zeros (and critical points) of random polynomials and holomorphic sections has its own intrinsic interest. Zeros of random holomorphic sections form the basis of statistical algebraic geometry– the probabalistic study of algebraic varieties when the coefficients of polynomial equations are viewed as random variables. We present a number of results on the geometry of zeros and critical points of random polynomials and sections in §2 - §3 and §6. They are motivated by the question, how do probabilities and correlations for random algebraic varieties behave as the complexity of the varieties increases? Here, complexity is usually measured by the degree of the defining equations but could also involve the number of variables (see [Z] for further background).

What are random polynomials or sections? A section may be expressed as a linear combination $S(z) = \sum_{j=1}^{N} c_j S_j(z)$ relative to a fixed basis $\{S_j\}$ of given space $\mathcal{S}$ of functions. We endow $\mathcal{S} \simeq \mathbb{C}^N$ with a probability measure P, so that the coefficients c_j become random variables. A 'random section' is an element of $(\mathcal{S}, P)$ which one thinks of as having expected (average) features.

Examples

Gaussian measures γ are very basic ones where $\mathcal{S}$ is equipped with an inner product $\langle , \rangle$, where $\{S_j\}$ is an orthonormal basis and where the coefficients are independent complex normal variables. They will be discussed extensively in §2 - §3. A basic invariant of Gaussian measures is the covariance kernel

$$\Pi_{\mathcal{S},\gamma}(z, w) = \mathbf{E}_{\gamma}(S(z)\overline{S(w)}, \tag{4}$$

which is the kernel of the orthogonal projection onto $\mathcal{S}$ (see Definition 3). It complex analysis it is often called the Szegö kernel. The asymptotic behavior of this kernel as the degree increases is often the most delicate aspect of the study of random zeros and critical points, and will be discussed in detail in §2 and below.

Non-Gaussian measures are of course of course also very important. One example of a non-Gaussian measure is the *spherical ensemble* defined by an inner product, whereby we endow the unit sphere $SH^0(M, L)$ with its $U(d)$-invariant probability measure, where $d = \dim \mathbf{H}^0(M, L)$. That is, we pick sections at random from the unit sphere with equal probability of picking any section. This ensemble is however only superficially different from the Gaussian ensemble. If one writes expected values or probabilities in polar coordinates, one easily relates spherical and Gaussian ensembles based on the same inner product. In the language of statistical mechanics, they are equivalent ensembles.

Another non-Gaussian example is the discrete ensemble where the coefficients c_j of z_j are independent ± 1 coefficients with equal probability of each sign. These are the ensembles studied by Bloch-Polya [BP] and Salem-Zygmund [SZ], among others. A more general discrete ensemble arises if one fixes a lattice L and a region Ω in the space of polynomials, and defines P by putting point-masses $a_n \delta_n$ at the lattice points $n \in L \cap \Omega$ so that the sum of the a_n equals one. For instance, one could choose the lattice to be the integral lattice in the coefficient space. This is the kind of measure which arises in the vacuum selection problem in string theory, although it is not normalized to be a probability measure. More generally, one could take the coefficients to be any kind of i.i.d. random variables with suitable regularity and moment conditions [IZ, DPSZ, BD, So].

Why study random polynomials in pure mathematics? One answer is that we can often obtain information about 'random polynomials' which is very difficult to obtain about any individual polynomial. How are its zeros distributed? How are its values distributed? How large are its (say, L^p) norms? Studying these aspects of individual polynomials inevitably forces one to spend most of the time on 'outliers' which exhibit extremal behavior. The extremals can be fascinating and beautiful. But the generic (average) polynomial demands equal time and is often what is more relevant. Indeed, concentration of measure phenomena take over when considering polynomials of large degree N, since then $\mathcal{P}_N^{(1)}$ is a Gaussian vector space of large dimension. Functionals of polynomials such as its norms become concentrated exponentially closely around their median values as $N \to \infty$ and outliers are very rare (see e.g. [SZ2] for results and references).

A second answer is that probabilistic information concerns the ensemble rather than the individual, and ensemble problems have their own fascination. To illustrate: "how likely is it to find two zeros of a polynomial of degree N which are within a disc of radius $1/N$ of each other? How likely is it to find no zeros in a given disc of radius 1?" are ensemble questions that make no sense for an individual polynomial. (See §6, and also [So], for discussions of these problems). As this answer suggests, the information one obtains reflects on both the objects involved (polynomials) and on the choice of probability measure P.

A physical answer to the question, 'why random polynomials' aside from string theory is that there is a long tradition in physics of modelling random fluctuations in physical systems by Gaussian random functions. For instance, the matter density fluctuations in the early universe are often modelled as Gaussian. The statistics of its peak points (local maxima) is considered to be relevant to the large scale distribution of galaxies in the universe [BBKS]. Of course, the matter density has since evolved into a very non-Gaussian form (cf. [BBKS, ABS]). The statistics of critical points of Gaussian random functions is also relevant to peak points of signals [Ri], speckle patterns [Hal], quantum chaotic eigenfunctions [B, Ze2, BBL, Han, NV, SZ], and metastable states in spin glasses (see [Fy] for references).

2. Zeros of Gaussian random polynomials of one variable

Random polynomials and more general random functions of one variable have a long history in mathematics, of which some representative early works are Bloch-Pòlya [BPo], Paley-Wiener-Zygmund [PWZ, PW], Littlewood-Offord [LO], Erdos-Turan [ET], Hammersley [Ham], M. Kac [K1, K2] and S.O. Rice [Ri].

Let us begin with the definition: Complex holomorphic polynomials in one variable form the complex vector space

$$\mathcal{P}_N^{(1)} = \{\sum_{j=1}^{N} c_j z^j, \quad c_j \in \mathbb{C}\} \simeq \mathbb{C}^N.$$

A 'random' polynomial is short for a probability measure P on the coefficients, and $(\mathcal{P}_N^{(1)}, P)$ is called an 'ensemble' of random polynomials. A random variable on the ensemble is simply a function $f(c)$ on $\mathcal{P}_N^{(1)}$ and its expected value is defined by

$$\mathbf{E}_P(f) = \int_{\mathcal{P}_N^{(1)}} f(c) dP(c).$$

Among the simplest and perhaps the most fundamental measures on $\mathcal{P}_N^{(1)}$ are the Gaussian measures. Let us postpone the general definition and consider one of the oldest and simplest examples: the 'Kac polynomial'

$$f(z) = \sum_{j=1}^{N} c_j z^j$$

where the coefficients c_j are independent complex Gaussian random variables of mean zero and variance one. That is, the Kac measure on $\mathcal{P}_N^{(1)}$ is the complex Gaussian measure for which the coefficients of the basis $\{z^j\}$ satisfy.

$$\mathbf{E}\,(c_j) = 0 = \mathbf{E}\,(c_j c_k), \quad \mathbf{E}\,(c_j \bar{c}_k) = \delta_{jk}. \tag{5}$$

Equivalently, in the coordinates c_j, it is the measure defined by

$$d\gamma_{KAC}(f) = e^{-|c|^2/2} dc.$$

(To be historically accurate, M. Kac [K1, K2] considered the analogous ensemble polynomials of one real variable and real coefficients. It was apparently Hammersley [Ham] who first considered the complex analogue.)

Most work on random polynomials concerns their zeros, so we begin with them. The distribution of zeros of a polynomial of degree N is the probability measure on $\mathbb{C}$ defined by

$$Z_f = \frac{1}{N} \sum_{z: f(z)=0} \delta_z, \tag{6}$$

where δ_z is the Dirac delta-function at z. We most often regard δ_z and other measures as linear functionals acting on test functions, and we then denote the pairing by $\langle \delta_z, \varphi \rangle = \varphi(z)$.

Given a probability measure P on $(\mathcal{P}_N^{(1)}$ we can define the expected distribution of zeros of random polynomials of degree N with measure P. It is the probability measure $\mathbf{E}_P Z_f$ on $\mathbb{C}$ defined by

$$\langle \mathbf{E}_P Z_f, \varphi \rangle = \int_{\mathcal{P}_N^{(1)}} \{ \frac{1}{N} \sum_{z: f(z)=0} \varphi(z) \} dP(f),$$

for $\varphi \in C_c(\mathbb{C})$. For most of the probability measures we study, it is a continuous (indeed, smooth) measure whose density relative to a given volume form measures the probability density of finding a zero near a point z.

How are zeros of complex Kac polynomials distributed? Kac studied real zeros of random real Kac polynomials, and found that the expected number of real zeros was of the very low order $\log N$. Hammersley [Ham] and Shepp-Vanderbei [SV] explained this low order by studying the expected distribution of complex zeros. They found that the complex zeros of random Kac polynomials of degree N concentrate in small annuli around the unit circle S^1. Since S^1 touches the real axis only at the two points ± 1, the real zeros therefore concentrate in thin intervals around these points. In the limit as the degree $N \to \infty$, the complex zeros asymptotically concentrate exactly on S^1, as follows:

THEOREM **2.1.** *[Kac, Hammersley, Shepp-Vanderbei] The expected distribution of zeros of polynomials of degree N in the Kac ensemble has the asymptotics:*

$$\mathbf{E}_{KAC}^N (Z_f^N) \to \delta_{S^1} \quad as \quad N \to \infty \text{ , where } \quad (\delta_{S^1}, \varphi) := \tfrac{1}{2\pi} \int_{S^1} \varphi(e^{i\theta})\, d\theta.$$

This concentration on S^1 may seem rather surprising. Do zeros of polynomials *really* tend to concentrate on S^1? The answer is a qualified 'yes': 'yes', for the polynomials which dominate the Kac measure $d\gamma^N_{KAC}$, 'no' for general polynomials. The concentration depends on the choice of the Kac measure: making the coefficients of $\{z^n\}$ independent complex normal variables had the effect of concentrating the expected distribution towards S^1.

The Kac measure achieved this by imposing an implicit choice of inner product on $\mathcal{P}_N^{(1)}$. This illustrates a basic fact: The choice of a Gaussian measure on a vector space $\mathcal{H}$ is equivalent to the choice of inner product on $\mathcal{H}$. The inner product induces an orthonormal basis $\{S_j\}$. The associated Gaussian measure dγ corresponds to random orthogonal sums

$$S = \sum_{j=1}^{d} c_j S_j,$$

where $\{c_j\}$ are independent complex normal random variables.

The inner product underlying the Kac measure on $\mathcal{P}_N^{(1)}$ is one for which the basis $\{z^j\}$ is orthonormal. They are orthonormal on S^1, and that is where the zeros concentrate. This suggests that if we orthonormalize the polynomials on the boundary $\partial\Omega$ of of another simply connected, bounded domain Ω, then the zeros will concentrate on $\partial\Omega$. This would justify our assertion that zeros of general polynomials do not concentrate on S^1.

To make this precise, we define the inner product on $\mathcal{P}_N^{(1)}$ by

$$\langle f, \bar{g}\rangle_{\partial\Omega} := \int_{\partial\Omega} f(z)\overline{g(z)}\,|dz| \ .$$

We let $\gamma^N_{\partial\Omega} =$ the Gaussian measure induced by $\langle f, \bar{g}\rangle_{\partial\Omega}$ and say that the Gaussian measure is adapted to Ω. We then denote the expectation relative to the ensemble $(\mathcal{P}_N,\, \gamma^N_{\partial\Omega})$ by $\mathbf{E}^N_{\partial\Omega}$.

Theorem **2.2.** [SZ1] *We have:*

$$\mathbf{E}^N_{\partial\Omega}(Z^N_f) = \nu_\Omega + O\left(1/N\right) \ ,$$

where ν_Ω is the equilibrium measure of $\bar{\Omega}$.

We recall that the equilibrium measure of a compact set K is the unique probability measure $d\nu_K$ which minimizes the energy

$$E(\mu) = -\int_K \int_K \log|z-w|\, d\mu(z)\, d\mu(w).$$

Thus, in the limit as the degree $N \to \infty$, random polynomials adapted to Ω act like electric charges in Ω.

We obtain the same asymptotic distribution of zeros if we use any analytic density $\rho|dz|$ on $\partial\Omega$ and, moreover, if we orthonormalize the polynomials in the interior of Ω relative to an analytic area form $\rho dz \wedge d\bar{z}$. This makes sense in the electron picture since electrons in a region would push each other to the boundary and distribute themselves uniformly with respect to $d\nu_K$.

As these results show, the asymptotic distribution of zeros of Gaussian random polynomials of degree N is highly sensitive on the choice of Gaussian measure (not to mention more general measures).

Let us give a few ingredients of the proofs which generalize to the line bundle setting. The first objects is the Szegö kernel of Ω with respect to a measure ρds on $\partial\Omega$, which is defined as the Schwartz kernel $S(z,w)$ of the orthogonal projection

$$S : \mathcal{L}^2(\partial\Omega, \rho ds) \to \mathcal{H}^2(\partial\Omega, \rho ds) \tag{7}$$

onto the Hardy space of boundary values of holomorphic functions in Ω which belong to $\mathcal{L}^2(\partial\Omega, ds)$. Since S is an orthogonal projection,we may express it in terms of any orthonormal basis $\{P_k\}$ as

$$S(z,w) = \sum_{k=0}^{\infty} P_k(z)\overline{P_k(w)}, \quad (z,w) \in \overline{\Omega} \times \overline{\Omega} \tag{8}$$

It is known that $S(z,z) < \infty$ for $z \in \Omega$, and thus $P_N \to 0$ on Ω.

We now choose the orthonormal basis to consist of the orthogonal polynomials of $\mathcal{L}^2(\partial\Omega, \rho ds)$. Since we are interested in polynomials in $\mathcal{P}_N$, we truncate the Szegö kernel to obtain the partial Szegö kernel

$$S_N(z,w) = \sum_{k=0}^{N} P_k(z)\overline{P_k(w)}. \tag{9}$$

A simple but basic calculation shows that it is the two-point function:

Proposition **2.3.** *Let f_N denote a random element of $\mathcal{P}_N$. Then*

$$S_N(z,w) = \mathbf{E}^N_{\partial\Omega}(f_N(z)\overline{f_N(w)}).$$

The proof is simple and completely general: write $f = \sum_{j=1}^n c_j P_j$, use (5) and obtain the right side of (9).

The next step, which is also valid in a very general context, is to express the distribution of zeros in terms of the two-point function:

Proposition **2.4.** *We have*

$$\mathbf{E}\,^N_{\partial\Omega,\rho}(Z_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log S_N(z,z).$$

The proof is very simple in this setting, and acquires new features in the line bundle setting. Since $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|z|^2 = \delta_0$, we have

$$Z_f = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|f|^2. \tag{10}$$

It follows that

$$\mathbf{E}^N_{\partial\Omega,\rho}(Z_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\mathbf{E}^N_{\partial\Omega,\rho}\left(\log|f|^2\right) \ . \tag{11}$$

To calculate the expectation, we write f in terms of the orthonormal basis $\{P_j\}$ of $\mathcal{P}_N$:

$$f(z) = \sum_{j=0}^{N} a_j P_j(z) = \langle a, p(z)\rangle \ ,$$

where $a = (a_0, \dots, a_N)$, $P = (P_0, \dots, P_N)$. Then,

$$\mathbf{E}^N_{\partial\Omega,\rho}(Z_f) = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\int_{\mathbb{C}^{N+1}} \log|\langle a, P(z)\rangle| \frac{1}{\pi^{N+1}} e^{-\|a\|^2}\, da.$$

We write

$$P(z) = \|P(z)\| u(z), \quad \|P(z)\|^2 = \sum_{j=0}^{N} |P_j(z)|^2 = S_N(z,z), \quad \|u(z)\| = 1 \; .$$

Then,

$$\log |\langle a, P(z)\rangle| = \log \|P(z)\| + \log |\langle a, u(z)\rangle| \; .$$

We observe that

$$\int_{\mathbb{C}^{N+1}} \log |\langle a, u(z)\rangle| \, e^{-\|a\|^2} \, da = \text{constant}$$

since for each z we may apply a unitary coordinate change so that $u(z) = (1, 0, \ldots, 0)$. Hence the derivative equals zero, and we have

$$\mathbf{E}^N_{\partial\Omega,\rho}(Z_f) = \frac{\sqrt{-1}}{\pi} \partial\bar\partial \log \|P(z)\| = \frac{\sqrt{-1}}{2\pi} \partial\bar\partial \log S_N(z,z) \; .$$

□

To complete the proof of Theorem 2.2, we have to understand the asymptotic behavior of the Szegö kernel $S_N(z,z)$. Up to this point, the arguments have been very general and did not really use the special features of the Kac-Hammersley ensemble. But at this point, the analysis of the two-point function $S_N(z,z)$, the argument becomes delicate and depends strongly on the context. In the proof of 2.2 one uses classical results of Szegö and Carleman relating the Szegö and Bergman kernels, as well as the equilibrium measure, to the exterior Riemann mapping function of Ω; since it does not generalize to the geometric setting, we stop the discussion here and refer the reader to [SZ1] for further discussion of the Kac-Hammersley case.

2.0.1. *$SU(2)$ polynomials.* Is there an inner product in which the expected distribution of zeros of polynomials of degree N is 'uniform' on $\mathbb{C}$, i.e. which doesn't concentrate anywhere? The answer is 'yes', if we take 'uniform' to mean uniform on $\mathbb{CP}^1$ with respect to the Fubini-Study area form ω_{FS}. The resulting ensemble is called the ensemble of $SU(2)$-polynomials.

The new feature is that we need to define an inner product on $\mathcal{P}^{(1)}_N$ which depends on N. We put:

$$\langle z^j, z^k \rangle_N = \frac{1}{\binom{N}{j}} \delta_{jk}. \tag{12}$$

Thus, a random $SU(2)$ polynomial has the form

$$f = \textstyle\sum_{j \le N} \; \lambda_j \, \sqrt{\binom{N}{j}} \; z^j,$$

$$\mathbf{E}(\lambda_j) = 0, \quad \mathbf{E}(\lambda_j \overline{\lambda}_k) = \delta_{jk}.$$

Proposition **2.5.** *In the $SU(2)$ ensemble, $\mathbf{E}_{SU(2)}(Z_f) = \omega_{FS}$, the Fubini-Study area form on $\mathbb{CP}^1$.*

This proposition is quite trivial if we make the right identifications. We recall that $\mathbb{CP}^1$ carries a tautological line bundle whose fiber at the point with homogeneous coordinates $[z_0, z_1] \in \mathbb{CP}^1$ is the line in $\mathbb{C}^2$ through (z_0, z_1). The dual line bundle is the hyperplane section line bundle $\mathcal{O}(1) \to \mathbb{CP}^1$ whose sections are linear functionals on the lines. The holomorphic sections of its Nth (tensor power) $\mathcal{O}(N) \to \mathbb{CP}^1$ are then polynomials of degree N, we we have an identification

$$\mathcal{P}_N^{(1)} \simeq H^0(\mathbb{CP}^1, \mathcal{O}(N)).$$

This space carries a natural $SU(2)$-invariant Fubini-Study inner product, defined by

$$\langle s_1, s_2 \rangle_{FS} = \int_{\mathbb{CP}^1} h_{FS}(s_1(z), s_2(z))_z \omega_{FS}(z).$$

Here, h_{FS} is the Fubini-Study hermitian metric on $\mathcal{O}(1)$ and ω_{FS} is its curvature $(1,1)$-form, the standard Fubini-Study area form. The integral can be put in a more familiar form if we homogenize $f(z) \in \mathcal{P}_N^{(1)}$ to $F(z_0, z_1) = z_0^N f(z_1/z_0)$: i.e. we introduce a new variable z_0 and homogenize each monomial z^j to $z_0^{N-j} z_1^j$ on $\mathbb{C}^2$. Sections of $\mathcal{O}(N)$ are equivalent to equivariant functions on the associated S^1 bundle $SU(2) \to \mathbb{CP}^1$, which in turn can be identified with the boundary of the unit ball $S^3 \subset \mathbb{C}^2$. The Fubini-Study inner product is then the usual inner product on the 3-sphere S^3. The inner product (12) can be easily calculated by homogenizing the monomials and calculating inner products on S^3.

Since the inner product, hence the Gaussian measure, are $SU(2)$ invariant, it follows that $\mathbf{E}_{SU(2)}(Z_f)$ is an $SU(2)$-invariant probability measure with a smooth density. There is a unique such measure of mass one, and therefore $\mathbf{E}_{SU(2)}(Z_f) = \frac{1}{Vol(\mathbb{CP}^1)} \omega_{FS}$.

3. Gaussian holomorphic sections of line bundles over Kähler manifolds

The ensemble of $SU(2)$ polynomials has an obvious generalization to m variables. We denote by $\mathcal{P}_N^m$ the space of complex polynomials

$$f(z_1, \dots, z_m) = \sum_{\alpha \in \mathbb{N}^m : |\alpha| \leq N} \lambda_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m},$$

of degree N in m complex variables with $c_\alpha \in \mathbb{C}$. As with one variable, we identify polynomials of degree N with holomorphic sections on the Nth power of the hyperplane section bundle:

$$\mathcal{P}_N^m \simeq H^0(\mathbb{CP}^m, \mathcal{O}(N)).$$

We further define the $SU(m+1)$- Gaussian measure γ_N^m by using the Fubini-Study inner product on the space $H^0(\mathbb{CP}^m, \mathcal{O}(N))$, which is defined as in the one-variable case by $||f||_{FS}^2 = \int_{S^{2m+1}} |F|^2 \, d\sigma$, where $F(z_0, \dots, z_m) = z_0^N f(z'/z_0)$ is the homogenization of f. A simple calculation gives

$$||z^\alpha||_{FS} = \binom{N}{\alpha}^{-1/2}, \;\; \langle z^\alpha, z^\beta \rangle = 0, \; \alpha \neq \beta.$$

We then define Gaussian random $SU(m+1)$ polynomials of degree N by

$$f = \textstyle\sum_{|\alpha| \leq N} \; \lambda_\alpha \sqrt{\binom{N}{\alpha}} \; z^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \overline{\lambda}_\beta) = \delta_{\alpha\beta}.$$

In coordinates λ_α:

$$d\gamma_N^m(f) = \frac{1}{\pi^{k_N}} e^{-|\lambda|^2} d\lambda \;\text{ on }\; \mathcal{P}_N^m.$$

By the same argument as for $m = 1$, the expected distribution of zeros is uniform with respect to the Fubini-Study volume form.

The $SU(m+1)$ ensemble in turn generalizes to any compact Kähler manifold of complex dimension m, and to any *positive* hermitian holomorphic line bundle $L \to M$. To define the term 'positive line bundle', we recall that in a local frame e, the hermitian metric is given by a positive function $h(z) = ||e||_z$. The curvature form is defined locally by

$$\Theta_h = \partial\bar{\partial}K, \qquad K = -\log h.$$

We say that (L, h) is positive if the (real) 2-form $\omega = \frac{\sqrt{-1}}{2}\Theta_h$ is positive, i.e., if ω is a Kähler form. Given one positive metric h_0 on L, the other metrics have the form $h_\varphi = e^\varphi h$. We denote the space of positive metrics by

$$P(M, L) = \{\varphi \in C^\infty(M) : \Theta_h = \Theta_{h_0} - \partial\bar{\partial}\varphi >> 0\}. \tag{13}$$

If (L, h) is positive, then $dV_h = \frac{\omega_h^m}{m!}$ defines a volume form for M, where $m = \dim_{\mathbb{C}} M$. We define the inner product

$$\langle s_1, s_2 \rangle_h = \int_M h(s_1(z), s_2(z)) dV_h(z).$$

This inner product determines a special Gaussian ensemble which depends only on the metric h.

Definition: Let $(L, h) \to M$ be a positive line bundle. We define the *Hermitian Gaussian measure* of (L, h) to be the Gaussian measure γ_h on $H^0(M, L)$ determined by $\langle, \rangle_h$, i.e.

$$s = \sum_j c_j S_j, \quad \langle S_j, S_k \rangle = \delta_{jk}$$

with $\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k)$, $\mathbf{E}(c_j \overline{c_k}) = \delta_{jk}$, where $\{S_j\}$ denotes an orthonormal basis of the space $H^0(M, L)$ relative to $\langle, \rangle_h$.

The Hermitian Gaussian measures seem the simplest and most natural ones in geometry. However, we caution the reader that the line bundle in string theory is *negative* and the Gaussian measures relevant to string/M theory are not Hermitian.

Given any inner product $\langle, \rangle$ on a subspace $\mathcal{S} \subset H^0(M, L)$, a key invariant is the following:

Definition: The two-point kernel of a Gaussian measure γ defined by $(\mathcal{S}, \langle, \rangle)$ is defined by

$$\Pi_{\mathcal{S}}(z, w) = \mathbf{E}_{\mathcal{S}}(s(z) \otimes \overline{s}(w)) \in L_z \otimes \overline{L_w}.$$

In a local frame e_L for L over $U \subset M$, we write

$$\Pi_{\mathcal{S}}(z, w) = F_{\mathcal{S}}(z, w) e_L(z) \otimes \overline{e_L}(w),$$

and call $F_{\mathcal{S}}(z, w)$ the *local two-point kernel*.

Here $\overline{L}$ denotes the complex conjugate of the line bundle L. $\Pi_{\mathcal{S}}$ can be written in the form

$$\Pi_{\mathcal{S}}(z, w) = \sum_{j=1}^n s_j(z) \otimes \overline{s_j(w)}\ ,$$

where $\{s_1, \dots s_n\}$ is an orthonormal basis for $\mathcal{S}$ with respect to the inner product $\langle , \rangle$ associated to the Gaussian measure γ. Indeed, as in Proposition 2.3 we have,

$$\mathbf{E}\left(s(z) \otimes \overline{s(w)}\right) = \mathbf{E}\left(\sum_{j,k=1}^{n} c_j \overline{c_k}\, s_j(z) \otimes \overline{s_k(w)}\right) = \sum_{j=1}^{n} s_j(z) \otimes \overline{s_j(w)}\ , \tag{14}$$

since the c_j are independent complex (Gaussian) random variables of variance 1.

3.1. **Density of zeros in Hermitian Gaussian ensembles.** The zero set Z_s of a holomorphic section is a hypersurface in M. Parallel to the definition (6) of the distribution of zeros of a random polynomial of one variable is the probability measure $|Z_s|$ on M defined by

$$\langle |Z_s|, \varphi \rangle = \frac{1}{\int_{Z_s} \omega_h^{m-1}} \int_{Z_s} \varphi \omega_h^{m-1}, \quad \varphi \in C(M). \tag{15}$$

We note that, by the Wirtinger formula, $\omega_h^{m-1}|_{Z_s}$ is the natural volume form on Z_s induced by the Kähler metric ω_h. The measure $|Z_s|$ is the total variation of the current of integration Z_s over the zero set, i.e.

$$\langle Z_s, \varphi \rangle = \int_{Z_s} \varphi, \quad \varphi \in \mathcal{D}^{m-1,m-1}.$$

The Poincaré-Lelong formula says that

$$Z_s = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log |f|^2 \tag{16}$$

where $s = fe$ locally in a frame e. Generalizing Proposition 2.4, we have:

PROPOSITION **3.1.** [BSZ2] *We have:*

$$\mathbf{E}_{\gamma_h}[Z_s] = \partial\bar{\partial} \log \Pi_h(z,z) - \Theta_h$$

where

$$\Pi_h : L^2(M, L) \to H^0(M, L)$$

is the orthogonal (Szegö) projector with respect to the inner product $\langle , \rangle_h$.

This formula leads to a kind of generalization of the asymptotic distribution result of Kac-Hammersley-Shepp-Vanderbei (Theorems 2.1 and 2.2) to positive holomorphic line bundles. In place of letting the degree N of the polynomial tend to infinity, we let the power $L^N := L^{\otimes N}$ of the line bundle tend to ∞. These limits are in fact of the same type, since $\mathcal{P}^{(N)} \simeq H^0(\mathbb{CP}^1, \mathcal{O}(N))$ and $\mathcal{O}(N) == \mathcal{O}(1)^{\otimes N}$. We choose the metric on L^N to have the form $h_N = h^N$.

THEOREM **3.2.** [SZ] *Let* $(L,h) \to M$ *be a positive line bundle, and endow* $H^0(M, L^N)$ *with its Hermitian Gaussian measures* γ_N *relative to* h^N. *Then the expected distribution of zeros w.r.t* γ_N *is given asymptotically by*

$$\frac{1}{N}\mathbf{E}_{\gamma_N}[Z_s] = \Theta_h + O(\frac{1}{N}).$$

Let us sketch the proof: By Proposition 3.1, we have

$$\frac{1}{N}\mathbf{E}_{\gamma_N}[Z_s] = \frac{1}{N}\partial\bar{\partial}\log\Pi_N(z,z) - \Theta_h,$$

where $\Pi_N = \Pi_{h^N}$. As is generally the case in distribution problems for zeros or critical points, the delicate step is the analysis of the behavior of the Szegö kernel $\Pi_N(z,w)$. In this case, we only need the diagonal asymptotics, which are known to have the form (cf. [Cat, Ze, Lu]).

$$\Pi_N(z,z) \sim N^m[1 + \frac{a_1(z)}{N} + \cdots]. \tag{17}$$

It follows that

$$\frac{1}{N}\partial\bar{\partial}\log\Pi_N(z,z) = O(\frac{1}{N}),$$

concluding the proof.

Thus zeros become equidistributed with respect to the curvature. A heuristic explanation of this reslt is that curvature causes sections to oscillate more rapidly that in flat regions, so zeros concentrate in the most highly curved regions.

We are more interested in the discrete set of critical points than zeros in this article, so we close this section by stating a result on the discrete zeros of m independent sections. Since the zeros of a full system of m polynomials (or holomorphic sections) in m variables form a discrete set, we define distribution of zeros of the system $(f_1,\dots,f_m)$ by

$$Z_{f_1,\dots,f_m} = \sum_{\{z_j : f_1(z_j) = \cdots = f_m(z_j) = 0\}} \delta_{z_j}.$$

We endow m-tuples of sections with the product measure so that they are independent random sections. We denote the expected distribution of the simultaneous zeros of a random system of m polynomials by $\mathbf{E}_N(Z_{f_1,\dots,f_m},\varphi)$. It is the average value of the measure $(Z_{f_1,\dots,f_m},\varphi)$ w.r.t. f. We thus define

$$\mathbf{E}_N(Z_{f_1,\dots,f_m})(U) = \int d\gamma_N(f_1)\cdots\int d\gamma_N(f_m) \times \left[\#\{z \in U : f_1(z) = \cdots = f_m(z) = 0\}\right],$$

for $U \subset \mathbb{C}^{*m}$, where the integrals are over $\mathcal{P}_N^{\mathbb{C}}$ or more generally over $H^0(M,L^N)$.

THEOREM **3.3.** [SZ] *In the Hermitian Gaussian ensemble on m independent copies of $H^0(M,L^N)$, we have:*

$$\frac{1}{N^m}\mathbf{E}_N(Z_{f_1,\dots,f_m}) \to \omega_h^m$$

in the sense of weak convergence; i.e., for any open $U \subset M$, we have

$$\tfrac{1}{(N)^m}\mathbf{E}_N\big(\#\{z \in U : f_1(z) = \cdots = f_m(z) = 0\}\big) \to m!\mathrm{Vol}_\omega(U).$$

The results surveyed above concern the distribution of zeros of one or several Gaussian random holomorphic sections. But the distribution of zeros is merely the simplest invariant of the zeros of an ensemble. Higher level invariants are given by the correlation functions of the zeros (or critical points), which measure whether zeros (or critical points) tend to repel, or clump together or ignore each other. We will meet them in §6, where we review a result from [BSZ3] which shows that this tendency depends on the dimension.

4. Critical points

Algebraic geometers are most interested in zeros of holmorphic sections, but with vacua of string theory in mind, we are more interested here in critical points. It turns out that the theory of critical points relative to metric connections is quite different from that of zeros. To understand why, let us recall the definition.

Definition: Let $(L, h) \to M$ be a Hermitian holomorphic line bundle over a complex manifold M, and let $\nabla = \nabla_h$ be its Chern connection. A critical point of a holomorphic section $s \in H^0(M, L)$ is defined to be a point $z \in M$ where $\nabla s(z) = 0$.

In a local frame e, the critical point equation for $s = fe$ reads:

$$\partial f(w) + f(w)\partial K(w) = 0, \tag{18}$$

where $K = -\log ||e(z)||_h$ and where $\partial f = \sum_{j=1}^m \frac{\partial f}{\partial z_j} dz_j$. The key difference with zeros is that the critical point equation is only C^∞ and not holomorphic since K is not holomorphic. In fact, the critical point equation is equivalent to $d \log ||s(z)||_h^2 = 0$, i.e. to the non-zero critical points of the metric norm $||s(z)||_h$ of the section.

The reader might wonder how such a notion of critical points relates to the more classical notion on $\mathbb{C}^m$,

$$\frac{\partial f}{\partial z_1}(w) = \cdots = \frac{\partial f}{\partial z_m}(w) = 0, \tag{19}$$

which is manifestly complex analytical [AGV, Mi]. There are actually two distinct ways to interpret this notion on manifolds. The first is that w is a *singular point.* For sections of line bundle, a singular point is a point where $s(w) = \nabla s(w) = 0$. For a polynomial f on $\mathbb{C}^m$ there is no change in geometry of the hypersurface $f = 0$ if one subtracts $f(w)$ so that $f(w) = 0$. Generic holomorphic sections $s \in H^0(M, L)$ have no singular points, but they are nevertheless interesting and important in both mathematics and physics (in string theory, they are the 'Minkowski vacua') and one can develop a conditional probability theory of them (in progress). The second interpretation is that the gradient ∇ in (19) compactifies to a *meromorphic flat connection* on $\mathbb{CP}^m$ and thus (19) genereralizes to the study critical points of sections relative to flat meromorphic connections. It is simpler than the C^∞ metric theory and is studied in [DSZ4].

But for supersymmetric vacua of string theory, the connection is a metric connection and the critical points are non-singular in general; and therefore (18) is the appropriate notion in this article.

4.1. **Critical points relative to Hermitian connections.**

The distribution of critical points of a fixed section s with respect to h (or ∇_h) is the measure

$$C_s^h := \sum_{z \in Crit(s,h)} \delta_z. \tag{20}$$

That is,

$$\langle C_s^h, \varphi \rangle := \sum_{z \in Crit(s,h)} \varphi(z). \tag{21}$$

We denote the set of critical points of s by $Crit(s,h)$ and its number by $\#Crit(s,h)$. Note that we do not normalize C_s^h to be a probability; this is because $\#Crit(s,h)$ is a complicated random variable.

As with the expected distribution of zeros, if we equip $H^0(M,L)$ with a probability measure γ, then we obtain a distribution of critical points. We need a rather general version for later application to string theory:

Definition: Let $\mathcal{S} \subset H^0(M,L)$ denote a subspace of holomorphic sections, let γ denote a Gaussian measure on $\mathcal{S}$, and let ∇_h denote a connection on L for a hermitian metric h. The (expected) distribution $\mathbf{E}_\gamma C_s^h$ of critical points $\nabla_h s(z) = 0$ of $s \in \mathcal{S} \subset H^0(M,L)$ with respect to ∇_h and γ is defined by

$$\langle \mathbf{E}_\gamma C_s^h, \varphi \rangle := \int_M \varphi(z) \mathcal{K}^{\mathrm{crit}}(z)\, dV(z) := \int_{\mathcal{S}} \left[\sum_{z : \nabla_h s(z) = 0} \varphi(z) \right] d\gamma(s)$$

A particularly important statistic is the **expected number** of critical points:

Definition:

$$\mathcal{N}^{crit}(h,\gamma) \quad = \quad \int_{\mathcal{S}} \#Crit(s,h) d\gamma(s).$$

If we use the Hermitian Gaussian ensemble, then $\mathbf{E}_{\gamma_h} C_s^h$ and $\mathcal{N}^{crit}(h,\gamma_h)$ are purely metric invariants; in that case, we omit the notation γ_h.

It is important to keep in mind that $\mathcal{N}^{crit}(h,\gamma)$ depends on both the connection ∇_h and the measure γ. This is in contrast to the number of simultaneous zeros of m generic holomorphic sections of a line bundle, or the number of critical points of random polynomials in the Kac-Hammersley ensembles, which are topological invariants; in the line bundle case, it is the Chern number $c_1(L)^m$ of $L \to M$. On the contrary, the number $\#Crit(s,h)$ of critical points of a section $s \in H^0(M,L)$–even a polynomial in $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ – is genuinely a random variable because the equation is non-holomorphic. The only topological invariant is the index sum $\sum_{z:\nabla s(z)=0} ind(z)$ where $ind(z)$ is the Morse index of the critical point.

The following questions about critical points seem fundamental:

- What is the maximum number
$$Max(L,h) = \max_{s \in H^0(M,L)} \#Crit(s,h)$$
of critical points of $s \in H^0(M,L)$ (one could add: of a given Morse index)? Is $Max(L,h)$ unbounded as h ranges over hermitian metrics on L? What about positive hermitian metrics? If we take powers (L^N, h^N), how does $Max(L^N, h^N)$ grow with N.
- To what extend is $\mathcal{N}^{crit}(h,\gamma)$ a topological invariant? I.e. although $\#Crit(s,h)$ depends on the metric h for an individual section, to what extend is the average number of critical points independent of h?
- Which metrics (if any) minimize $\mathcal{N}^{crit}(h,\gamma)$?

Intuitively, the number of critical points reflects the degree of the Kähler potential in (18) as well as the degree of the section. Unless there is some limit on the degree of K, one should expect $Max(L,h)$ to be unbounded as h varies over hermitian metrics. Requiring

$h = e^{\varphi} h_0 \in P(M, L)$ (see (13)) to be positively curved constrains the second derivative of h and possibly $Max(L, h)$, though this is unknown. Metrics minimizing $\mathcal{N}^{crit}(h, \gamma)$ are in some sense 'of least complexity'.

Before we can attempt to answer such questions, we need to calculate the expected distribution of critical points. We need a very general version in which the subspace $\mathcal{S} \subset H^0(M, L)$ can be a proper subspace, and in which the Gaussian measure and connection are completely independent.

4.2. **The expected distribution of critical points in dimension one.** We first give the formula for the expected distribution of critical points in dimension one.

THEOREM **4.1.** [DSZ1] *Let $(L, h) \to M$ be a Hermitian line bundle on a (possibly non-compact) Riemann surface M with area form dV. Let $\mu_1 = \mu_1(z)$, $\mu_2 = \mu_2(z)$ denote the eigenvalues of $\Lambda(z) Q_r$, where $r = r(z) = \frac{i}{2}\Theta_h / dV$. Then $\mathbf{K}^{\rm crit}_{\mathcal{S},\gamma,\nabla} = \mathcal{K}_{\mathcal{S},h,V}\, dV$, where*

$$\mathcal{K}^{\rm crit}_{\mathcal{S},h,V} = \frac{1}{\pi A} \frac{\mu_1^2 + \mu_2^2}{|\mu_1| + |\mu_2|} = \frac{1}{\pi A} \frac{Tr\, \Lambda^2}{Tr |\Lambda^{\frac{1}{2}} Q_r \Lambda^{\frac{1}{2}}|} \, .$$

Here,

$$Q_r = \begin{pmatrix} 1 & 0 \\ 0 & -r^2 \end{pmatrix} \, ,$$

and

$$A(z_0) = \left(\frac{\partial^2}{\partial z \partial \bar{w}} F_{\mathcal{S}}(z, w) \Big|_{(z,w)=(z_0,z_0)} \right) \tag{22}$$

and

$$\Lambda(z_0) = C(z_0) - B(z_0)^* A(z_0)^{-1} B(z_0) \, , \tag{23}$$

where

$$B(z_0) = \left[\left(\frac{\partial^3}{\partial z \partial \bar{w} \partial \bar{w}} F_{\mathcal{S}}(z, w) \right) \quad \left(\frac{\partial}{\partial z} F_{\mathcal{S}}(z, w) \right) \right] \Big|_{(z,w)=(z_0,z_0)} \, , \tag{24}$$

$$C(z_0) = \left[\begin{array}{cc} \left(\frac{\partial^4}{\partial z \partial z \partial \bar{w} \partial \bar{w}} F_{\mathcal{S}}(z, w) \right) & \left(\frac{\partial^2}{\partial z \partial z} F_{\mathcal{S}}(z, w) \right) \\ \left(\frac{\partial^2}{\partial \bar{w} \partial \bar{w}} F_{\mathcal{S}}(z, w) \right) & F_{\mathcal{S}}(z, z) \end{array} \right] \Bigg|_{(z,w)=(z_0,z_0)} \, , \tag{25}$$

To our knowledge, the first formula of this kind is due to S. O. Rice [Ri, Ria] in the case of real polynomials in one real variable. Related formulae for critical points of real random functions are given in [BBKS] and [Fy], with references to the earlier literature. The expression in Theorem 4.1 is one of several proved in [DSZ1, DSZ2], based on our earlier work on zeros [BSZ1].

4.2.1. *Exact formula for $\mathcal{N}^{crit}(h_{FS}, \gamma_{FS})$ on $\mathbb{CP}^1$.* In the special case of $\mathbb{CP}^1$ we can give a more precise formula:

THEOREM **4.2.** [DSZ1] *The expected number of critical points of a random section $s_N \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$ (with respect to the Gaussian measure γ_{FS} on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ induced from the Fubini-Study metrics on $\mathcal{O}(N)$ and $\mathbb{CP}^1$) is*

$$\frac{5N^2 - 8N + 4}{3N - 2} = \frac{5}{3}N - \frac{14}{9} + \frac{8}{27}N^{-1} \cdots .$$

This should be compared to the number $N-1$ of critical points of a polynomial of degree N in the classical sense of $f'(z) = 0$. That number is topological, since (as mentioned above) d/dz defines a flat meromorphic connection on $\mathbb{CP}^1$ with pole at infinity. The equation $f'(z) = 0$ is then holomorphic rather than C^∞, all of the critical points are saddle points, and their number is given by a Chern class. We find that when a smooth metric of positive $(1,1)$ curvature is used, then the number of critical points goes up: There are $\frac{N}{3}$ new local maxima and $\frac{N}{3}$ new saddles. Thus, the positive curvature of the Fubini-Study hermitian metric and connection causes sections to oscillate much more than the flat connection.

4.3. **Higher dimensions.** We now generalize the density formula to higher dimensions. For simplicity of notation, we assume that the holomorphic Hessian maps

$$H_z : \mathcal{S}_z \to \mathrm{Sym}(m, \mathbb{C}), \quad \text{with } \mathcal{S}_z = \{s \in \mathcal{S} : \nabla s(z) = 0\} \tag{26}$$

are surjective for all z, where $\mathrm{Sym}(m, \mathbb{C})$ denotes the space of complex symmetric matrices of rank m, and only indicate the modification at the end of the statement of the theorem when this does not hold. The modification is important because the Hessian is not surjective in the case of string vacua and black holes.

THEOREM **4.3.** [DSZ1] *Let $(L, h) \to M$ denote a holomorphic hermitian line bundle. Assume that $H_z : \mathcal{S}_z \to \mathrm{Sym}(m, \mathbb{C})$ is surjective for all z. Then there exist positive-definite Hermitian operators*

$$A(z) : \mathbb{C}^m \to \mathbb{C}^m \ , \quad \Lambda(z) : \mathrm{Sym}(m, \mathbb{C}) \oplus \mathbb{C} \to \mathrm{Sym}(m, \mathbb{C}) \oplus \mathbb{C}\ , \textit{such that}$$

$$\mathcal{K}^{\mathrm{crit}}(z) = \tfrac{1}{\det A(z) \det \Lambda(z)} \times \int_{\mathbb{C}} \int_{\mathrm{Sym}(m,\mathbb{C})} \left| \det \begin{pmatrix} H' & x\,\Theta(z) \\ \bar{x}\,\bar{\Theta}(z) & \bar{H}' \end{pmatrix} \right| e^{-\left\langle \Lambda(z)^{-1}(H' \oplus x),\, H' \oplus x \right\rangle} dH'\, dx\,.$$

Here, dH and dx denote Lebesgue measure, and $\Theta_h(z_0) = \sum_{j,q} \Theta_{jq} dz_j \wedge d\bar{z}_q$ is the curvature $(1,1)$ form.

If $H_z : \mathcal{S}_z \to \mathcal{H}_z$ maps to a proper subspace, then $\mathrm{Sym}(m, \mathbb{C}) = \mathcal{H}_z \oplus \mathcal{H}_z^\perp$ is the orthogonal decompositon into Λ_z-invariant subspaces for which $\Lambda_z >> 0$ and $\Lambda_z = 0$. The integral is then over $\mathcal{H}_z \oplus \mathbb{C}$ and Λ_z should be everywhere replaced by $\Lambda_z|_{\mathcal{H}_z \oplus \mathbb{C}}$.

When h is a positive or negative hermitian metric, one can choose coordinates and frames in which $\Theta = \pm I$ to obtain:

COROLLARY **4.4.** [DSZ1] *Let $(L, h) \to M$ denote a positive or negative holomorphic line bundle, and give M the volume form $dV = \frac{1}{m!}\left(\pm\frac{i}{2}\Theta_h\right)^m$ induced from the curvature of L. Then if $H_z : \mathcal{S}_z \to \mathcal{H}_z$ is surjective, we have*

$$K^{\mathrm{crit}}_{h,\mathcal{S}}(z) = \tfrac{1}{\det A \det \Lambda} \int_{\mathrm{Sym}(m,\mathbb{C}) \times \mathbb{C}} \left|\det(H'H'^* - |x|^2 I)\right| e^{-\langle \Lambda(z)^{-1}(H',x),(H',x)\rangle} dH'\, dx\,.$$

We make the same modification as above if H_z is not surjective.

By integrating over M, we obtain an explicit formula for the expected number $\mathcal{N}^{\rm crit}(h)$ of critical points of a Gaussian random holomorphic section. For the sake of simplicity we only state it for positive or negative metrics.

COROLLARY **4.5.** [DSZ1]

$$\mathcal{N}^{\rm crit}(h) = \int_M \{\tfrac{1}{\det A \det \Lambda} \int_{{\rm Sym}(m,\mathbb{C})\times\mathbb{C}} |\det(H'H'^* - |x|^2 I)|\, e^{-\langle \Lambda(z)^{-1}(H',x),(H',x)\rangle}\, dH'\, dx\} dV_h \,.$$

When γ is the Hermitian Gaussian measure, $\mathcal{N}^{\rm crit}(h)$ is a *purely metric* invariant of (L,h).

Above, the operators $A(z), \Lambda(z)$ depend only on ∇ and on the Szegö kernel $\Pi_{\mathcal{S}}$ and are defined as follows in the surjective case: Let $H = (H_{jq}) \in {\rm Sym}(m,\mathbb{C})$, and write

$$H = \sum_{1\le j\le q\le m} \widehat{H}_{jq}\, E^{jq}\,, \qquad \widehat{H}_{jq} = \tau_{jq} H_{jq}\,, \tag{27}$$

where $\tau_{jq} = \sqrt{2}$ if $j<q$, resp. $\tau_{jj} = 1$ for $1 \le j \le m\,,\ 1\le j\le q\le m\,,\ 1\le j'\le q'\le m$. Then denote an operator on ${\rm Sym}(m,\mathbb{C})\oplus\mathbb{C}$ by

$$\Lambda = \begin{bmatrix} \left(\Lambda_{jq}^{j'q'}\right) & \left(\Lambda_{jq}^{0}\right) \\ \left(\Lambda_{0}^{j'q'}\right) & \Lambda_0^0 \end{bmatrix}, \tag{28}$$

and define

$$\langle \Lambda(z)^{-1}(H,x),(H,x)\rangle = \sum (\Lambda^{-1})_{jq}^{j'q'}\, \widehat{H}_{jq}\overline{\widehat{H}_{j'q'}} + 2{\rm Re}\sum (\Lambda^{-1})_{jq}^{0}\widehat{H}_{jq}\bar{x} + (\Lambda^{-1})_0^0|x|^2. \tag{29}$$

As in Definition 3, let $F_{\mathcal{S}}(z,w)$ be the local expression for $\Pi_{\mathcal{S}}(z,w)$ in the frame e_L. Then $\Lambda = C - B^*A^{-1}B$, where

$$\begin{array}{l} A = \left(\frac{\partial^2}{\partial z_j \partial \bar{w}_{j'}} F_{\mathcal{S}}(z,w)|_{z=w} \right), \\ B = \left[\left(\frac{\partial^3}{\partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}}) F_{\mathcal{S}}|_{z=w} \right) \quad \left((\frac{\partial}{\partial z_j} F_{\mathcal{S}}|_{z=w} \right) \right], \\ C = \begin{bmatrix} \left(\frac{\partial^4}{\partial z_q \partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}}|_{z=w} \right) & \left(\frac{\partial^2}{\partial z_j \partial z_q} F_{\mathcal{S}} \right) \\ \left(\frac{\partial^2}{\partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}} \right)|_{z=w} & F_{\mathcal{S}}(z,z) \end{bmatrix}, \\ 1\le j\le m\,, 1\le j\le q\le m\,, 1\le j'\le q'\le m\,. \end{array}$$

In the above, A, B, C are $m\times m$, $m\times n$, $n\times n$ matrices, respectively, where $n = \frac{1}{2}(m^2+m+2)$.

The space ${\rm Sym}(m,\mathbb{C})\oplus\mathbb{C}$ arises in the surjective case since it parametrizes the possible Hessians of holomorphic sections at critical points. To explain this, we locally write Hessians in the form

$$D'\nabla' s(z_0) = \sum_{j,q} H'_{jq} dz_q \otimes dz_j \otimes e_L, \qquad D''\nabla' s(z_0) = \sum_{j,q} H''_{jq} d\bar{z}_q \otimes dz_j \otimes e_L\,. \tag{30}$$

In the local frame, the hermitian metric is represented by the function

$$|e_L(z)|_h^2 = e^{-K(z)}\,, \tag{31}$$

and thus for a section $s = f e_L \in H^0(M,L)$,

$$\nabla s = \sum_{j=1}^m \left(\frac{\partial f}{\partial z_j} - f \frac{\partial K}{\partial z_j} \right) dz_j \otimes e_L = \sum_{j=1}^m e^K \frac{\partial}{\partial z_j} \left(e^{-K} f \right) dz_j \otimes e_L\,. \tag{32}$$

Differentiating (32), we then obtain:

$$H'_{jq} = \frac{\partial^2 f}{\partial z_j \partial z_q}(z_0) \in \mathrm{Sym}(m, \mathbb{C}) \,, \tag{33}$$

$$H''_{jq} = -f \frac{\partial^2 K}{\partial z_j \partial \bar{z}_q}\Big|_{z_0} = -f(z_0)\Theta_{jq} \in \mathbb{C}\Theta_{jq} \,. \tag{34}$$

We assemble the blocks into the following 'complex Hessian':

$$H^c := \begin{pmatrix} H' & -f(z_0)\Theta \\ -\overline{f(z_0)\Theta} & \overline{H'} \end{pmatrix} .. \tag{35}$$

Under the surjectivity assumption, the image of the Hessian map is thus $\mathrm{Sym}(m, \mathbb{C}) \oplus \mathbb{C}$. As mentioned above, when the Hessian maps to a proper subspace, one simply restricts Λ_z to its positive spectral subspaces.

The absolute value surrounding the determinant in the density formula of Theorem 4.3 makes this formula difficult to analyze. In particular, Wick's formula for computing moments of Gaussian measures does not apply. In [DSZ2] an alternative Itzykson-Zuber type formula for the expected distribution is given which reduces the calculation to repeated residue integral calculations. In §6 we discuss the asymptotics of the distribution of critical points which reveals the dependence on the geometry of (L, h) of the integral.

We briefly indicate the proof of Theorem 4.3. By definition,

$$\int_M \varphi(z) \mathcal{K}^{\mathrm{crit}}(z)\, dV(z) := \int_{\mathcal{S}} \left[\sum_{z: \nabla_h s(z) = 0} \varphi(z) \right] d\gamma(s). \tag{36}$$

The integral on the right hand side may be regarded as an integral over the incidence relation

$$\mathcal{I} = \{(s, z) : s \in \mathcal{S}_z\} \subset \mathcal{S} \times M \tag{37}$$

where $\mathcal{S}_z$ is defined in (26). The integral over $\mathcal{I}$ is performed as an iterated integral with respect to the 'fibration' $\mathcal{I} \to \mathcal{S}$. The integration measure on $\mathcal{I}$ is sometimes written $\delta(\nabla s(z))|\det D\nabla s(z)|$. Here, $\delta(\nabla s(z)) = \nabla s(z)^* \delta_0$ stands for the Leray form in $\mathcal{I}$, namely the quotient form $\frac{\gamma \times dV(z)}{d\mathcal{E}}$, where $\mathcal{E}$ is the evaluation map $\mathcal{E}(s, z) = \nabla s(z)$. The factor $|\det D\nabla s(z)|$ is needed to cancel out the denominators which appear in $\nabla s(z)^* \delta_0$ but do not appear in $\left[\sum_{z: \nabla_h s(z) = 0} \varphi(z)\right]$. To obtain the expression in Theorem 4.3, it is only necessary to switch the order of integration to integrate first over the fibers of $\mathcal{I} \to M$.

4.4. **Morse index.** We are ultimately interested in local minima of sections of negative line bundles, so we want to take the Morse index of a critical point into account. To do this, it is only necessary to integrate over the subclass of symmetric matrices represented this Morse index.

We define the *topological index* of a section s at a critical point z_0 to be the index of the vector field ∇s at z_0 (where ∇s vanishes). The critical points of a section s are the critical points of $\log |s|_h$, and for positive line bundles L, we have

$$\mathrm{index}_{z_0}(\nabla s) = (-1)^{m + \text{Morse index}_{z_0}(\log |s|)} \,, \tag{38}$$

at (nondegenerate) critical points z_0.

THEOREM **4.6.** [DSZ1] *Let $(L,h)\to M$ be a positive holomorphic line bundle over a complex manifold M with volume form $dV=\frac{1}{m!}(\frac{i}{2}\Theta_h)^m$. Suppose that $H^0(M,L)$ contains a finite-dimensional subspace $\mathcal{S}$ with the 2-jet spanning property, and let γ be the Hermitian Gaussian measure on $\mathcal{S}$. Then the expected density with respect to dV of critical points of* $\log|s|_h$ *of Morse index q is given by*

$$\mathcal{K}^{\rm crit}_{\mathcal{S},h,q}(z)=\frac{\pi^{-\binom{m+2}{2}}}{\det A(z)\det\Lambda(z)}\int_{\mathbf{S}_{m,q-m}}\left|\det(SS^*-|x|^2I)\right|e^{-\langle\Lambda(z)^{-1}(S,x),(S,x)\rangle}\,dS\,dx\,.$$

where

$$\mathbf{S}_{m,k}=\{S\in\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}:\mathrm{index}(SS^*-|x|^2I)=k\}\ .$$

Let us consider the case of Riemann surfaces. The critical points of s of index 1 are the saddle points of $\log|s|_h$ (or equivalently, of $|s|_h^2$), while those of index -1 are local maxima of $\log|s|_h$ in the case where L is positive, and are local minima of $|s|_h^2$ if L is negative. (If L is negative, the length $|s|_h$ cannot have local maxima; if L is positive, the only local minima of $|s|_h$ are where s vanishes.) Thus, in dimension 1, topological index 1 corresponds to $\log|s|_h$ having Morse index 1, while topological index -1 corresponds to Morse index 2 if L is positive.

COROLLARY **4.7.** [DSZ1] *Let $(L,h)\to(M,dV)$, μ_1,μ_2 be as in Theorem 4.1. Then:*

- *The expected density of critical points of topological index* 1 *(where $|s|_h^2$ has a saddle point) is given by*
$$\mathcal{K}^{\rm crit}_{+}(z)=\frac{1}{\pi A(z)}\,\frac{\mu_1^2}{|\mu_1|+|\mu_2|}\ ,$$
- *The expected density of critical points of topological index -1 (where $|s|_h^2$ has a local maximum) is*
$$\mathcal{K}^{\rm crit}_{-}(z)=\frac{1}{\pi A(z)}\,\frac{\mu_2^2}{|\mu_1|+|\mu_2|}\ .$$
- *Hence, the index density is given by*
$$\mathcal{K}^{\rm crit}_{\rm index}:=\mathcal{K}^{\rm crit}_{+}(z)-\mathcal{K}^{\rm crit}_{-}(z)=\frac{1}{\pi A(z)}\,(\mu_1+\mu_2)=\frac{1}{\pi A(z)}\,Tr[\Lambda(z)Q]\ .$$

5. STRING THEORY

Having gained some experience with metric critical points of Gaussian random holomorphic sections, we return to vacuum statistics problem in string theory. As discussed in the introduction (see §1.1), the main problem is to count the number of vacua coming from all flux superpotentials satisfying a certain tadople constraint which are consistent with the known values of physical constants such as the cosmological constant. We will not discuss the physics of string vacua further, and refer the reader to M. R. Douglas' articles and lectures on the subject for movation and background (see also [D, AD, DD, DD2, DGKT, GKT, CQ, Gid]). We will confine our discussion to the mathematics of the vacuum selection problem. As a counting problem, it differs in several key aspects from the problems studied in §4 which we should highlight at the outset:

- The ensembles of sections relevant to string theory are discrete, i.e. the coefficients only take on discrete values lying in a set of lattice points. We mainly discuss the asymptotic regime in which the (scaled) values become dense and the ensembles are well approximated by Gaussian ensembles;
- The measure we put on the lattice of superpotentials is not normalized to be a probability measure: the purpose is to count all vacua coming from the collection of superpotentials which satisfy physical constraints, not to estimate probabilities of vacua satisfying constraints.

As this suggests, the distribution of vacua in string theory is a hybrid problem involving both lattice point problems and the complex geometry of critical points. The main result we describe, Theorem 5.1, is an asymptotic formula for the density of critical points of the physically relevant flux superpotentials as a certain *tadpole number* L increases. Its purpose is to provide a simple but reasonably accurate 'order of magnitude' estimate for numbers of vacua in the high tadpole number regime. More refined and difficult results will be briefly discussed at the end of this section.

5.1. **Type IIb compactifications on Calabi-Yau manifolds (or orientifolds) with flux.** Our first object is to describe the string theory models. The specific models we work with are called type IIb compactifications on Calabi-Yau manifolds (or orientifolds) with flux, which are described in a down-to-earth manner in [GKTT, GKT, AD] and which apparently originate in [GVW, GKP].

As mentioned in the introduction, the vacuum in such a model has the form $M^{3,1} \times X$ where X is a Calabi-Yau 3-fold. We recall that a Calabi-Yau d-fold is a compact complex Kähler manifold X of dimension d with trivial canonical bundle K_X, i.e. $c_1(X) = 0$. By the Calabi-Yau theorem, there exists a unique Ricci flat Kähler metric in each Kähler class on X. Roughly speaking, the dimension $10D$ of the space-time is needed for a consistent supersymmetric string theory, the fact that X is Kähler is needed for supersymmetry, and the Ricci flatness of X follows from the Einstein vacuum equations (cf. [CHSW], or [St]). In these models, any Calabi-Yau 3-fold could play the role of vacuum, or in other words, the vacuum degeneracy consists in the moduli space $\mathcal{M}_{CY}$ of Calabi-Yau (i.e. Ricci flat Kähler) metrics on a 3-fold X.

String theory makes contact with 'reality' in the effective supergravity theory it induces. Roughly speaking, effective supergravity is derived by 'integrating out' or neglecting the massive modes (positive eigenvalues) of various operators (cf. [St]). The data of effective supergravity consists of $(\mathcal{C}, \mathcal{L}, W)$ where:

1. $\mathcal{C}$ is the configuration space;
2. $\mathcal{L} \to \mathcal{C}$ is a holomorphic line bundle.
3. the superpotential W is a holomorphic section of $\mathcal{L}$.

In type IIb flux compactifications, the configuration space has the form

$$\mathcal{C} = \mathcal{M}_{CY} \times \mathcal{E}, \tag{39}$$

where $\mathcal{E} = \mathcal{H}/SL(2,\mathbb{Z})$ is the moduli space of elliptic curves. One may view $\mathcal{C}$ as a moduli space of Calabi-Yau metrics on the 4-fold $X \times T^2$, which are elliptic fibrations over X as in 'F theory' [V, MV]. The parameter $\tau \in \mathcal{H}/SL(2,\mathbb{Z})$ is known as the dilaton-axion parameter, and it has an important impact on the critical point equation.

The moduli space $\mathcal{M}_{CY}$ contains two sorts of moduli: Kähler class moduli and complex structure moduli. In what follows, as in [AD, DD, DSZ3] and elsewhere, we will simplify the problem by fixing the Kähler class. Once the Kähler class is fixed, the Calabi-Yau metrics are parametrized by complex structures on X. To make this point clear, we denote by $\mathcal{M}_{\mathbb{C}}$ the moduli space of complex structures on X and henceforth replace $\mathcal{M}_{CY}$ by $\mathcal{M}_{\mathbb{C}}$.

5.2. **Calabi-Yau geometry.** We now explain the Calabi-Yau moduli objects in more detail, following the references [Can1, Can2, Gr, GHJ, LS].

Let X be a complex 3-fold. The Teichmüller space of X is defined by

$$\mathcal{T}eich(X) = \{\text{complex structures on } \ X\}/Diff_0$$

where $J \sim J'$ if there exists a diffeomorphism $\varphi \in Diff_0$ isotopic to the identity satisfying $\varphi^* J' = J$. The moduli space $\mathcal{M}_{\mathbb{C}}$ of complex structures on X is the quotient of $\mathcal{T}eich(X)$ by the mapping class group

$$\Gamma_X := Diff(X)/Diff_0(X).$$

We often work on $\mathcal{T}eich(X)$ and treat $\mathcal{M}_{\mathbb{C}}$ as a fundamental domain $\mathcal{D}_{\Gamma_X}$ for Γ_X in $\mathcal{T}eich(X)$. The full symmetry group of $\mathcal{C}$ or more precisely its universal cover $\mathcal{T}eich(X) \times \mathcal{H}$ is then $\Gamma = \Gamma_X \times SL(2,\mathbb{Z})$.

The mapping class group Γ_X has a representation on middle-dimensional cycles or co-cycles $H^3(X,\mathbb{R})$ which preserves the intersection form

$$\eta(\alpha,\beta) = \int_X \alpha \wedge \beta.$$

The action of Γ_X on 3-cycles defines a homomorphism

$$\varphi : \Gamma_X \to Sp(2b_3,\mathbb{Z}).$$

According to a theorem of D. Sullivan, $\varphi(\Gamma_X)$ is an (arithmetic) subgroup of finite index in $Sp(2b_3,\mathbb{Z})$ and the kernel of φ is a finite subgroup.

Each choice of complex structure $z \in \mathcal{M}_{\mathbb{C}}$ gives rise to a Hodge decomposition

$$H^3(X,\mathbb{C}) = H^{3,0}_z(X) \oplus H^{2,1}_z(X) \oplus H^{1,2}_z(X) \oplus H^{0,3}_z(X) \tag{40}$$

into forms of type (p,q). We put as usual $h^{pq} = \dim_{\mathbb{C}} H^{p,q}$ and $b_3 = \dim_{\mathbb{R}} H^3(X,\mathbb{R})$. Thus, $h^{3,0} = h^{0,3} = 1$, $h^{1,2} = h^{2,1}$ and $b_3 = 2 + 2h^{2,1}$. We also have $h^{2,1} = \dim_{\mathbb{C}} \mathcal{M}_{\mathbb{C}}$, so that the high dimensionality of the configuration space is due to the topological complexity of X.

The intersection form defines a non-degenerate pairing of $H^{p,q}$ with $H^{q,p}$ whose sign depends only on the parity of p. One can modify η to make the real symmetric bilinear form

$$Q(\psi,\varphi) = i^3 \int_X \psi \wedge \bar{\varphi}. \tag{41}$$

The Hodge-Riemann bilinear relations for a 3-fold say that the form Q is definite in each $H^{p,q}_z(X)$ for $p+q=3$ with sign alternating $+-+-$ as one moves left to right in (40).

We now specify the line bundle $\mathcal{L}$. On a Calabi-Yau 3-fold, $\dim H^{3,0}_z(X) = 1$, and $H^{3,0}_z(X) \to \mathcal{M}_{\mathbb{C}}$ is a (holomorphic) line bundle known as the Hodge bundle. It carries a natural Hermitian metric

$$h_{WP}(\Omega_z,\Omega_z) = \int_X \Omega_z \wedge \overline{\Omega_z}$$

known as the Weil-Petersson metric. Thus, the Kähler potential on $\mathcal{M}_{\mathbb{C}}$ is

$$K(z,\bar{z}) = -\log \int_X \Omega_z \wedge \overline{\Omega_z}. \tag{42}$$

We denote the associated Chern metric connection by ∇_{WP}. We then define $\mathcal{L}_X$ to be the dual line bundle to the Hodge bundle, and endow it with the dual Weil-Petersson metric and connection. The hermitian line bundle $(H^{3,0}, h_{WP}) \to \mathcal{M}_{\mathbb{C}}$ is a positive line bundle, and it follows that $\mathcal{L}_X$ is a negative line bundle. We make a similar construction for the moduli space $\mathcal{E}$ of complex structurs on the complex 1-torus and then tensor the two bundles together to obtain a holomorphic line bundle $\mathcal{L} \to \mathcal{C}$ with its product Weil-Petersson metric.

5.3. **Flux superpotentials and tadpole constraint.** Type IIb flux compactifications contain two non-zero harmonic 3-forms $F, H \in H^3(X,\mathbb{Z})$ which are known respectively as the RR (Ramond-Ramond) and NS (Neveu-Schwartz) 3-form field strengths. These forms induce a superpotential a potential $V(Z)$ on $\mathcal{C}$ which 'stabilizes moduli', i.e. only the local minima of V can be vacua. We now define these objects.

On a compact manifold, a negative line bundle has no holomorphic sections. But $\mathcal{C}$ is non-compact and $H^0(\mathcal{C},\mathcal{L})$ has many holomorphic sections. The physically relevant sections are the *quantized, or integral, flux superpotentials*, which correspond to complex integral co-cycles ('fluxes') $G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$. Given $G = F + iH \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$, and $\tau \in \mathcal{H}$, physicists define the superpotential corresponding to G, τ by:

$$W_G(z,\tau) = \int_X (F - \tau H) \wedge \Omega_z. \tag{43}$$

To be more precise, we form the 4-form on $X \times T^2$

$$\tilde{G} = F \wedge dy + H \wedge dx$$

and define a linear functional on $H^{3,0}_z(X) \otimes H^{1,0}_\tau(T^2)$ by

$$\langle W_G(z,\tau), \Omega_z \wedge \omega_\tau \rangle = \int_{X \times T^2} \tilde{G} \wedge \Omega_z \wedge \omega_\tau. \tag{44}$$

When $\omega_\tau = dx + \tau dy$ we obtain the original formula. As $Z = (z,\tau) \in \mathcal{C}$ varies, (44) defines a holomorphic section of the line bundle $\mathcal{L}$ dual to $H^{3,0}_z \otimes H^{1,0}_\tau \to \mathcal{C}$. The same definition makes sense for any $G \in H^3(X,\mathbb{C})$. We denote by $\mathcal{F} \subset H^0(\mathcal{C},\mathcal{L})$ the space of all complex-valued flux superpotentials with dilaton-axion and by $\mathcal{F}_{\mathbb{Z}} \subset \mathcal{F}$ the lattice of sections corresponding to integral $G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$.

Without any constraint on G, there would be an infinite number of possible superpotentials, hence an infinity of vacua. This would destroy any vestige of predicitivity to string theory. However, there is a constraint on G which limits the number of vacua to a finite number, namely the *tadpole constraint*

$$Q[G] = \int F^{RR} \wedge H^{NS} \leq L \tag{45}$$

where as in (41), Q is the Hodge-Riemann form on $H^3(X,\mathbb{C})$. The number L is not a free parameter but is fixed by X. When there exists an involution g of X (an "orientifolding") and a Calabi-Yau 4-fold Z which is an elliptic fibration over X/g, then

$$L = \chi(Z)/24. \tag{46}$$

Since Q is indefinite, the integral fluxes G satisfying the constraint are lattice points in the hyperbolic shell (45) in $H^3(X,\mathbb{C})$.

5.4. **Critical points and Hodge decomposition.** As mentioned in the introduction, a superpotential W_G adds a scalar potential (1) to the Lagrangian of effective supergravity. Vacua are critical points of this real valued function. In this article, we only consider the special class of supersymmetric vacua where $\nabla_{WP} W(Z) = 0$. There is a similar but more complicated theory of non-supersymmetric vacua [DD2].

At a point $Z = (z,\tau)$, the supersymmetric critical point equation reads:

$$\nabla_{WP}(W_F(z) - \tau W_H(z)) = 0, \qquad W_H(z) - \tfrac{1}{\tau-\bar\tau}(W_F - \tau W_H) = 0. \tag{47}$$

Let us write this system of equations down explicitly in a local frame $\Omega_z \otimes \omega_\tau$ of the Hodge bundle with $\omega_\tau = dx + \tau dy$. We denote by $\Omega_z^* \otimes \omega_\tau^*$ the dual co-frames of $\mathcal{L}$. A holomorphic section of $\mathcal{L}$ can then be expressed as $W = f\Omega_z^* \otimes \omega_\tau^*$ where $f \in \mathcal{O}(\mathcal{C})$ is a local holomorphic function. When $W = W_G$, the coefficient as $f = \Pi_G$ is given by (43). It is essentially an integral linear combinations of periods of X with coefficient τ. More precisely, let us choose a symplectic basis $\{\alpha_j, \beta_k\}$ of $H_3(X,\mathbb{R})$ of A and B cycles with respect to the intersection form η, and let $\{\hat\alpha_j, \hat\beta_j\}$ be their Poincaré duals. Write $G = \sum_{j=1}^{b_3/2}[N_j^A \hat\alpha_j + iN_j^B \hat\beta_j]$. Then f_G may be written as

$$f_G(z,\tau) = \sum_{j=1}^{b_3/2} [N_j^A \Pi_{\alpha_j}(z) - \tau N_j^B \Pi_{\beta_j}(z)],$$

where $\Pi_\gamma(z) = \int_\gamma \Omega_z$. These period functions satisfy a Picard-Fuchs equation along curves and in special cases are given by generalized hypergeometric functions [Can1, Can2, Mor, BCDFHJQ]. The supersymmetric critical point equations then read:

$$\begin{cases} \sum_{i=1}^{b_3}\{(N^i_{RR} + \tau N^i_{NS})(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_j}K)\Pi_i(z) = 0, \\ \\ \sum_{i=1}^{b_3}(N^i_{NS} - \frac{1}{\tau-\bar\tau}(N^i_{RR} + \tau N^i_{NS}))\Pi(z) = 0. \end{cases} \tag{48}$$

where K is the Kähler potential (42).

An interesting and rather surprising feature of the critical point equation emerges if one fixes $(z,\tau) \in \mathcal{C}$ and views the critical point equation as an equation for a complex flux $G \in H^3(X,\mathbb{C})$. That is, for $(z,\tau) \in \mathcal{C}$, we define the vector space

$$\mathcal{F}_{z,\tau} := \{W_G : \ \nabla_{WP} W_G(\tau, z) = 0\}. \tag{49}$$

The spaces $\mathcal{F}_{z,\tau}$ are the fibers of the map $\pi : \mathcal{I} \to \mathcal{M}$ in (53). Then we have:

$$\nabla_{WP} W_G(z,\tau) = 0 \iff G \in H_z^{2,1} \oplus H_z^{0,3}, \tag{50}$$

i.e. $\mathcal{F}_{z,\tau} \simeq H_z^{2,1} \oplus H_z^{0,3}$.

In the language of complex symplectic geometry, the critical point equation is picking out a moving positive complex polarization of $H^3(X,\mathbb{C})$ for each (z,τ) . We recall that if (V,ω) is a real symplectic vector space and if $(V_\mathbb{C}, \omega_\mathbb{C})$ is its complexification, a complex Lagrangian subspace $F \subset V_\mathbb{C}$ is called a polarization. The polarization is complex if $F \cap \overline{F} = \{0\}$, and *positive* if $i\omega(v, \bar{w})$ is positive definite on F. In our case, $(V, \omega) = (H^3(X, \mathbb{R}), \eta)$, and a complex structure z on X determines the polarization

$$H^3(X, \mathbb{C}) = F \oplus \overline{F}, \;\; F = H^{3,0} \oplus H^{1,2}, \;\; \overline{F} = H^{2,1} \oplus H^{0,3}.$$

Clearly, F is complex Lagrangian, and by the alternating sign of the Hodge-Riemann form Q (41),

$$Q(v, w) = -i\eta(v, \overline{w}), \;\; v, w \in F$$

is a positive definite real quadratic form on $H^{2,1}_z \oplus H^{0,3}_z$. Hence F is a positive polarization. It is curious that the derivative in the additional parameter τ is needed in the critical point equation to obtain a positive polarization of $H^3(X, \mathbb{C})$.

Our main goal is to count solutions of the system of equations (47)- (48) in a region of moduli space as G varies over fluxes satisfying the tadpole constraint. Equivalently, to count inequivalent vacua in Teichmüller space. That is, Γ acts on the pairs (W, Z) of superpotentials and moduli by

$$\gamma \cdot (G, Z) = (\varphi(\gamma) \cdot G, \gamma \cdot Z),$$

Therefore Γ acts on the incidence relation (37), or more precisely on

$$\mathcal{TI} := \{(G, Z) \in (H^3(X, \mathbb{C}) \times (\mathcal{T} \times \mathcal{H}) : \nabla_{WP} W_G(Z) = 0\}. \tag{51}$$

We only wish to count critical points modulo the action of Γ. To do this, there are two choices: we could break the symmetry by fixing a fundamental domain $\mathcal{D}_\Gamma$ for Γ in $\mathcal{T} \times \mathcal{H}$, i.e. only count critical points in a fundamental domain. Or second, we could fix a fundamental domain for $\varphi(\Gamma)$ in $H^3(X, \mathbb{C})$ and count all critical points of these special flux superpotentials. We follow the first course.

The total number of critical points is a combination of two effects:

(1) The number of fluxes $G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$ satisfying $Q[G] \leq l$ for which W_G has a critical point in $\mathcal{D}_\Gamma$;
(2) The number of critical points of each such W_G.

It seems rather difficult to separate out these counting problems. As will be seen below, there are a lot of critical points if L is large, but they mainly come from (1) rather than from (2), and it is hard to determine how large (2) is. Intuitively, the number of critical points of W_G measures its 'complexity'. This number should be its number of critical points in all of $\mathcal{T}eich(X)$. If we only count critical points in $\mathcal{D}_\Gamma$, then the complexity of W_G is measured by the number of critical points in in $\mathcal{D}_\Gamma$ of the Γ-orbit of W_G. It seems difficult to tell from (48) alone how many critical points each W_G should have.

5.5. **Rigid Calabi-Yau's.** To illustrate the issues, let us consider the simplest case where the Calabi-Yau manifold X is rigid, i.e. $\mathcal{M}_{\mathbb{C}} = \{pt\}$ [AD]. Then only the parameter $\tau \in \mathcal{H}$ varies, and quantized flux superpotentials have the form

$$W_{A,B}(\tau) = A\tau + B, \;\; A = a_1 + ia_2, B = b_1 + ib_2 \in \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}.$$

The Kähler potential is given by $K = -\frac{1}{\tau - \bar{\tau}}$. Hence the critical point equation reads

$$\begin{array}{rcl} W'(\tau) - \frac{1}{\tau - \bar{\tau}} W & = & 0 \iff A\bar{\tau} + B = 0 \\ & & \iff \tau = -\overline{\frac{A}{B}} \end{array}.$$

We observe that each superpotential $A\tau + B$ has a unique critical point, which may or may not lie in a fundamental domain for $SL(2,\mathbb{Z})$ in $\mathcal{H}$. There is a unique superpotential in its $SL(2,\mathbb{Z})$-orbit whose critical point lie in the fundamental domain, so the union of the critical sets of all superpotentials in an $SL(2,\mathbb{Z})$-orbit contains one point in the fundamental domain.

Thus, counting critical points is equivalent to counting $SL(2,\mathbb{Z})$ orbits of superpotentials satisfying the tadpole constraint. The Hodge-Riemann form may be identified with the indefinite quadratic form

$$Q[(A,B)] = a_1 b_2 - b_2 a_1$$

on $\mathbb{R}^4$. The pair (A,B) corresponds to the element $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in GL(2,\mathbb{Z})$ and the quadratic form is its the determinant. The modular group $SL(2,\mathbb{Z})$ acts by the standard diagonal action on $(A,B) \in \mathbb{R}^2 \times \mathbb{R}^2$ preserving $Q[(A,B)]$ or equivalently by left multiplication preserving det. Thus, the set of superpotentials satisfying the tadpole constraint is thus parametrized by:

$$\{\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in GL(2,\mathbb{Z}) : \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \leq L\},$$

and we want to count the number of $SL(2,\mathbb{Z})$-orbits in this set. A fundamental domain for the $SL(2,\mathbb{Z})$ action on $GL(2,\mathbb{R})$ is:

$$\mathcal{D} = \{\begin{pmatrix} a_1 & b_1 \\ 0 & b_2 \end{pmatrix} \in GL(2,\mathbb{R}) : |b_1| \leq |b_2| \ \}.$$

Now let $\mathcal{D}_L$ denote this fundamental domain intersected with $\det < L$, or equivalently $0 \leq a_1 b_2 \leq L$ and count integral solutions. Since $|a_1| \geq 1$, $b_2 \leq L$ and then $|b_1| \leq L$. Counting the number of $SL(2,\mathbb{Z})$ orbits in $\mathcal{D}_L$ is thus equivalent to determining the average order of the classical divisor function $\sigma(m)$, see for instance Hardy-Wright [HW], Theorem 324:

$$\mathcal{N}^{\text{crit}}(L) = \sum_{m=1}^{L} \sum_{k|m} k = \sum_{m=1}^{L} \sigma(m) \sim \frac{\pi^2}{12} L^2. + O(L \log L). \tag{52}$$

5.6. **Statistics of vacua.** We now state the counting problems precisely and present our main results. We put counting measure on all lattice points $N \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}))$ satisfying $Q[N] \leq L$. Note that the number of lattice points in this region could be infinite. As in (37), we define the 'incidence relation'

$$\mathcal{I} = \{(W; z, \tau) \in \mathcal{F} \times \mathcal{C} : \nabla W(z,\tau) = 0\}. \tag{53}$$

Fix $\psi \in C_0^\infty(\mathcal{I})$, and define

$$N_\psi(L) = \sum_{N \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}) : Q[N] \leq L} \langle C_N, \psi \rangle, \tag{54}$$

where

$$\langle C_N, \psi \rangle = \sum_{(z,\tau):\nabla N(z,\tau)=0} \psi(N, z, \tau).$$

Problem Estimate $N_\psi(L)$ where L is the tadpole number.

An important example is the cosmological constant $\psi(N, z, \tau) = V_N(z, \tau)$, i.e. the value of the potential at the vacuum. Or, to count critical points in a compact subset $\mathcal{K} \subset \mathcal{C}$ of moduli space, we would put $\psi = \chi_K(z, \tau)$.

A nasty complication is the existence of a *real discriminant hypersurface* $\mathcal{D}$ of sections $W \in \mathcal{F}$ which have degenerate critical points, i.e the Hessian $D\nabla W(\tau)$ is degenerate. The number of critical points and the summand $\langle C_W, \psi \rangle$ jump across $\mathcal{D}$, so in $N_\psi(L)$ we are summing a discontinuous function. However, this complication seems to be irrelevant for the physical application: the Hessian $D\nabla W(z, \tau)$ of the superpotential at a critical point is the 'fermionic mass matrix'. A degenerate critical point would be one where massless fermions are observed. Such vacua are non-physical since there are no massless fermions. Hence, we will assume that *supp*ψ is disjoint from $\mathcal{D}$.

As mentioned above, L depends on X and can have a broad range of values. In this article and in [DSZ3], we assume that $L >> 1$ and thus consider the asymptotics of $N_\psi(L)$ as $L \to \infty$. We then have:

THEOREM **5.1.** *Suppose $\psi(W, z, \tau) \in C_b^\infty(\mathcal{F} \times \mathcal{C})$, with $\psi(W, z, \tau) = 0$ for $W \in \mathcal{D}$. Then*

$$\mathcal{N}_\psi(L) = L^{b_3} \left[\int_{\mathcal{C}} \int_{\mathcal{F}_{z,\tau}} \psi(W, z, \tau) |\det D\nabla W(z, \tau)| \chi_{Q_{z,\tau}}(W) dW dV_{WP}(z, \tau) + O\left(L^{-\frac{2b_3}{2b_3+1}}\right) \right].$$

Here, $b_3 = \dim H_3(X, \mathbb{R}), Q_{z,\tau} = Q|_{\mathcal{F}_{z,\tau}}$, and $\chi_{Q_{z,\tau}}(W)$ is the characteristic function of $\{Q_{z,\tau} \leq 1\} \subset \mathcal{F}_{z,\tau}$. Also C_b^∞ denotes bounded smooth functions.

Here, $D\nabla W(z, \tau)$ denotes the Hessian in the sense of (35). The integrand a priori includes the additional factor of $\frac{1}{\det A(Z)}$ as in Theorem 4.3 (for $A(Z)$ see (30)), but it happens to equal 1 due to the special geometry of the moduli space. We observe that the integral over $\mathcal{F}_{z,\tau}$ converges since $Q_{z,\tau}$ is a positive definite form. It is dual under the Laplace transform to the expected distribution of critical points with respect to the Gaussian measure $e^{-\langle Q_{z,\tau} W, W \rangle} dW$ on $\mathcal{F}_{z,\tau}$. In this sense, the Gaussian ensemble is a good approximation to the lattice ensemble for large L. Also, we note that there are further results of this kind for more general ψ, e.g. for $\psi = \chi_K(z, \tau)$ as above. However, the size of the remainder term then depends on K and for simplicity of exposition we confine ourselves to smooth test functions.

What does this imply about the number of vacua? As mentioned above, typical values of L are in the range $100 - 300$. Thus, it would be reasonable to estimate $L^{2b_3} \sim 100^{200}$, which is similar to the number 10^{500} quoted in the introduction. However, this is an over-simplification. To obtain a good estimate of the number of vacua one would need to estimate the size of the integral in the leading term and the dependence on b_3 and other parameters of the remainder.

Comparison with the expression in Theorem 4.3 shows that

$$\mathcal{K}^{\text{crit}}(z, \tau) = \int_{\mathcal{F}_{z,\tau}} \psi(W, z, \tau) |\det D\nabla W(z, \tau)| \chi_{Q_{z,\tau}}(W) dW \tag{55}$$

is analogous to the expected density of critical points with respect to the measure $\psi\chi_Q(W)dW$ in the hyperbolic shell $0 \leq Q[W] \leq 1$. The term 'expected density' is only formal since $\chi_Q(W)dW$ is not a probability measure and indeed has infinite volume. But in attempting to separate out the contributions of the number of fluxes and the number of critical points per flux, this density is a measure of the latter.

Further expressions for the leading coefficient are possible. First, as in Theorem 4.3, one can 'push forward' the integral (integrate over the fibers) with respect to the Hessian maps $W \to D\nabla W(z,\tau)$. As mentioned above, $\mathcal{F}_{z,\tau} = H^{3,0}_z \oplus H^{1,2}$ and the Hessian map on this space takes its values in a proper subspace of symmetric matrices which depends on the *prepotential* of the Calabi-Yau moduli space [DD, DSZ3]. Moreover, there is an Itzykson-Zuber type formula which seems best for numerical computations [DSZ3]. In the analogous problem for extremal black holes, it is easy to compute the resulting integrals, and one finds that the leading coefficient is bounded in the relevant parameters [DD, DSZ3]. However, it is not clear whether the same is true in the string/M vacuum problem.

5.7. **Statistics of vacua as a lattice point problem.** We briefly discuss the proof of Theorem 5.1. It involves summing over lattice points and then summing over the critical points of the holomorphic section corresponding to each lattice point. We now explain the lattice point aspect of Theorem 5.1.

A key property of the critical point equation $\nabla W = 0$ is that it is homogeneous in W, i.e. the critical points of a multiple cW are the same as those of W. Hence the summand $\langle C_N, \psi\rangle$ is homogeneous. We now observe that summing a homogeneous function (of degree 0) over lattice points is tantamount to studying the equidistribution of radial projections of lattice points on the surface $Q[W] = 1$. Equivalently, it amounts to counting lattice points in cones through regions of $Q[W] = 1$.

The simplest problem of this kind is to radially project all lattice points $N \in \mathbb{R}^n$ in the ball of radius $\sqrt{L}$ onto the unit sphere S^{n-1}. As $L \to \infty$, the lattice points become uniformly distributed on the sphere with respect to its usual $SO(n)$- invariant probability measure $d\omega$. The question is then, what is the rate of equidstribution? More generally, we could replace the unit sphere by any ellipsoid, i.e. the quadratic form $|x|^2$ by any elliptic quadratic form $Q[x]$.

Our problem actually involves a hyperbolic analogue of this problem: take a hyperbolic quadratic form $Q[x]$, and divide $\mathbb{R}^n$ into the interior $Q[x] > 0$ of the 'light-cone' $Q[x] = 0$ and the exterior $Q[x] < 0$. Radially project the lattice points in the hyperbolic shell $0 < Q[x] < L$ onto the surface $Q[x] = 1$ and determine their equidistribution law. Hyperbolic lattice point problems are much harder than the elliptic problem and the number of lattice points in the shell could be infinite. In the string problem, this problem is cured by the special nature of the summand $\langle C_W, \psi\rangle$: it vanishes unless the lattice point W lies within a sub-cone C of $Q > 0$ that depends on the support of ψ, namely the sub-cone of flux superpotentials with critical points in the support of ψ.

Thus, one of the two main ingredients in Theorem 5.1 is the following model problem: Let $\mathbf{Q} \subset \mathbb{R}^n$ ($n \geq 2$) be a smooth, star-shaped set with $0 \in \mathbf{Q}^\circ$ and whose boundary has a non-degenerate second fundamental form. Let $|X|_{\mathbf{Q}}$ denote the norm of $X \in \mathbb{R}^n$ defined by

$$\mathbf{Q} = \{X \in \mathbb{R}^n : |X|_{\mathbf{Q}} < 1\}.$$

Let $f \in C_0^\infty(\partial \mathbf{Q})$ and consider the sums

$$S_f(L) = \sum_{k \in \mathbb{Z}^n \cap L\mathbf{Q} \backslash \{0\}} f\left(\frac{k}{|k|_{\mathbf{Q}}}\right), \quad \text{with } \; L > 0.$$

We extend f to $\mathbb{R}^n$ as a homogeneous function of degree 0, so that $f(k) = f\left(\frac{k}{|k|_{\mathbf{Q}}}\right)$.

Theorem 5.2. [DSZ3] *If f is homogeneous of degree 0 and $f|_{\partial Q} \in C_0^\infty(\partial Q)$, then*

$$S_f(L) = L^n \int_{\mathbf{Q}} f \, dX + O(L^{n - \frac{2n}{n+1}}), \quad L \to \infty.$$

Theorem 5.2 is reminiscent of the estimate of van der Corput, Hlawka, Herz and Randol on the number of lattice points in dilates of a convex set [Ho, Ran]. It can be proved by a related harmonic analysis method (Poisson summation and stationary phase) combined with a dyadic decomposition to handle the singularity of the homogeneous function at 0. Despite its classical seeming nature, we have not found prior results on this equidistribution problems; the ones we know concern the more difficult problem of determining the distribution of lattice points of fixed 'height' $Q[x] = L$ (cf. [Pom, D, DO]), which involves delicate number theory and larger remainder estimates.

Theorem 5.2 can be generalized (in work in progress) to $f|_{\partial Q} = \chi_K$ where K is a smooth domain in ∂Q. However, the remainder estimate then depends on K. For instance, if $\partial Q = S^2$ and K is a polar cap, then the remainder measures the concentration of lattice points near ∂K, a lattitude circle. There is very little concentration compared to the main term when the lattitude circle has positive height above the $x - y$ plane, but $\sim L^2$ lattice points concentrate at the equator.

Applying Theorem 5.2 to the string/M problem gives that

$$\mathcal{N}_\psi(L) = L^{b_3}\left[\int_{\{Q[W] \le 1\}} \langle C_W, \psi \rangle \, dW + O\left(L^{-\frac{2b_3}{2b_3+1}}\right)\right]. \tag{56}$$

As discussed above, we then write (56) as an integral over the incidence relation (53) and change the order of integration to obtain the leading coefficient

$$\int_{\{Q[W] \le 1\}} \langle C_W, \psi \rangle \, dW = \int_{\mathcal{C}} \int_{\mathcal{F}_{z,\tau}} \psi(W, z, \tau) |\det D\nabla W(z,\tau)| \chi_{Q_{z,\tau}} dW dV_{WP}(z,\tau) \tag{57}$$

in Theorem 5.1. In general, this interchange brings in the additional Jacobian factor $\frac{1}{\det A(Z)}$ but here, as mentioned above, it equals one.

5.8. **Problems.** We close this section by highlighting some important problems on statistics of vacua which deserve further attention.

(1) How are the order of magnitudes of $b_3(X)$ and L of (46) related as X varies over Calabi-Yau manifolds?
(2) Obtain an effective estimate of the leading coefficient and remainder in Theorem 5.1, in particular their b_3-dependence. How large does L need to be to ensure that there exists a vacuum consistent with the standard model? Find examples of Calabi-Yau manifolds where it is certain that such a vacuum exists.

(3) Estimate the remainder if ψ does not vanish near the discriminant variety $\mathcal{D}$, or if ψ is a characteristic function of a smooth region $K \subset \mathcal{C}$.

(4) Separate out the number of fluxes which contribute to the sum and the number of critical points per flux. The first quantity is measured by the sum

$$\Theta_K(L) = \sum_{G \in H^3(X, \mathbb{Z} \oplus i\mathbb{Z}): Q[G] \leq L} \theta(\sum_{Z: \nabla W_G(Z)=0} \chi_K(Z)),$$

where $\theta(x) = 1$ for $x > 0$ and $= 0$ for $x \leq 0$. Find the asymptotics of $\Theta_K(L)$ as $L \to \infty$. The second quantity is the ration $\mathcal{N}_K(L)/\Theta_K(L)$.

6. Geometric asymptotics of zeros and critical points

In this final section, we return to the pure mathematics of critical points and discuss some results on density of and correlations between critical points which are suggested by statistics of vacua but which can be studied more effectively in model geometric settings that have nothing apparent to do with string theory. As discussed in the last section, the complications of the formula for the density of critical points given by Theorem 4.3 makes it difficult to estimate the leading order term of Theorem 5.1 on statistics of vacua. In fact, it is difficult to estimate on any manifold, not just the intrinsically difficult moduli space of Calabi-Yau metrics. We would like to understand how, in principle, the number of critical points of a 'random' holomorphic section depends on the geometry and dimensionality of the underlying manifold.

These dependences simplify in asymptotic regimes where the number of critical points grows quickly, i.e. in the limit of 'high complexity'. Two natural regimes of this kind suggest themselves: one is on a fixed manifold, where the 'degree' of the line bundle and sections is taken to infinity. The second is where one lets the dimension tend to infinity and fixes the degree of the line bundle. The former problem is a kind of semi-classical limit which has no apparent physical meaning in string theory, but is undertaken simply to reveal the key geometric features underlying the density of (and correlations between) critical points. The latter problem has not yet been studied in the setting of this paper. A recent paper of Fyodorov [Fy] studies a simpler but analogous model problem on $\mathbb{R}^N$ and shows that the expected number of critical points grows exponentially in the dimension. We therefore concentrate on high degree asymptotics of the density of and correlations between critical points.

6.1. Kac-Hammersley for critical points. Let us begin by proving the analogue of Theorem 2.2 for critical points. We would like to determine the limit distribution of critical points in the classical $\frac{d}{dz}$ sense in the Kac-Hammersley type ensembles of §2 as the degree $N \to \infty$. The distribution of critical points is then

$$C_f^N = \sum_{z: f'(z)=0} \delta_z.$$

THEOREM **6.1.** *Suppose that Ω is a simply-connected bounded $\mathcal{C}^\omega$ domain and ρ is a positive $\mathcal{C}^\omega$ density on $\partial\Omega$. Then the expected distribution of critical points for the Kac-Hammersley ensembles has the asymptotics,*

$$\mathbf{E}^N_{\partial\Omega,\rho}(\frac{1}{N-1} C_f^N) = \nu_\Omega + O\left(1/N\right) \ ,$$

where ν_Ω is the equilibrium measure of $\bar{\Omega}$.

Theorem 6.1 is new, but its proof is essentially the same as that of Theorem 2.2 for zeros. Analogously to Proposition 2.4,we have:

PROPOSITION **6.2.** *The expected distribution of critical points in the Kac-Hammersley ensemble is given by:*

$$\mathbf{E}\,^N_{\partial\Omega,\rho}(C_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log[\Delta S_N(z,z)].$$

Indeed, since critical points of f are zeros of f' we have $C_f^N = Z_{f'}^{N-1}$, and

$$C_f^N = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|f'|^2,$$

and so

$$\mathbf{E}^N_{\partial\Omega,\rho}(C_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\mathbf{E}^N_{\partial\Omega,\rho}\left(\log|f'|^2\right) .$$

Following through the calculation in the case of zeros, we get

$$\mathbf{E}^N_{\partial\Omega,\rho}(C_f) = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log\|\partial\bar{\partial}P(z)\| = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Delta S_N(z,z)\ .$$

□

The formula of Theorem 4.1 becomes identical with the 'flat' formula if the curvature Θ and r vanish.

As in the distribution of zeros, the Szegö kernel S_N comes up, but instead of analyzing its values we need to analyze the asymptotics of its derivatives. As mentioned above, the analysis of the Szegö kernel is the delicate aspect of the proof. In the case at hand, the asymptotic formula follows by tracing through the estimates in [SZ1] in the case of zeros. It turns out that the same estimates hold for the derivatives of the Szegö kernel, and hence the expected distribution of critical points has the same asymptotics as the distribution of zeros.

The asymptotic equality of the distributions of critical points and of zeros in these ensembles could be explained by the facts that critical points of random elements of $\mathcal{P}_N$ are zeros of random elements of $\frac{d}{dz}\mathcal{P}_N$, and that the linear map $\frac{d}{dz}:\mathcal{P}_N\to\mathcal{P}_{N-1}$ does not distort the Gaussian measures too much. Furthermore, the number $N-1$ of critical points (counted with multiplicity) is a topological invariant. None of these special features holds for smooth critical points.

6.2. **Positive line bundles.** We now consider the deeper problem of distribution of zeros on curved line bundles $L\to M$ over an m-dimensional complex manifold. We will assume that $L\to M$ is a positive line bundle (cf. §3). The space $H^0(M,L^N)$ of the Nth (tensor) power of L then has many holomorphic sections, $\dim H^0(M,L^N)\sim\frac{c_1(L)^m}{m!}N^m$. Moreover, the sections behave like polynomials of degree $Nc_1(L)$ and hence oscillate quickly. The following result determines the asymptotic number of critical points of random sections of the Nth power of such bundles:

THEOREM **6.3.** [DSZ2] *Let (L, h) be a positive hermitian line bundle. Let $\mathcal{N}^{\rm crit}(h^N)$ denote the expected number of critical points of random $s \in H^0(M, L^N)$ with respect to the Hermitian Gaussian measure (cf. Definition 3). Then,*

$$\mathcal{N}(h^N) = \tfrac{\pi^m}{m!}\Gamma^{\rm crit}_m\, c_1(L)^m\, N^m + \textstyle\int_M \rho dV_\omega N^{m-1} + C_m \int_M \rho^2 dV_\omega N^{m-2} + O(N^{m-3})\,.$$

Here, ρ is the scalar curvature of $\omega_h = \frac{i}{2}\Theta_h$, the curvature of h. As for $\mathbb{CP}^1$, the leading coefficient $\Gamma^{\rm crit}_m\, c_1(L)^m$ is larger than for a flat connection. It depends only on the dimension and hence is universal. A rather surprising feature of the asymptotic expansion is that the first two terms are topological invariants of a positive line bundle, i.e. independent of the metric (both are Chern numbers of L). But the third term

$$C_m \int_M \rho^2 dV_\Omega N^{m-2}$$

is a non-topological invariant as long as $C_m \neq 0$. It is a multiple of the Calabi functional. These calculations are based on the Tian-Yau-Zelditch (-Catlin) expansion of the Szegö kernel (cf. [Ze]) and on Zhiqin Lu's calculation of the coefficients in that expansion [Lu].

As one can see from (18), the expected number of critical points reflects both the complexity (degree) of the sections and the complexity of the Hermitian metric. A metric which minimizes the expected number of critical points is in this sense least complex. Which hermitian metrics minimize the expected number of critical points? This seems difficult to determine for a fixed bundle, but has an interesting answer in an asymptotic sense.

Definition: We say that $h \in P(M, L)$ (cf. (13)) is *asymptotically minimal* if

$$\exists N_0 :\ \forall N \geq N_0,\ \ \mathcal{N}(h^N) \leq \mathcal{N}(h_1^N), \quad \forall h_1 \in P(M, L). \tag{58}$$

From Theorem 6.3, we have:

THEOREM **6.4.** [DSZ2] *If $C_m > 0$, then the Calabi extremal metric in $c_1(L)$ is the unique asymptotically minimal metric.*

We recall that Calabi extremal metrics are metrics which are critical for $\int_M \rho^2 dVol_h$. When they exist, they are unique. We have proved $C_m > 0$, hence the conclusion of the conjecture, in dimensions $m \leq 5$ and conjecture $C_m > 0$ in all dimensions. A conjectural new formula for C_m has been proposed by B. Baugher on the basis of numerical calculations which easily implies that $C_m > 0$ in all dimensions. The formula is correct for $m \leq 5$ and that gives strong circumstantial evidence that it is true in all dimensions.

6.3. **Zero point and critical point correlations.** So far we have only discussed the distribution of zeros and critical points, but deeper and more interesting are their correlations, i.e. their tendency to repel or attract. In this section, we briefly review scaling asymptotics of correlations of zeros [BSZ1, BSZ3] and state the analogue for critical points in dimension one. We believe that analogues for critical points should exist in all dimensions.

As we saw in §2, zeros of random holomorphic polynomials in one dimension behave in some ways like electrically charged particles. But how far does this analogy go? Do the simultaneous zeros of m sections in dimension m repel each other like charged particles? Or behave independently like particles of an ideal gas? Or attract like gravitating particles?

Exactly the same questions can be posed for critical points. For these, the defining equations $\nabla W(w) = 0$ are dependent; does that make a difference?

These questions involve correlation functions, which may be defined intuitively as follows: Simultaneous zeros of m independent sections or critical points of one random holomorphic section on a manifold M define a *point process* on M, that is, a measure on the configuration space $Conf(X)$ of finite subsets of M. The critical point process is the measure on $Conf(M)$ which gives the probability distribution of $X \subset M$ being the critical point set of a holomorphic section. It is determined by its n-point correlations $\mathcal{K}(z_1, \dots, z_n)$ which give the probabilities of critical points occurring at the points $z_1, \dots, z_n \in M$. They determine whether critical points tend to cluster or to repel each other. More precisely, the n-point zero correlation function for zeros of m independent sections in dimension m is defined by

$$K^N_{nm}(z^1, \dots, z^n)dz = \mathbf{E}\, |Z_s|^n,$$

where $|Z_s|^n$ denotes the product of the measures $|Z_s|$ (cf. (15) on the punctured product $M_n = \{(z^1, \dots, z^n) \in M \times \cdots \times M : z^p \neq z^q \text{ for } p \neq q\}$ and where dz denotes the product volume form on M_n. In the case of m sections in dimension m, Z_s is of course a discrete measure given by summing delta-functions at the simultaneous zeros of the sections.

In [BSZ1, BSZ3] with P. Bleher and B. Shiffman, we proved that simultaneous zeros of m independent polynomials or sections of degree N behave almost independently if they are of distance $\geq \frac{D}{\sqrt{N}}$ apart for $D \gg 1$, i.e. only interact on distance scales of size $\frac{1}{\sqrt{N}}$. We therefore rescale the zeros in the $1/\sqrt{N}$-ball $B_{1/\sqrt{N}}(z_0)$ by a factor of $\sqrt{N}$ to get configurations of zeros with a constant density as $N \to \infty$. In [BSZ1]-[BSZ3], we proved that the rescaled correlation functions have *scaling limits*

$$\widetilde{K}^{\infty}_{nm}(z^1, \dots, z^n) = \lim_{N \to \infty} K^N_{1k}(z_0)^{-n} K^N_{nm}(z_0 + \frac{z^1}{\sqrt{N}}, \dots, z_0 + \frac{z^n}{\sqrt{N}}) \tag{59}$$

which are universal, i.e. independent of M, L, ω, h and depend only on the dimension m of the manifold. There exist explicit formulae for the pair correlation functions, i.e. $n = 2$. In this case, $\widetilde{K}^{\infty}_{2m}(z^1, z^2)$, depends only on the distance between the points z^1, z^2, since it is universal and hence invariant under rigid motions. Hence it may be written as:

$$\widetilde{K}^{\infty}_{2km}(z^1, z^2) = \kappa_{km}(|z^1 - z^2|)\,. \tag{60}$$

THEOREM **6.5.** *The pair correlation functions of zeros when $k = m$ are given by*

$$\kappa_{mm}(r) = \begin{cases} \frac{m+1}{4} r^{4-2m} + O(r^{8-2m})\,, & \text{as } r \to 0 \\ \\ 1 + O(e^{-Cr^2}), \ \ (C > 0) & \text{as } r \to \infty. \end{cases} \tag{61}$$

When $m = 1, \kappa_{mm}(r) \to 0$ as $r \to 0$ and one has "zero repulsion." When $m = 2$, $\kappa_{mm}(r) \to 3/4$ as $r \to 0$ and one has a kind of neutrality. With $m \geq 3$, $\kappa_{mm}(r) \nearrow \infty$ as $r \to 0$ and there is some kind of attraction between zeros; i.e., zeros tend to clump together in high dimensions.

We conjecture that the correlation functions of critical points also have scaling limits with a similar small distance behavior. The calculations are of a similar nature to zeros, but are much more complicated. They are a good deal easier for meromorphic flat connections, and it appears that in dimension one the scaling limit of the pair correlation function for critical points relative to a meromorphic connection is the same as for zeros. It is natural to conjecture that the similarity persists in higher dimensions and for smooth as well as meromorphic connections. Thus we conjecture that critical points tend to cluster together once the complex dimension of the configuration space is > 2, as tends to be the case for string theory models.

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