

LARGE DEVIATIONS OF EMPIRICAL ZERO POINT MEASURES ON RIEMANN SURFACES, I: $g = 0$

OFER ZEITOUNI AND STEVE ZELDITCH

ABSTRACT. We prove a large deviation principle for empirical measures

$$Z_s := \frac{1}{N} \sum_{\zeta: s(\zeta)=0} \delta_\zeta, \quad (N := \#\{\zeta : s(\zeta) = 0\})$$

of zeros of random polynomials in one variable. By random polynomial, we mean a Gaussian measure on the space $\mathcal{P}_N = H^0(\mathbb{CP}^1, \mathcal{O}(N))$ determined by inner products $G_N(h, \nu)$ induced by any smooth Hermitian metric h on $\mathcal{O}(1) \rightarrow \mathbb{CP}^1$ and any probability measure $d\nu$ on \mathbb{CP}^1 satisfying the weighted Bernstein-Markov inequality. The speed of the LDP is N^2 and the rate function is closely related to the weighted energy of probability measures on \mathbb{CP}^1 , and in particular its unique minimizer is the weighted equilibrium measure.

1. INTRODUCTION AND STATEMENT OF RESULTS

The purpose of this article is to establish a large deviations principle for the empirical measure

$$Z_s := d\mu_\zeta := \frac{1}{N} \sum_{\zeta: s(\zeta)=0} \delta_\zeta, \quad N := \#\{\zeta : s(\zeta) = 0\} \quad (1)$$

of zeros of a random polynomial s of degree N . Here, δ_ζ is the Dirac point measure at $\zeta \in \mathbb{C}$. We define random polynomials of degree N by putting geometrically defined Gaussian probability measures $d\gamma_N$ on the space \mathcal{P}_N of holomorphic polynomials of degree N , or equivalently, Fubini-Study measures dV_N^{FS} on the projective space $\mathbb{P}\mathcal{P}_N$ of polynomials (see §2 for background). The measures $d\gamma_N, dV_N^{FS}$ are determined by a pair $(h = e^{-\varphi}, \nu)$ consisting of a ‘weight’ φ or (globally) a Hermitian metric h on the hyperplane line bundle $\mathcal{O}(1) \rightarrow \mathbb{CP}^1$, and a probability measure ν on \mathbb{CP}^1 satisfying the Bernstein-Markov condition (9). The Gaussian measure on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ and the Fubini-Study measure on $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$ are induced from the Hermitian inner products

$$\|s\|_{G_N(h, \nu)}^2 := \int_C \|s(z)\|_{h,N}^2 d\nu(z), \quad (s \in \mathcal{P}_N). \quad (2)$$

The zeros then become equidistributed with high probability in the large N limit according to an equilibrium measure $d\nu_{h,K}$ depending on h and the support K of ν , which reflects the competition between the repulsion of nearby zeros and the force of the external electric field (curvature form) ω_h of h (see [SZ, Ber1, Ber2, BB]). The large deviations results show that the empirical measures (1) are concentrated exponentially closely (with speed N^2) to $\nu_{h,K}$ as $N \rightarrow \infty$, with rate given by a rate function $\tilde{I}^{h,K}$ that is minimized by $\nu_{h,K}$.

Date: April 26, 2009.

Research partially supported by NSF grant DMS-0804133 and by a grant from the Israel Science Foundation (first author), and by NSF grant DMS-0603850 (second author).

The large deviations rate function is determined from the joint probability density $D_N(\zeta_1, \dots, \zeta_N)$ of zeros, which measures the likelihood of a given configuration of N points arising as zeros of $s \in \mathbb{P}H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$. The joint probability density is the density of a joint probability current on the configuration space

$$(\mathbb{C}\mathbb{P}^1)^{(N)} = \text{Sym}^N \mathbb{C}\mathbb{P}^1 := \underbrace{\mathbb{C}\mathbb{P}^1 \times \dots \times \mathbb{C}\mathbb{P}^1}_N / S_N \quad (3)$$

of N points of $\mathbb{C}\mathbb{P}^1$. Here, S_N is the symmetric group on N letters. The joint probability current is by definition the pushforward

$$\vec{K}_n^N(\zeta_1, \dots, \zeta_N) := \mathcal{D}_* dV_N^{FS} \quad (4)$$

of the Fubini-Study measure on $\mathbb{P}\mathcal{P}_N$ under the ‘zero set’ or divisor map

$$\mathcal{D} : \mathcal{P}_N \rightarrow (\mathbb{C}\mathbb{P}^1)^{(N)}, \quad \mathcal{D}(p) = \zeta_1 + \dots + \zeta_N, \quad (5)$$

where $\{\zeta_1, \dots, \zeta_N\}$ is the zero set of s . Following a standard notation in algebraic geometry, we are writing an unordered set of points $\{\zeta_1, \dots, \zeta_N\}$ as a formal sum (i.e. a divisor) $\zeta_1 + \dots + \zeta_N \in (\mathbb{C}\mathbb{P}^1)^{(N)}$. In the case of polynomials, \mathcal{D} is obviously surjective (any N -tuple of points is the zero set of some polynomial of degree N); one may identify $\mathbb{P}\mathcal{P}_N \simeq (\mathbb{C}\mathbb{P}^1)^{(N)}$.

The zero set can also be encoded by the probability measure (1) on $\mathbb{C}\mathbb{P}^1$. This identification defines a map

$$\mu : (\mathbb{C}\mathbb{P}^1)^{(N)} \rightarrow \mathcal{M}(\mathbb{C}\mathbb{P}^1), \quad d\mu_{\zeta_1 + \dots + \zeta_N} = d\mu_\zeta := \frac{1}{N} \sum_{j=1}^N \delta_{\zeta_j}, \quad (6)$$

where $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$ is the (Polish) space of probability measures on $\mathbb{C}\mathbb{P}^1$, equipped with the weak-* topology (i.e. the topology induced by weak, or equivalently vague, convergence of measures). In general, for any closed subset $F \subset \mathbb{C}\mathbb{P}^1$ we denote by $\mathcal{M}(F)$ the probability measures supported on F . Thus, the zero sets can all be embedded as elements of the space $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$ of probability measures on $\mathbb{C}\mathbb{P}^1$. This point of view is ideal for taking large N limits, and has been previously used in many similar situations, for instance in analyzing the eigenvalues of random matrices [BG, BZ, HP].

Under the map $p \rightarrow \mu \circ \mathcal{D}(p)$ we further push forward the joint probability current to obtain a probability measure

$$\mathbf{Prob}_N = \mu_* \mathcal{D}_* dV_{FS}^N \quad (7)$$

on $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$. Our main results show that this sequence of measures \mathbf{Prob}_N satisfies a large deviations principle with speed N^2 and with a rate function $I^{h,K}$ reflecting the choice of (h, ν) . Roughly speaking this means that for any Borel subset $E \subset \mathcal{M}(\mathbb{C}\mathbb{P}^1)$,

$$\frac{1}{N^2} \log \mathbf{Prob}_N \{\sigma \in \mathcal{M} : \sigma \in E\} \rightarrow - \inf_{\sigma \in E} I^{h,K}(\sigma).$$

Before stating our results, we recall some notation and background. Throughout this article we use the language of complex geometry, and in particular we identify $\mathcal{P}_N = H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$, i.e. we identify polynomials of degree N with holomorphic sections of the N th power of the hyperplane line bundle [GH] (Chapter I.3). In the affine chart $U = \mathbb{C}\mathbb{P}^1 \setminus \{\infty\}$, and in the standard holomorphic frame $e : U \rightarrow \mathbb{C}\mathbb{P}^1$, the Hermitian metric h is represented by the function $\|e\|_h^2 = e^{-\varphi}$.

In weighted potential theory, the function φ is referred to as a weight [ST, B, B2]. Several authors have generalized weighted potential theory to Kähler manifolds, and we will use their geometric language [GZ, Ber1, Ber2, B, B2, BS]. In the local frame any holomorphic section may be written $s = fe$ where $f \in \mathcal{O}(U)$ is a local holomorphic function. The inner product (2) then takes the form,

$$\|s\|_{G_N(h,\nu)} = \int_{\mathbb{C}} |f(z)|^2 e^{-N\varphi} d\nu(z). \quad (8)$$

The measure ν is assumed to satisfy the weighted Bernstein-Markov condition (see [B2] (3.2) or [BB], Definition 4.3 and references):

For all $\epsilon > 0$ there exists $C_\epsilon > 0$ so that

$$\sup_K \|s(z)\|_{h^N} \leq C_\epsilon e^{\epsilon N} \|s\|_{G_N(h,\nu)}, \quad s \in H_0(\mathbb{CP}^1, \mathcal{O}(N)). \quad (9)$$

Here, and throughout this article, we write

$$K = \text{supp } \nu. \quad (10)$$

We further assume that

$$K \text{ is regular} \quad (11)$$

in the sense that every point of ∂K is a regular point. For this paper, the operative meaning of regular is that for any $z^* \in \partial K$, the capacity $\text{Cap}_h(D(z^*, \epsilon) \cap K) > 0$ for every $\epsilon > 0$, where Cap_h is the Green's capacity (see Definition 86). Here, $D(z^*, \epsilon)$ is a metric disc of radius ϵ ; the condition is independent of the choice of metric. We refer to §5.3 and to [Lan] for background on regular and irregular points of compact sets.

We then define the Gaussian probability measures $\gamma_{h^N, \nu}$ on $\mathcal{P}_N = H^0(\mathbb{CP}^1, \mathcal{O}(N))$ as the Gaussian measure determined by the inner product (2) (the definition is reviewed in §2.3). The associated Fubini-Study measures are denoted by $dV_{h^N, \nu}^{FS}$ on $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$ (see §3 for the definition).

In order to take advantage of the compactness of \mathbb{CP}^1 , we now introduce some notions of Kähler potential theory which will be discussed further in §4. Let ω_h be the curvature (1, 1) form of a smooth Hermitian metric h on \mathbb{CP}^1 . The Green's function G_h relative to ω_h is defined to be the unique solution $G_h(z, \cdot) \in \mathcal{D}'(\mathbb{CP}^1)$ of

$$\left\{ \begin{array}{l} (i) \quad dd_w^c G_h(z, w) = \delta_z(w) - \omega_h(w), \\ (ii) \quad G_h(z, w) = G_h(w, z), \\ (iii) \quad \int_{\mathbb{CP}^1} G_h(z, w) \omega_h(w) = 0, \end{array} \right. \quad (12)$$

where the equality in the top line is in the sense of (1, 1) forms. Existence of G_h is guaranteed by the $\partial\bar{\partial}$ Lemma even when ω_h is non-positive, i.e. is not a Kähler form; uniqueness follows from condition (iii). As shown in Lemma 8 of §4, in the frame $e(z)$ over the affine chart \mathbb{C} in which $h = e^{-\varphi}$ and $\omega_h = dd^c\varphi$, the Green's function has the local expression,

$$G_h(z, w) = 2 \log |z - w| - \varphi(z) - \varphi(w) + E(h), \quad (13)$$

where

$$E(h) := \left(\int_{\mathbb{CP}^1} \varphi(z) \omega_h + \rho_\varphi(\infty) \right), \quad (14)$$

ρ_φ being a certain Robin constant (see (109) of §8.2, and also (115) and (62)).

The Green's potential of a measure μ (with respect to ω_h) is defined by

$$U_h^\mu(z) = \int_{\mathbb{CP}^1} G_h(z, w) d\mu(w), \quad (15)$$

and the Green's energy by

$$\mathcal{E}_h(\mu) = \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} G_h(z, w) d\mu(z) d\mu(w). \quad (16)$$

We will see, c.f. Lemma 26 and Proposition 27, that under Assumptions (9) and (11), for any $\mu \in \mathcal{M}(\mathbb{CP}^1)$, $\mathcal{E}_h(\mu) < \infty$ and $|\sup_K U_h^\mu| < \infty$. In particular, the function

$$I^{h,K}(\mu) = -\frac{1}{2}\mathcal{E}_h(\mu) + \sup_K U_h^\mu, \quad \mu \in \mathcal{M}(\mathbb{CP}^1) \quad (17)$$

is well-defined (with $+\infty$ as possible value). Set

$$E_0(h) = \inf_{\mu \in \mathcal{M}(\mathbb{CP}^1)} I^{h,K}(\mu), \quad \tilde{I}^{h,K} = I^{h,K} - E_0(h). \quad (18)$$

The infimum $\inf_{\mu \in \mathcal{M}(\mathbb{CP}^1)} I^{h,K}(\mu)$ is achieved at the Green's equilibrium measure $\nu_{h,K}$ with respect to (h, K) , and $E_0(h) = \frac{1}{2} \log \text{Cap}_{\omega_h}(K)$, where $\text{Cap}_{\omega_h}(K)$ is the Green's capacity. See Lemma 4 (proved in §7.4). By the Green's equilibrium measure we mean the minimizer of $-\mathcal{E}_h$ on $\mathcal{M}(K)$. We refer to §5 for definitions and discussion of $\nu_{h,K}$ and of $\text{Cap}_{\omega_h}(K)$ (See (86)).

1.1. Statement of results. Our main result is the following:

THEOREM 1. *Let h be a smooth Hermitian metric on $\mathcal{O}(1) \rightarrow \mathbb{CP}^1$ and let $d\nu \in \mathcal{M}(\mathbb{CP}^1)$ satisfy the Bernstein-Markov property (9) and the regularity assumption (11). Then $\tilde{I}^{h,K}$ of (18) is a convex rate function and the sequence of probability measures $\{\mathbf{Prob}_N\}$ on $\mathcal{M}(\mathbb{CP}^1)$ defined by (7) satisfies a large deviations principle with speed N^2 and rate function $\tilde{I}^{h,K}$. Further, there exists a unique measure $\nu_{h,K} \in \mathcal{M}(\mathbb{CP}^1)$ minimizing $\tilde{I}^{h,K}$, namely the Green's equilibrium measure of K with respect to h .*

(Recall that a function $I : \mathcal{M}(\mathbb{CP}^1) \rightarrow \mathbb{R}$ is a rate function if it is lower semicontinuous and non-negative.)

Theorem 1 shows that the empirical measures $d\mu_\zeta$, see (1), concentrate near $\nu_{h,K}$ at an exponential rate. More precisely, if $B(\sigma, \delta)$ denotes the ball of radius δ around $\sigma \in \mathcal{M}(\mathbb{CP}^1)$ in the Wasserstein metric, and $B^\circ(\sigma, \delta)$ (respectively, $\overline{B(\sigma, \delta)}$) denote its interior (respectively, its closure), then

$$\begin{aligned} -\inf_{\mu \in B^\circ(\sigma, \delta)} \tilde{I}^{h,K}(\mu) &\leq \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) \leq -\inf_{\mu \in \overline{B(\sigma, \delta)}} \tilde{I}^{h,K}(\mu). \end{aligned} \quad (19)$$

This implies afortiori that the expected value of $d\mu_\zeta$ tends to $\nu_{h,K}$, refining the result of [SZ] on the equilibrium distribution of zeros in the unweighted case and the more general results in the subsequent articles [B, BS, Ber1, Ber2]. Intuitively, in the unweighted case, zeros repel each other like electrons to the outer boundary of K . A Hermitian metric or weight $h = e^{-\varphi}$

with $\omega_\varphi > 0$ behaves like an uphill potential which pushes electrons back into the interior of K and gives rise to an equilibrium potential which charges the interior of K , with extra accumulation along ∂K .

The inner product (2) depends only on the restriction of the metric h to K , see (10), and consequently the rate function should only depend on this restriction. To see this, we rewrite it in the standard affine chart \mathbb{C} and frame for $\mathcal{O}(1)$ in the form

$$-\frac{1}{2}\mathcal{E}_h(\mu) + \sup_K U_h^\mu = -\Sigma(\mu) + \sup_{z \in K} \left\{ 2 \int_{\mathbb{C}} \log |z - w| d\mu(w) - \varphi(z) \right\}, \quad (20)$$

where

$$\Sigma(\mu) = \int_{\mathbb{C} \times \mathbb{C}} \log |z - w| d\mu(z) d\mu(w) \quad (21)$$

is the logarithmic energy or entropy function. In the large deviations analysis, it is more convenient to use the formula in Theorem 1 which uses the ‘compactification’ of the metric to \mathbb{CP}^1 .

1.2. Examples. As an illustration of the methods and results, we observe that Theorem 1 applies to the Kac-Hammersley ensemble as in [SZ], where $d\nu = \delta_{S^1}$ (the invariant probability measure on the unit circle), and where the weight $e^{-\varphi} = 1$. Hence, the inner product is simply $\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$. It is simple to verify that $d\nu = \delta_{S^1}$ satisfies the Bernstein-Markov property, i.e. that for holomorphic polynomials of degree N , $\|p_N\|_{S^1} \leq C_\epsilon e^{\epsilon N} \frac{1}{2\pi} \left(\int |p_N(e^{i\theta})|^2 d\theta \right)^{1/2}$. Indeed, we let $\Pi_N(z, w) = \sum_0^N z^n \bar{w}^n$ denote the Szegő reproducing kernel for \mathcal{P}_N with this measure. Then by the Schwarz inequality,

$$\sup_{z \in S^1} |p_N(z)| \leq \sup_{z \in S^1} \sqrt{\Pi_N(z, z)} \|p_N\|_{L^2(\nu)} \leq \sqrt{N} \|p_N\|_{L^2(\nu)}.$$

On S^1 we are taking the weight to be ‘flat’, i.e. the Hermitian metric to be $\equiv 1$. We are free to choose a smooth extension of this Hermitian metric to $\mathcal{O}(1) \rightarrow \mathbb{CP}^1$. For instance, we may take $h = e^{-\varphi}$ to be S^1 invariant, equal 1 in a neighborhood of \mathbb{CP}^1 and to equal the Fubini-Study metric in a neighborhood of ∞ . There is of course no unique choice of the smooth extension. With any of these extensions, δ_{S^1} is easily seen to satisfy the condition (11).

At the opposite extreme, the methods and results apply to the case where $d\nu = \omega_{FS}$, the Fubini-Study Kähler form, and where $h = h_{FS} = e^{-\log(1+|z|^2)}$. The Bernstein-Markov property follows from the same calculation as in the Kac-Hammersley example, except that the Szegő reproducing kernel is different (but still equals $N + 1$ on the diagonal; see [SZ, SZ2, SZ3] for further background). The regularity condition is obviously satisfied.

1.3. An application - hole probabilities. The large deviation results give an accurate upper bound for ‘hole probabilities’ for our ensembles of Gaussian random polynomials of one complex variable. A hole probability for an open set U is the probability that the random polynomial has no zeros in U . Large deviations estimates for hole probabilities for balls $U = B_R$ of increasing radius were proved in [SoTs] for certain random analytic functions. More in line with the present paper are asymptotic hole probabilities as the degree $N \rightarrow \infty$ of random holomorphic sections of powers $L^N \rightarrow M$ of positive line bundles in [SZZr].

The results there hold in all dimensions, but the stronger assumption is made that h is a Hermitian metric with positive curvature $(1, 1)$ form.

We now state a hole probability for our general Gaussian ensembles on \mathbb{CP}^1 , where the hole is an open set $U \subset \mathbb{CP}^1$. We consider the

$$A_U = \{\mu \in \mathcal{M}(\mathbb{C}) : \mu(U) = 0\}.$$

The following hole probability has the same speed of exponential decay as in [SZZr].

COROLLARY 1. *For any of the Gaussian ensembles $G_N(h, \nu)$ and for any open set U ,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(A_U) \leq - \inf_{\mu \in A_U} \tilde{I}^{h,K}(\mu). \quad (22)$$

Proof. If $\mu_n \rightarrow \mu$ weakly in $\mathcal{M}(\mathbb{C})$ then $\liminf_{n \rightarrow \infty} \mu_n(U^c) \leq \mu(U^c)$. Thus, A_U is a closed set, both in $\mathcal{M}(\mathbb{C})$ and (with a slight abuse of notation) in $\mathcal{M}(\mathbb{CP}^1)$. The upper bound is then immediate from Theorem 1. \square

Unfortunately, the large deviation principle is not quite strong enough to provide complementary lower bounds. Indeed, the set

$$A_U^o = \{\mu \in \mathcal{M}(\mathbb{C}) : \mu(U) = 1\}$$

has empty interior for any set $U \neq \mathbb{CP}^1$, and the large deviations lower bound is $-\infty$. The best one can obtain from the LDP is that, with U closed and the set

$$A_U^{p,o} = \{\mu \in \mathcal{M}(\mathbb{C}) : \mu(U) > p\},$$

one has by a similar analysis

$$\lim_{p \nearrow 1} \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(A_U^{p,o}) \geq - \inf_{\mu \in A_U^o} \tilde{I}^{h,K}(\mu). \quad (23)$$

The constrained infimum $\inf_{\mu \in A_U} \tilde{I}^{h,K}(\mu)$ is achieved by a measure $\nu_{U,h,K}$, which may be regarded as a relative weighted equilibrium measure with respect to the two independent sets U, K . In general it is impossible to evaluate numerically. In a special case, we can however evaluate it. With $r < 1$, let $U^c = \bar{B}_r \subset \mathbb{C}$ be the closed ball of radius r centered at the origin. Set

$$A_r = \{\mu \in \mathcal{M}(\mathbb{C}) : \mu(\bar{B}_r) = 1\}.$$

COROLLARY 2. *For the Kac-Hammersley ensemble, and for $r < 1$, we have*

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(A_r) \leq \log r. \quad (24)$$

Proof. By Corollary 1,

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(A_r) \leq - \inf_{\mu \in A_r} \tilde{I}^{h,K}(\mu). \quad (25)$$

We specialize the last expression in the case of the Kac-Hammersley ensemble: written in the affine chart around 0, we have

$$\tilde{I}^{h,K}(\mu) = -\Sigma(\mu) + 2 \sup_{z \in S^1} \int_{\mathbb{C}} \log |z - w| d\mu(w),$$

where we used that for any $R \geq 1$,

$$\inf_{\mu \in A_R} I^{h,K}(\mu) = -\Sigma(\nu) + 2 \int_{\mathbb{C}} \log |1 - w| d\nu(w) = 0,$$

with $\nu = \delta_{S^1}$ the uniform distribution on S^1 .

Fix $r < 1$. For given $\mu \in A_r$, let $\tilde{\mu}$ denote the radial symmetrization of μ , that is, for any measurable $A \subset \mathbb{C}$,

$$\tilde{\mu}(A) = \frac{1}{2\pi} \int_0^{2\pi} \int \mathbf{1}_{ze^{i\theta} \in A} d\mu(z) d\theta.$$

Due to the convexity of A_r and of $I^{h,K}(\cdot)$, the minimizer μ^* in the right side of (25) is radially symmetric, i.e. $\tilde{\mu}^* = \mu^*$, and supported in A_r . Using the identity, valid for any $s \leq 1$,

$$\int_0^{2\pi} \log |1 - se^{i\theta}| d\theta = 0, \quad (26)$$

we thus obtain

$$\inf_{\mu \in A_r} \tilde{I}^{h,K}(\mu) = \left[\inf_{\mu \in A_r, \mu = \tilde{\mu}} -\Sigma(\mu) \right] + \Sigma(\delta_{S^1}) = \inf_{\mu \in A_r, \mu = \tilde{\mu}} -\Sigma(\mu).$$

For $\mu \in A_r$ with $\mu = \tilde{\mu}$, write $\mu = \rho(dr) \times d\theta$, with $\rho \in \mathcal{M}([0, r])$. Then,

$$\begin{aligned} \Sigma(\mu) &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^r \int_0^r \log |se^{i\theta} - s'e^{i\theta'}| \rho(ds) \rho(ds') \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^r \int_0^r \log |s - s'e^{i\theta'}| \rho(ds) \rho(ds') \\ &= - \int_0^r \int_0^r (\log s') [2\mathbf{1}_{s' > s} + \mathbf{1}_{s=s'}] \rho(ds) \rho(ds'), \end{aligned}$$

where we used (26) in the last equality. The last expression is maximized (over $\rho \in \mathcal{M}([0, r])$) at $\rho_r = \delta_r$. □

1.4. Discussion of the proof. Functions somewhat similar to (17) or (20) arise as rate functions in large deviations problems for empirical measures of eigenvalues of random matrices (see e.g. [BG, BZ]). In particular, much of the analysis of the energy term $\mathcal{E}_h(\mu)$ can be carried over from the eigenvalue setting and from known results in classical weighted potential theory on \mathbb{C} (see [ST]). We recall that the weighted equilibrium measure of K with respect to the weight e^{-Q} is the unique maximizer in $\mathcal{M}(K)$ of the weighted energy function,

$$\Sigma_{Q,K}(\mu) = \int_K \int_K \log (|z - w| e^{-Q(z)} e^{-Q(w)}) d\mu(w) d\mu(z). \quad (27)$$

We observe that $Q = \frac{\varphi}{2}$ in the global setting, i.e. the weight is essentially a Hermitian metric.

However, the (non-differentiable) sup function $J^{h,K}(\mu) := \sup_K U_h^\mu$ is quite different from, and somewhat more difficult than, the linear functions such as $\int x^2 d\mu$ which occur in the eigenvalue setting. Under Assumption (11), we show that it is convex and continuous on $\mathcal{M}(\mathbb{CP}^1)$ with respect to weak convergence (which, due to the compactness of \mathbb{CP}^1 , is equivalent to vague convergence) of probability measures (Proposition 27). The continuity also uses the fact that the Green's function G_h is bounded above on \mathbb{CP}^1 .

It is not obvious that the minimizer of $I^{h,K}$ should be the same as the minimizer of (27). This is proved in Proposition 30. The main differences are that (i) \mathcal{E}_h is not constrained to measures supported on K ; (ii) the second \sup_K term is additional to the energy. In Proposition 30, we show that $\nu_{h,K}$ minimizes both $-\mathcal{E}_h$ and $J^{h,K}$.

Besides potential theory, an important ingredient in the proof of Theorem 1 is a formula for the joint probability current of zeros when \mathcal{P}_N is endowed with the Gaussian measure derived from the inner product (2). A novelty of our presentation is that we derive the joint probability current in a natural way from the associated Fubini-Study probability measure on $\mathbb{P}\mathcal{P}_N$. In the following, we work in the standard chart, i.e. $(\mathbb{C})^{(N)}$. Let $\Delta(\zeta) = \prod_{i < j} (\zeta_i - \zeta_j)$ denote the Vandermonde determinant, $s_\zeta(\cdot) \in \mathcal{P}_N$ the polynomial with zero set $\{\zeta_1, \dots, \zeta_N\}$, and $d^2\zeta = d\zeta \wedge \bar{d}\zeta$ on \mathbb{C} .

In the following, and hereafter, we often use the following identity:

$$\int_{\mathbb{C}\mathbb{P}^1} G_h(z, w) dd^c \log \|s_\zeta(w)\|_{h^N}^2 = N \int G_h(z, w) d\mu_\zeta(w) = NU_h^{\mu_\zeta}(z), \quad (28)$$

which follows from the definitions (1) and (12) and from the Poincaré-Lelong formula (see (38)).

PROPOSITION 3. *The joint probability current (4) (see also (52)) is given in the affine chart $(\mathbb{C})^N \subset (\mathbb{C}\mathbb{P}^1)^N$ by*

$$\vec{K}_n^N(\zeta_1, \dots, \zeta_N) = \frac{1}{Z_N(h)} \frac{|\Delta(\zeta_1, \dots, \zeta_N)|^2 d^2\zeta_1 \cdots d^2\zeta_N}{\left(\int_{\mathbb{C}\mathbb{P}^1} \prod_{j=1}^N |(z - \zeta_j)|^2 e^{-N\varphi(z)} d\nu(z) \right)^{N+1}} \quad (29)$$

$$= \frac{1}{\hat{Z}_N(h)} \frac{\exp\left(\sum_{i < j} G_h(\zeta_i, \zeta_j)\right) \prod_{j=1}^N e^{-2N\varphi(\zeta_j)} d^2\zeta_j}{\left(\int_{\mathbb{C}\mathbb{P}^1} e^N \int_{\mathbb{C}\mathbb{P}^1} G_h(z, w) d\mu_\zeta d\nu(z) \right)^{N+1}}. \quad (30)$$

where G_h is the Green's function (12). Also, $Z_N(h)$ and $\hat{Z}_N(h)$ are normalizing constants, defined so that the measure on the left side has mass one.

The second expression (30) is invariantly defined. We will see that

$$\begin{cases} Z_N(h) = |\det \mathcal{A}_N(h)|^{-2} = \text{Vol}(\mathcal{B}^2(L^N, h^N)), & \text{see (47)} \\ \hat{Z}_N(h) = |\det \mathcal{A}_N(h)|^{-2} e^{-(-\frac{1}{2}N(N-1) + N(N+1))E(h)}, & \text{see (14)}. \end{cases} \quad (31)$$

Here, $\mathcal{A}_N(h)$ is the change of basis matrix from the monomials z^j to an orthonormal basis for the inner product $G_N(h, \nu)$ on $H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$. In Lemma 18, we further rewrite the expression for $\vec{K}_n^N(\zeta_1, \dots, \zeta_N)$ as a functional $I_N^{h, \nu}(\mu_\zeta)$ on the measures μ_ζ . The rate function I is then extracted from I_N as $N \rightarrow \infty$.

To complete the calculation, we need to determine the logarithmic asymptotics of $\hat{Z}_N(h)$.

LEMMA 4. *We have,*

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \hat{Z}_N(h) = \frac{1}{2} \log \text{Cap}_{\omega_h}(K).$$

This limit formula gives an alternative approach to the asymptotics of $|\det \mathcal{A}_N(h)|^2$ from that in [BB], in this one dimensional setting.

1.5. **Sketch of proof for Kac-Hammersley.** We now sketch the proof of Theorem 1 in the case of the Kac-Hammersley example. In this case, we do not need the geometric language used in the rest of the paper.

Consider the polynomial $P_N(z) = \sum_{i=0}^N a_i z^i$, where the a_i are independent Gaussian circular standard complex random variables. We have $P_N(z) = a_N \prod_{i=1}^N (z - z_i)$. Further, conditioned on a_N , the variables $\{b_i\}_{i=0}^{N-1}$ are independent, circular normal, of variance $|a_N|^{-2}$.

Let $\Delta = \prod_{i < j} |z_i - z_j|$. Then, the Jacobian of the transformation $\{b_i\}_{i=0}^{N-1} \mapsto \{z_i\}_{i=1}^N$ is $|\Delta|^2$. On the other hand, with $d\mu_\xi \in \mathcal{M}(\mathbb{C})$ denoting the empirical measure of the zeros of P_N ,

$$\begin{aligned} |a_N|^2 + \sum_{i=0}^{N-1} |a_N b_i|^2 &= \sum |a_i|^2 \\ &= (2\pi)^{-1} \int_{S^1} P_N(z) P_N^*(z) dz \\ &= \frac{|a_N|^2}{2\pi} \int_{S^1} \prod |z - z_i|^2 dz = \frac{|a_N|^2}{2\pi} \int_{S^1} e^{2N \int \log |z-x| d\mu_\xi(x)} dz := |a_N|^2 e^{N \mathcal{J}_N(\mu_\xi)}, \end{aligned} \quad (32)$$

where the integrals are path integral along the unit circle, and we used the fact that the integrand is real to express it as an exponential of a real function. We then have, for any measurable set $A \subset \mathcal{M}(\mathbb{C})$,

$$\begin{aligned} \mathbf{Prob}_N(d\mu_\xi \in A) &= \frac{1}{Z_N} \int_{z_i: \Delta \neq 0} dz_1 \dots dz_N \mathbf{1}_{\{L_N \in A\}} \int_0^\infty y^N e^{N^2 \Sigma(L_N) - y e^{N \mathcal{J}_N(L_N)}} dy \\ &= \frac{1}{\widetilde{Z}_N} \int_{z_i: \Delta \neq 0} dz_1 \dots dz_N \mathbf{1}_{\{L_N \in A\}} e^{N^2 \Sigma(L_N)} e^{-N^2 \mathcal{J}_N(L_N)}. \end{aligned} \quad (33)$$

Here, $L_N = N^{-1} \sum_{i=1}^N \delta_{z_i}$, Σ is as in (21) (with the convention $\log(0) = 0$), and Z_N, \widetilde{Z}_N are normalization constants.

Now, for each fixed μ for which $\log |z - \cdot|$ is uniformly integrable for $z \in S^1$, we have that

$$\mathcal{J}_N(\mu) = N^{-1} \log \left(\frac{1}{2\pi} \int_{S^1} \exp(2N \langle \mu, \log |z - \cdot| \rangle) dz \right) \xrightarrow{N \rightarrow \infty} 2J(\mu), \quad (34)$$

where

$$J(\mu) := \max_{z \in S^1} \int \log |z - x| d\mu(x).$$

One thus expects, as in [BG, BZ], to obtain the large deviation principle, with speed N^2 and rate function

$$2J(\mu) - \Sigma(\mu) - \inf_{\nu \in \mathcal{M}(\mathbb{C})} [2J(\nu) - \Sigma(\nu)].$$

(Compare with (20), noting that $K = S^1$ and $\varphi = 1$ on S^1 for the Kac-Hammersley example.) The technical details of the derivation, however, are best handled in a more general geometric framework, where relevant properties of the rate function are more transparent.

1.6. Generalizations. We close the introduction with some comments on the generalization of the results of this article to other Kähler manifolds. In the sequel [Z], we use the method of this article to give an explicit formula for the joint probability current in the more difficult higher genus cases. In higher genus, the relation between configuration spaces and sections of line bundles of degree N is the subject of Abel-Jacobi theory, and the formula for the joint probability current involves such objects as the prime form. Some of the geometric discussion of this paper is intended to set the stage for the higher genus sequel. The results can also be generalized from Gaussian ensembles to non-linear ensembles of Ginzburg-Landau type. For the sake of brevity, we do not carry out the generalization here.

Another type of generalization to consider is to ensembles of random holomorphic functions, for instance random holomorphic functions in the unit disc with various weighted norms or random entire functions on \mathbb{C} . The random analytic functions have an infinite number of zeros and one apparently needs to make a finite dimensional approximation to obtain a useful configuration space and a map to empirical measures.

An interesting question is whether one can generalize the large deviations results to higher dimensions. One could consider the hypersurface zero set of a single random section, or the joint zero set of a full system of m sections in dimension m . The rate function $\tilde{I}^{h,K}$ has a generalization to all dimensions, and so the large deviations result might admit a generalization. But the approach of this article, to extract the large deviations rate function from the joint probability current of zeros, does not seem to generalize well to higher dimensions. In dimension one, there exists a simple configuration space of possible zero sets of sections, but in higher dimensions there is no manageable analogue. It is possible that one can avoid this impasse by working on the space of potentials $\frac{1}{N} \log \|s\|_{h^N}$ rather than on the configuration space of zeros. But it appears that one would have to extract the rate function directly from the potentials without using zeros coordinates. This circle of problems fits in very naturally with the Kähler potential theory of [GZ, BB, Ber1, Ber2], and it would be interesting to explore it further.

Finally, the authors would like to thank R. Berman, T. Bloom, A. Dembo, A. Guionnet and B. Shiffman for helpful conversations.

2. BACKGROUND

Polynomials of degree N on \mathbb{C} may be viewed as meromorphic functions on \mathbb{CP}^1 with a pole of order N at ∞ , or equivalently as holomorphic sections of the N power $\mathcal{O}(N) \rightarrow \mathbb{CP}^1$ of the hyperplane line bundle $\mathcal{O}(1) \rightarrow \mathbb{CP}^1$, or again as homogeneous holomorphic polynomials of degree N on \mathbb{C}^2 . It is useful to employ the geometric language of line bundles, Hermitian metrics and curvature, and in [Z] this language is indispensable. We briefly recall the relevant definitions, referring the reader to [GH] for further details.

We use the following standard notation: $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Also, $\partial f = \frac{\partial f}{\partial z} dz$ and similarly for $\bar{\partial} f$. The Euclidean Laplacian is given by $\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$ and $\partial \bar{\partial} = \frac{\partial^2}{\partial z \partial \bar{z}} dz \wedge d\bar{z}$. It is often convenient to use the real operators $d = \partial + \bar{\partial}$, $d^c := \frac{i}{4\pi}(\bar{\partial} - \partial)$ and $dd^c = \frac{i}{2\pi} \partial \bar{\partial}$. Thus, $dd^c f = \frac{i}{8\pi} \Delta f dz \wedge d\bar{z} = \frac{1}{4\pi} \Delta f dx \wedge dy$. We will often need the classical formula,

$$\Delta \left(\frac{1}{2\pi} \log |z| \right) = \delta_0 \iff dd^c(2 \log |z|) = \delta_0 dx \wedge dy. \quad (35)$$

Henceforth, we regard δ_0 as a $(1, 1)$ current so that δ_0 and $\delta_0 dx \wedge dy$ have the same meaning.

A smooth Hermitian metric h on a holomorphic line bundle L is a smooth family h_z of Hermitian inner products on the one-dimensional complex vector spaces L_z . Its Chern form is defined by

$$c_1(h) = \omega_h := -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \|e_L\|_h^2, \quad (36)$$

where e_L denotes a local holomorphic frame (= nonvanishing section) of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h -norm of e_L . We say that h is positive if the (real) 2-form ω_h is a positive $(1, 1)$ form, i.e. defines a Kähler metric. For any smooth Hermitian metric h and local frame e_L for L , we write $\|e_L\|_h^2 = e^{-\varphi}$ (or, $h = e^{-\varphi}$), and

$$\omega_h = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi = dd^c \varphi. \quad (37)$$

We refer to $\varphi = -\log \|e_L\|_h^2$ as the potential of ω_h in U , or as the Kähler potential when ω_h is a Kähler form. We are interested in general smooth metrics, not only those where ω_h is positive; for instance, our methods and results apply in the case where $\varphi = 0$ (i.e. the metric is flat) on the support of $d\nu$. The metric h induces Hermitian metrics h^N on $L^N = L \otimes \cdots \otimes L$ given by $\|s^{\otimes N}\|_{h^N} = \|s\|_h^N$. The N -dependent factor $e^{-N\varphi}$ is then the local expression of h^N in the local frame e^N . We will only be considering the line bundles $\mathcal{O}(N) \rightarrow \mathbb{CP}^1$ in this paper.

We now specialize to the hyperplane line bundle $\mathcal{O}(1) \rightarrow \mathbb{CP}^1$ and its powers. We recall that \mathbb{CP}^1 is the set of lines through 0 in \mathbb{C}^2 . The line through (z_0, z_1) is denoted $[z_0, z_1]$, which are the homogeneous coordinates of the line. In the case of \mathbb{CP}^1 there exists a single holomorphic line bundle L^N of each degree. One writes $L = \mathcal{O}(1)$ and $L^N = \mathcal{O}(N)$. The bundle $\mathcal{O}(1)$ is dual to the tautological line bundle $\mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ whose fiber at a point $[z_0, z_1] \in \mathbb{CP}^1$ is the line $[z_0, z_1]$ in \mathbb{C}^2 . The line bundle $\mathcal{O}(1)$ is defined by two charts $U_1 = \mathbb{CP}^1 \setminus \{\infty\}$ ($z_0 \neq 0$) and $U_2 = \mathbb{CP}^1 \setminus \{0\}$ ($z_1 \neq 0$). A frame (nowhere vanishing holomorphic section) of $\mathcal{O}(-1)$ over U_1 is given by $e_1^*([z_0, z_1]) = (1, \frac{z_1}{z_0})$, and over U_2 by $e_2^*([z_0, z_1]) = (\frac{z_0}{z_1}, 1)$. The dual frames are the homogeneous polynomials on \mathbb{C}^2 defined by $e_1(z_0, z_1) = z_0$, resp. $e_2(z_0, z_1) = z_1$.

The potential φ is only defined relative to a frame, and we will need to know how it changes under a change of frame. Suppose that φ_1 is the potential of ω_h in the frame e_1 , i.e. $\|e_1([z_0, z_1])\|_h^2 = e^{-\varphi_1}$. We assume that h , hence φ_1 is smooth in U_1 and we may (with a slight abuse of notation) regard it as a function on U_1 or on \mathbb{C} in the standard coordinate $[z_0, z_1] \rightarrow \frac{z_0}{z_1} = w$. In the frame e_2 we have the local potential $\|e_2([z_0, z_1])\|_h^2 = e^{-\varphi_2}$ for some $\varphi_2 \in C^\infty(\mathbb{CP}^1 \setminus \{0\})$, which we identify with a function on \mathbb{C} . On the overlap $\mathbb{CP}^1 \setminus \{0, \infty\}$ the frames e_1, e_2 are related by $e_2([z_0, z_1]) = \frac{z_1}{z_0} e_1([z_0, z_1])$, so $\|e_2([z_0, z_1])\|_h^2 = |\frac{z_1}{z_0}|^2 \|e_1([z_0, z_1])\|_h^2$. It follows that $\varphi_2([z_0, z_1]) = \varphi_1([z_0, z_1]) - 2 \log |\frac{z_1}{z_0}|$. If we use $w = \frac{z_0}{z_1}$ as a local coordinate, then $\varphi_2(w) = \varphi_1(\frac{1}{w}) + \log |w|^2$. As an illustration, the Kähler potential of the Fubini-Study metric on $\mathcal{O}(1)$ is given by $\log(1 + |w|^2) = \log(1 + \frac{1}{|w|^2}) + \log |w|^2$ in the two charts.

An important observation in understanding the global nature of (30) is the following:

LEMMA 5. *The $(1, 1)$ form $e^{-2\varphi_1(z)} dz \wedge d\bar{z}$ in the chart U_1 extends to a global smooth $(1, 1)$ form κ on \mathbb{CP}^1 . In the chart U_2 it equals $e^{-2\varphi_2(z)} dz \wedge d\bar{z}$.*

Proof. We need to check its invariance under the change of variables $\sigma(z) = \frac{1}{z}$. We have,

$$\sigma^* e^{-2\varphi_1(z)} dz \wedge d\bar{z} = e^{-2\varphi_1(\frac{1}{z})} \frac{dz \wedge d\bar{z}}{|z|^4}.$$

Since $\varphi_1(\frac{1}{z}) = \varphi_2(z) - \log|z|^2$, this is

$$e^{-2\varphi_2(z)} e^{2\log|z|^2} \frac{dz \wedge d\bar{z}}{|z|^4} = e^{-2\varphi_2(z)} dz \wedge d\bar{z}.$$

□

2.1. Poincaré-Lelong formula for the empirical measure of zeros. The empirical measure of zeros Z_s (1) is given by (one-dimensional) Poincaré-Lelong formula,

$$\begin{aligned} Z_s &= \frac{i}{\pi N} \partial \bar{\partial} \log |f| = \frac{i}{N\pi} \partial \bar{\partial} \log \|s\|_{h^N} + \omega_h \\ &= \frac{2}{N} dd^c \log \|s\|_{h^N} + \omega_h. \end{aligned} \quad (38)$$

It is completely elementary in dimension one.

2.2. dd^c Lemma. We will need the dd^c Lemma on not-necessarily-positive $(1, 1)$ currents. The dd^c Lemma on forms (cf. [Dem], Lemma 8.6 of Chapter VI) asserts that on a compact Kähler manifold, a d -closed (p, q) form u may be expressed as $u = dd^c v$ where v is a $(p-1, q-1)$ form. The same Lemma is true for currents, with the change that v is only asserted to be a current.

When ω, ω' are two cohomologous positive closed $(1, 1)$ currents (which on $\mathbb{C}\mathbb{P}^1$ simply means $\int_{\mathbb{C}\mathbb{P}^1} \omega = \int_{\mathbb{C}\mathbb{P}^1} \omega'$), then one has a regularity theorem: $\omega - \omega' = dd^c \psi$ where $\psi \in L^1(\mathbb{C}\mathbb{P}^1, \mathbb{R})$. We refer to [GZ], Proposition 1.4.

2.3. Hermitian inner products and Gaussian measures on $H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$. We denote by $H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$ the space of holomorphic sections of $\mathcal{O}(N)$. It is well-known that they correspond to polynomials of degree N , which are their local expressions in the affine chart $U = \mathbb{C}\mathbb{P}^1 \setminus \{\infty\}$ (see [GH]).

As mentioned in the introduction, the data (h, ν) determine inner products $G_N(h, \nu)$ on the complex vector spaces $H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$ (see (2) and (8)). An inner product on $H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$ induces a Gaussian measure on this complex vector space by the formula,

$$d\gamma_N(s_N) := \frac{1}{\pi^{d_N}} e^{-|c|^2} dc, \quad s_N = \sum_{j=1}^{d_N} c_j S_j^N, \quad c = (c_1, \dots, c_{d_N}) \in \mathbb{C}^{d_N}, \quad (39)$$

where $d_N = N+1$, $\{S_1^N, \dots, S_{d_N}^N\}$ is an orthonormal basis for $H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$, and dc denotes $2d_N$ -dimensional Lebesgue measure. The measure γ_N is characterized by the property that the $2d_N$ real variables $\Re c_j, \Im c_j$ ($j = 1, \dots, d_N$) are independent Gaussian random variables with mean 0 and variance 1/2; equivalently,

$$\mathbf{E}_N c_j = 0, \quad \mathbf{E}_N c_j c_k = 0, \quad \mathbf{E}_N c_j \bar{c}_k = \delta_{jk},$$

where \mathbf{E}_N denotes the expectation with respect to the measure γ_N .

In §3, we will define an essentially equivalent Fubini-Study volume form on the projective space of sections $\mathbb{P}H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$.

3. JOINT PROBABILITY CURRENT OF ZEROS AND THE FUBINI-STUDY VOLUME FORM

In this section, we define the principal object of this article, the joint probability current of zeros. We then prove the first part (29) of Proposition 3, giving the formula for the joint probability current of zeros as the pull back to configuration space of the Fubini-Study volume form on the projective space of sections. We re-write the formula in terms of the Green's function in the next section.

3.1. The joint probability current of zeros. The joint probability current of zeros is defined by

$$\vec{K}_N^N(z^1, \dots, z^N) := \mathbf{E}(Z_s(z^1) \otimes Z_s(z^2) \otimes \dots \otimes Z_s(z^N)). \quad (40)$$

It is a current on the configuration space $(\mathbb{CP}^1)^{(N)}$ of N points. It is the extreme case $n = N$ of the n -point zero correlation current

$$\vec{K}_n^N(z^1, \dots, z^n) := \mathbf{E}(Z_s(z^1) \otimes Z_s(z^2) \otimes \dots \otimes Z_s(z^n)) \quad (41)$$

on the configuration space $(\mathbb{CP}^1)^{(n)}$. Recall that by a current we mean a linear functional on test forms, i.e. for any test function $\varphi_1(z^1) \otimes \dots \otimes \varphi_n(z^n) \in C((\mathbb{CP}^1)^{(n)})$,

$$(\vec{K}_n^N(z^1, \dots, z^n), \varphi_1(z^1) \otimes \dots \otimes \varphi_n(z^n)) = \mathbf{E}[(Z_s, \varphi_1)(Z_s, \varphi_2) \dots (Z_s, \varphi_n)]. \quad (42)$$

3.2. Fubini-Study formula. We now present the most useful approach to the joint probability current of zeros in the case of genus zero.

It is a classical fact that the projective space of sections $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$ may be identified with the configuration space $(\mathbb{CP}^1)^{(N)}$ of N points of \mathbb{CP}^1 . This essentially comes down to the elementary fact that a set $\{\zeta_1, \dots, \zeta_N\}$ determines a line of polynomials $[P_\zeta] \in \mathbb{P}\mathcal{P}_N$ of degree N , at least when none of the zeros occur at ∞ . Viewed as holomorphic sections of $\mathcal{O}(N) \rightarrow \mathbb{CP}^1$ one can also allow ∞ to be a zero and then N points of \mathbb{CP}^1 corresponds to a line of holomorphic sections.

The correspondence $\zeta \rightarrow [P_\zeta]$ defines a line bundle

$$\mathcal{Z}_N \rightarrow (\mathbb{CP}^1)^{(N)}, \quad (\mathcal{Z}_N)_\zeta = \{[p] \in \mathcal{P}_N : \mathcal{D}(p) = \zeta\}, \quad (43)$$

i.e. the fiber of \mathcal{Z}_N at $\zeta_1 + \dots + \zeta_N$ is the line $\mathbb{C}P_\zeta$ of holomorphic sections of $\mathcal{O}(N)$ with the divisor $\zeta = \zeta_1 + \dots + \zeta_N$. It is isomorphic to the bundle $\mathcal{O}(1) \rightarrow \mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$ under the identification $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N)) = (\mathbb{CP}^1)^{(N)}$. One can construct a form representing the first Chern class $c_1(\mathcal{Z}_N)$ using a Hermitian inner product on \mathcal{Z}_N or equivalently a Hermitian inner product on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$: at a point $\zeta \in (\mathbb{CP}^1)^{(N)}$, the G_ζ -norm of a vector $P_\zeta \in \mathcal{Z}_\zeta$ is $\|P_\zeta\|_{G_\zeta}$, the norm of P_ζ as an element of $H^0(\mathbb{CP}^1, \mathcal{O}(N))$. This is the Fubini-Study Hermitian metric determined by the inner product.

Let us recall the basic definitions and formulae in the case of the standard inner product on \mathbb{C}^{d+1} and $\mathbb{C}\mathbb{P}^d$. Let $Z \in \mathbb{C}^{d+1}$ and let $\|Z\|^2 = \sum_{j=1}^{d+1} |Z_j|^2$. In the open dense chart $Z_0 \neq 0$, and in affine coordinates $w_j = \frac{Z_j}{Z_0}$, the Fubini-Study volume form is given by,

$$dVol_I = \frac{\prod_i dw_i \wedge d\bar{w}_i}{(1 + \|w\|^2)^{d+1}}.$$

For our purposes, it is more useful to lift this form to \mathbb{C}^{d+1} under the natural projection, $\pi : \mathbb{C}^{d+1} - \{0\} \rightarrow \mathbb{C}\mathbb{P}^d$. A straightforward calculation shows that

$$\pi^* dVol_I = \|Z_0\|^2 \frac{\prod_{j=1}^d dZ_j \wedge d\bar{Z}_j}{\|Z\|^{2(d+1)}}, \quad (44)$$

in the sense that

$$\frac{dZ_0 \wedge d\bar{Z}_0}{\|Z_0\|^2} \wedge \frac{\prod_{i=1}^d dw_i \wedge d\bar{w}_i}{(1 + \|w\|^2)^{d+1}} = \frac{\prod_{j=0}^d dZ_j \wedge d\bar{Z}_j}{\|Z\|^{2(d+1)}}.$$

We need a more general formula where the inner product $\|Z\|^2$ is replaced by any Hermitian inner product on \mathbb{C}^{d+1} . We recall that the space of Hermitian inner products on V is the symmetric space $GL(d+1, \mathbb{C})/U(d+1)$. If we identify $V = \mathbb{C}^d$ and fix the standard inner product (v, w) , then any other inner product has the form $G(v, w) = (Pv, w)$ where P is a positive Hermitian matrix. It has the form $P = A^*A$ where $A \in GL(d+1, \mathbb{C})$.

Suppose, then, that instead of the standard inner norm $\|Z\|$ on \mathbb{C}^{d+1} we are given the norm $\|AZ\|$ where $A \in GL(d+1, \mathbb{C})$. Then the Fubini-Study metric becomes $\partial\bar{\partial} \log \|AZ\|^2$. Since the linear transformation defined by A is holomorphic, the associated volume form dV_A is simply the pull-back by A of the previous form,

$$\begin{aligned} \pi^* dVol_A &= A^* \frac{(\partial\bar{\partial} \log \|Z\|^2)^{d+1}}{\frac{dZ_0 \wedge d\bar{Z}_0}{\|Z_0\|^2}} = \frac{\|(AZ)_0\|^2}{\|AZ\|^{2(d+1)}} A^* \left(\frac{\prod_{j=0}^d dZ_j \wedge d\bar{Z}_j}{dZ_0 \wedge d\bar{Z}_0} \right) \\ &= |\det A|^2 |(AZ)_0|^2 \cdot \left(\frac{\partial}{\partial Z_0} \wedge \frac{\partial}{\partial \bar{Z}_0} \perp A^*(dZ_0 \wedge d\bar{Z}_0) \right)^{-1} \left(\frac{\prod_{j=1}^d dZ_j \wedge d\bar{Z}_j}{\|AZ\|^{2(d+1)}} \right). \end{aligned} \quad (45)$$

We now prove the first part of Proposition 3.

3.3. Proof of (29) in Proposition 3. To prove (29), we use (45) and change variables to zeros coordinates.

We first consider the change of variables in local coordinates on $\mathbb{C}\mathbb{P}^1$. We fix the usual affine chart $U \subset \mathbb{C}$ and let z be the local coordinate. We then have a corresponding local coordinate system $(\zeta_1, \dots, \zeta_N)$ on $(\mathbb{C}\mathbb{P}^1)^N$ which is defined in the chart $(\mathbb{C})^N$.

We have defined the joint probability current (41) as an (N, N) form on configuration space $(\mathbb{C}\mathbb{P}^1)^N$. It pulls back under the S_N cover $(\mathbb{C}\mathbb{P}^1)^N \rightarrow (\mathbb{C}\mathbb{P}^1)^{(N)}$ and we wish to express it in the local coordinate system $(\zeta_1, \dots, \zeta_N)$ to obtain the formula in Proposition 3. We then write down its density with respect to the local Lebesgue volume form $d^2\zeta_1 \cdots d^2\zeta_N$ of the chart.

To prove the Proposition, we start with the Newton-Vieta's formula:

$$\prod_{j=1}^N (z - \zeta_j) = \sum_{k=0}^N (-1)^k e_{N-k}(\zeta_1, \dots, \zeta_N) z^k. \quad (46)$$

Here, the elementary symmetric functions are defined by

$$e_j = \sum_{1 \leq p_1 < \dots < p_j \leq N} z_{p_1} \cdots z_{p_j}.$$

As mentioned above, the formula (46) defines a map $(\mathbb{C}\mathbb{P}^1)^{(N)} \rightarrow \mathcal{P}_N$, which is a section of the line bundle \mathcal{Z}_N (43). It is the section taking its values in the polynomials $\sum_{i=0}^N a_i z^i$

for which $a_N = 1$. Since $e_0(\zeta) \equiv 1$, the linear coordinates are affine coordinates in the chart $c_0 = 1$, where c_j are coordinates with respect to the basis $\{z^j\}$. We then change variables from the Lebesgue volume form $da_1 \wedge d\bar{a}_1 \wedge \cdots \wedge da_N \wedge d\bar{a}_N$ in the affine chart to a volume form in the coordinates $(\zeta_1, \dots, \zeta_N)$. It is well-known (see e.g. [LP]) that this change of variables has Jacobian $|\Delta(\zeta)|^2$ where as above, $\Delta(\zeta_1, \dots, \zeta_N) = \prod_{1 \leq j < k \leq N} (\zeta_k - \zeta_j)$ is the Vandermonde determinant.

We now express the Fubini-Study probability measure on $\mathbb{P}H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$ in the coordinates ζ_j . The first problem we face is that the right side of (46) expresses the polynomial on the left side in coordinates with respect to the basis $\{z^j\}_{j=0}^N$, which is usually not an orthonormal basis with respect to the inner product (2). We need to make the additional change of variables from coordinates E_j with respect to an orthonormal basis $\{\psi_j\}$ for our inner product $G_N(h, \nu)$, $\prod_{j=1}^N (z - \zeta_j) = \sum_{\ell=0}^d \mathcal{E}_{N-\ell} \psi_\ell$ to coordinates $Z_j = (-1)^{N-j} e_{N-j}$ with respect to the monomial basis $\{z^j\}$. With no loss of generality, we assume that the orthogonal polynomials $\{\psi_j\}$ are enumerated according to degree, so that ψ_N is the unique polynomial in the basis with a z^N term. The change of basis matrix $\mathcal{A}_N(h, \nu)Z = \mathcal{E}$ is given by,

$$\left(\mathcal{A}_N^{jk} \right)_{j,k=0}^N = \left(\langle z^j, \psi_k \rangle_{G_N(h, \nu)} \right)_{j,k=0}^N. \quad (47)$$

Next we observe that

$$\frac{\partial}{\partial Z_0} \wedge \frac{\partial}{\partial \bar{Z}_0} \perp \mathcal{A}_N(h, \nu)^* dZ_0 \wedge d\bar{Z}_0 = \|\mathcal{A}_N^{00}\|^2. \quad (48)$$

Indeed, $\mathcal{A}_N^* dZ_0 = \sum_j \mathcal{A}_N^{0j} dZ_j$ and the desired expression is the coefficient of $dZ_0 \wedge d\bar{Z}_0$ in $d(\mathcal{A}_N^* Z)_0 \wedge d(\overline{\mathcal{A}_N^* Z})_0$. We further observe that $\|\mathcal{A}_N^{00}\|^2$ is a constant independent of ζ . By our ordering, $\psi_N = k_N z^N + k_{N-1} z^{N-1} \cdots$ for some $k_N \neq 0$. Since

$$\prod_j (z - \zeta_j) = \sum_j \mathcal{E}_{N-j} \psi_j = z^N + e_1(\zeta) z^{N-1} + \cdots, \quad (49)$$

it follows that

$$\mathcal{A}_N^{00} = k_N^{-1}, \quad \text{and that } \|(\mathcal{A}_N(h, \nu)Z)_0\|^2 = k_N^{-2}. \quad (50)$$

Hence, (48) equals k_N^{-2} , and the factors $\|(\mathcal{A}_N(h, \nu)Z)_0\|^2 \cdot k_N^2$ cancel.

Combining this evaluation with (45), we see that the pull back of the Fubini-Study volume form with respect to $G_N(h, \nu)$ to \mathbb{C}^{N+1} is given by

$$|\det \mathcal{A}_N(h, \nu)|^2 \left(\frac{\prod_{j=1}^N dZ_j \wedge d\bar{Z}_j}{\|\mathcal{A}_N(h, \nu)Z\|^{2(N+1)}} \right). \quad (51)$$

We now change variables to zeros coordinates. As mentioned above, $\prod_{j=1}^N dZ_j \wedge d\bar{Z}_j = |\Delta(\zeta)|^2 \prod_j d^2 \zeta_j$. The denominator in (51) equals the sum of the squares of the components of $\mathcal{A}_N(h, \nu)Z$, which is L^2 norm-squared of $\prod_{j=1}^N (z - \zeta_j)$ with respect to $G_N(h, \nu)$, i.e.

$$\|\mathcal{A}_N(h, \nu)Z\|^{2(N+1)} = \left(\int_{\mathbb{C}\mathbb{P}^1} \prod_{j=1}^N |(z - \zeta_j)|^2 e^{-N\varphi} d\nu(z) \right)^{N+1}.$$

Further,

$$\begin{aligned}
|(\mathcal{A}_N(h, \nu)Z)_0|^2 &= |\mathcal{E}_0(\zeta)|^2 = |\langle \prod_{j=1}^N (z - \zeta_j), \psi_N \rangle|^2 \\
&= \left| \int_{\mathbb{C}} \prod_{j=1}^N (z - \zeta_j) \overline{\psi_N(z)} e^{-N\varphi(z)} d\nu(z) \right|^2 \\
&= e^{2N \int \varphi d\mu_\zeta} \int_{\mathbb{C}} \psi_N(z) e^{N \int_M G_h(z, w) d\mu_\zeta} e^{-N\varphi(z)} d\nu(z)
\end{aligned}$$

This completes the proof of (29). \square

We refer to the coefficient of $d^2\zeta_1 \cdots d^2\zeta_N$ in (29) as the joint probability density (JPD) of zeros:

$$D_N(\zeta_1, \dots, \zeta_N) = |\det \mathcal{A}_N(h, \nu)|^2 \frac{|\Delta(\zeta_1, \dots, \zeta_N)|^2}{\left(\int_{\mathbb{C}} \prod_{j=1}^N |z - \zeta_j|^2 e^{-N\varphi} d\nu(z) \right)^{N+1}}. \quad (52)$$

Remark: The elementary symmetric functions $e_j(\zeta)$ of $\zeta = (\zeta_1, \dots, \zeta_N)$ are natural coordinates in $\mathbb{C}^{(N)}$, and a natural holomorphic volume form is given by

$$\Omega_{\mathbb{C}^{(N)}} = de_1 \wedge \cdots \wedge de_N \quad (53)$$

while the corresponding (N, N) form is

$$\Omega_{\mathbb{C}^{(N)}} \wedge \overline{\Omega}_{\mathbb{C}^{(N)}} = de_1 \wedge d\bar{e}_1 \wedge \cdots \wedge de_N \wedge d\bar{e}_N = |\Delta(\zeta_1, \dots, \zeta_N)|^2 d^2\zeta_1 \cdots d^2\zeta_N. \quad (54)$$

3.4. Intrinsic formula for the joint probability current. The Fubini-Study form has an intrinsic geometric interpretation as the curvature form (36) for the Hermitian line bundle $\mathcal{Z}_N \rightarrow (\mathbb{C}\mathbb{P}^1)^{(N)}$ equipped with its metric $G(h^N, \nu)$. This is of independent geometric interest and we pause to consider it.

A local frame for \mathcal{Z}_N (henceforth we drop the N for notation simplicity) is a non-vanishing holomorphic selection of a polynomial P_ζ from the line $\mathbb{C}P_\zeta$ of polynomials (or more generally, holomorphic sections of $\mathcal{O}(N) \rightarrow \mathbb{C}\mathbb{P}^1$) with divisor ζ . The standard choice is to trivialize \mathcal{Z} over $(\mathbb{C})^N$ using the section $P_\zeta(z) = \prod_{j=1}^N (z - \zeta_j) e^N(z)$ where $e(z)$ is the standard affine frame of $\mathcal{O}(1) \rightarrow \mathbb{C}\mathbb{P}^1$ over \mathbb{C} . In this article, the inner product $G = G_N(h, \nu)$ is defined by (8). It follows that the curvature $(1, 1)$ form of \mathcal{Z} is given by

$$\omega_{\mathcal{Z}} = \frac{i}{2} \partial \bar{\partial} \log \|P_\zeta\|_{G(h, \nu)}, \quad (55)$$

where $\partial \bar{\partial}$ is the operator on $(\mathbb{C}\mathbb{P}^1)^{(N)}$. Thus,

$$\Phi_N(\zeta) := \log \|P_\zeta\|_{G(h, \nu)} \quad (56)$$

is the Kähler potential for the Kähler form of configuration space, and the volume form is given by

$$dV_{FS, G_N(h, \nu)} = \left(\frac{i}{2} \partial \bar{\partial} \log \|P_\zeta\|_{G_N(h, \nu)} \right)^N, \quad (57)$$

the (N, N) form defined as the top exterior power of (55). What (29) asserts is thus equivalent to

PROPOSITION 6. *We have,*

$$\left(\frac{i}{2}\partial\bar{\partial}\Phi_N\right)^N = |\det \mathcal{A}_N(h, \nu)|^2 |\Delta(\zeta)|^2 e^{-(N+1)\Phi_N(\zeta)} \prod_{j=1}^N d^2\zeta_j.$$

This Proposition clarifies in what sense the right hand side is a well-defined volume form on $(\mathbb{CP}^1)^{(N)}$. Namely, it corresponds to the choice of the Kähler potential Φ_N , i.e. the expression of the Hermitian metric G on \mathcal{Z} in the local frame P_ζ .

4. GREEN'S FUNCTIONS AND THE JOINT PROBABILITY CURRENT: COMPLETION OF THE PROOF OF PROPOSITION 3

As discussed in the introduction, it is very helpful to express the joint probability current and rate function in terms of global objects on \mathbb{CP}^1 . In the statement of Theorem 1, we expressed $I^{h,K}$ in terms of the Green's function G_h . In this section, we give background on the definition and properties of Green's function that are needed in the proof of Theorem 1. The main result is Proposition 17, in which we express the joint probability current in terms of Green's functions, and thus complete the proof of Proposition 3.

4.1. **Green's function for ω_h .** The Green's function $G_h(z, w)$ is defined in (12). We now verify that G_h is well-defined, that it is smooth outside of the diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and that its only singularity is a logarithmic singularity on the diagonal. We sometimes write $g_z(w) = G(z, w)$ to emphasize that the derivatives in (12) are in the w variable. When ω_h is a Kähler metric, $g_z(w)$ is a special case of the notion of Green's current for the divisor $\{z\}$. For background we refer to [He], although it only discusses the case where ω_h is a Kähler form. We also refer to [ABMNV] for background on global analysis on Riemann surfaces.

When we express the Green's function in the charts $U_1 \times U_1$, resp. $U_2 \times U_2$ of $\mathbb{CP}^1 \times \mathbb{CP}^1$, we subscript G_h accordingly. We also drop the subscript h for simplicity of notation when the metric is understood.

PROPOSITION 7. *There exists a unique function $G_h(z, w) \in L^1(\mathbb{CP}^1 \times \mathbb{CP}^1)$ solving the system of equations (12). When $z \neq \infty$, in the local affine chart \mathbb{C} it is given by (13). Under the holomorphic map $z \rightarrow \frac{1}{z}$, we have*

$$G_1\left(\frac{1}{z}, \frac{1}{w}\right) = G_2(z, w).$$

Proof. Given any $z \in \mathbb{CP}^1$, there exists a section $s_z \in H^0(\mathbb{CP}^1, \mathcal{O}(1))$ which vanishes at z . There exists a distinguished section (denoted $\mathbf{1}_z(w)$ in [ABMNV]) which has the Taylor expansion $w - z$ in the standard affine frame and which corresponds to the meromorphic function $w - z$. When $z = \infty$, $s_\infty(w)$ corresponds to the meromorphic function 1. As a homogeneous polynomial of degree one in each variable on $\mathbb{C}^2 \times \mathbb{C}^2$ it is given by $w_1 z_0 - z_1 w_0$. We view the two-variable section $s_w(z)$ as a section of $\pi_1^* \mathcal{O}(1) \boxtimes \pi_2^* \mathcal{O}(1) \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ and equip the line bundle with the product Hermitian metric $h_z \boxtimes h_w$ (here and in what follows, \boxtimes denotes the exterior tensor product on $\mathbb{CP}^1 \times \mathbb{CP}^1$). We then claim that (with $E(h)$ defined in (14)),

$$G_h(z, w) = \log \|s_z(w)\|_{h_z \boxtimes h_w}^2 - E(h) \tag{58}$$

satisfies (i)- (iii) of (12) for all z . Both (i) and (ii) are clear from the formula and from (38).

To prove (iii) and the identity claimed in the Proposition, it is convenient to use the local affine frames e_j of $\mathcal{O}(1) \rightarrow \mathbb{C}\mathbb{P}^1$ over the affine charts U_j (see §2 for notation).

LEMMA 8. *There exists a constant $E(h)$ so that, in the affine chart U_j ($j = 1, 2$) and all $z \in \mathbb{C}$,*

$$G_j(z, w) = 2 \log |z - w| - \varphi_j(z) - \varphi_j(w) + E(h), \quad (59)$$

and $\int_{\mathbb{C}} G_j(z, w) dd^c \varphi_j = 0$.

Indeed, in U_1 we put $z_0 = w_0 = 1$ and $z_1 = z, w_1 = w$, and then

$$\log \|s_z(w)\|_{h_z \boxtimes h_w}^2 = 2 \log |z - w| - \varphi_1(z) - \varphi_1(w). \quad (60)$$

In U_2 we put $z_1 = w_1 = 1$ and $z_0 = z, w_0 = w$ and obtain the same expression with φ_2 replacing φ_1 . On the overlap, the stated identity follows from the fact that

$$2 \log \left| \frac{1}{z} - \frac{1}{w} \right| - \varphi_1\left(\frac{1}{z}\right) - \varphi_1\left(\frac{1}{w}\right) = 2 \log |z - w| - \varphi_1\left(\frac{1}{z}\right) - \varphi_1\left(\frac{1}{w}\right) - 2 \log |z| - 2 \log |w|,$$

and the fact that $\varphi_2(w) = \varphi_1\left(\frac{1}{w}\right) + \log |w|^2$ (see §2).

To complete the proof, we need to show that $\int_{\mathbb{C}\mathbb{P}^1} \log \|z - w\|_{h_z \boxtimes h_w}^2 \omega_h$ is a constant in z . In fact we claim that when $z, w \in U_1$, then

$$\int_{\mathbb{C}\mathbb{P}^1} \log \|z - w\|_{h_z \boxtimes h_w}^2 \omega_h = - \int \varphi \omega_h - 4\pi \rho_\varphi(\infty). \quad (61)$$

The calculation of this integral can be done by the integration by parts formulae in §8.2. We use (60) to break up the integrand into three terms. The second integrates to $-\varphi(z) \int_{\mathbb{C}\mathbb{P}^1} \omega_h = -\varphi(z)$, while the third integrates to $-\int \varphi \omega_h = -\int \varphi dd^c \varphi$. The first (logarithmic) term is of the form (115):

$$\int_{\mathbb{C}} 2 \log |z - w| dd^c \varphi_1 = \varphi_1(z) - 4\pi \rho_{\varphi_1}(\infty). \quad (62)$$

In the full sum, the $\varphi_1(z)$ terms cancel, leaving the stated expression. The same integral holds with φ_2 replaced by φ_1 if $z, w \in U_2$ by the identity in the Proposition. This proves the integral formula in all cases. \square

As an example of the calculation, the Fubini-Study Green's function is given in the chart $U_1 \times U_1$ by $G_{FS}(z, w) = 2 \log [z, w]^2 - C$, where $[z, w] = \frac{|z-w|}{\sqrt{1+|z|^2} \sqrt{1+|w|^2}}$. The constant C is determined by the condition (iii). To study its behavior when $z = \infty$ we change coordinates $\sigma : z \rightarrow \frac{1}{z}, w \rightarrow \frac{1}{w}$ and study the behavior at 0. The distance $[z, w]$ and Green's function are invariant under the isometry σ , so we obtain the same expression after the change of coordinates. In particular, in these coordinates, $G_{FS}(\infty, u) = 2 \log |u| - \log(1 + |u|^2) = \varphi\left(\frac{1}{u}\right)$, where $\varphi_{FS}(w) = \log(1 + |w|^2)$.

Remark: We note that a local Kähler potential φ (or a global relative Kähler potential) is only unique up to an additive constant. One may normalize φ by the condition $\int_{\mathbb{C}\mathbb{P}^1} \varphi \omega_h = 0$. However, in the above formula we have not done so. We observe that the Green's function is (as it must be) invariant under addition of a constant to φ .

4.2. Green's potential of a measure. We now return to the Green's potential (15) and Green's energy (16) of the introduction. Given a real $(1, 1)$ form ω on $\mathbb{C}\mathbb{P}^1$, we define

$$SH(\mathbb{C}\mathbb{P}^1, \omega) := \{u \in L^1(\mathbb{C}\mathbb{P}^1, \mathbb{R} \cup \{-\infty\}) : dd^c u + \omega \geq 0\}. \quad (63)$$

For any closed $(1, 1)$ form, the $\partial\bar{\partial}$ Lemma implies that the map

$$\psi \rightarrow \omega_\psi := \omega + dd^c \psi \in \mathcal{M}(\mathbb{C}\mathbb{P}^1) \quad (64)$$

is surjective and has only constants in its kernel, i.e.

$$SH(\mathbb{C}\mathbb{P}^1, \omega) \simeq \mathcal{M}(\mathbb{C}\mathbb{P}^1) \oplus \mathbb{R}. \quad (65)$$

The Green's potential (15) of a measure defines a global inverse to (64) and is uniquely characterized as the solution of

$$\begin{cases} dd^d U_\omega^\mu = \mu - \omega; \\ \int_{\mathbb{C}\mathbb{P}^1} U_\omega^\mu \omega = 0. \end{cases} \quad (66)$$

Any smooth integral $(1, 1)$ form $\omega \in H^2(\mathbb{C}\mathbb{P}^1, \mathbb{Z})$ is the curvature $(1, 1)$ form of a smooth Hermitian metric h (see §2), and we subscript the potential by h rather than ω . Thus,

$$dd^c U_h^\mu(z) = \mu - \omega_h. \quad (67)$$

We illustrate Green's potentials in the important case where $\mu = \mu_\zeta$. In Lemma 15, we essentially wrote the ω_h -subharmonic function $\frac{1}{N} \log \|s_\zeta(z)\|_{h^N}$ with $s_\zeta = \prod_{j=1}^N (z - \zeta_j) e^N(z)$ as a Green's potential. To tie the discussions together, we note that the special case $\omega = \omega_h$ of Lemma 15 can be reformulated in terms of Green's potentials as follows:

LEMMA 9. *We have,*

- $\frac{1}{N} \log \|s_\zeta(z)\|_{h^N} - \frac{1}{N} \int_{\mathbb{C}\mathbb{P}^1} \log \|s_\zeta\|_{h^N}^2 \omega_h = U_h^{\mu_\zeta}(z)$.
- $\int \log \|s_\zeta(w)\|_{h^N}^2 \omega_h = \int_{\mathbb{C}} \log \|s_\zeta(w)\|_{h^N}^2 dd^c \varphi = N(\int \varphi d\mu_\zeta - E(h))$.
- Hence

$$\|s_\zeta(z)\|_{h^N} e^{-\frac{1}{N} \int_{\mathbb{C}\mathbb{P}^1} \log \|s_\zeta\|_{h^N}^2 \omega_h} = e^{U_h^{\mu_\zeta}}.$$

Proof. Since $d\mu_\zeta = dd^c \frac{1}{N} \log \|s_\zeta(z)\|_{h^N}^2 + \omega_h$,

$$\begin{aligned} U_h^{\mu_\zeta}(z) &:= \int_{\mathbb{C}\mathbb{P}^1} G_h(z, w) d\mu_\zeta(w) \\ &= \int_{\mathbb{C}\mathbb{P}^1} G_h(z, w) \left(\frac{1}{N} dd^c \log \|s_\zeta(w)\|_{h^N}^2 + \omega_h \right) \\ &= \int_{\mathbb{C}\mathbb{P}^1} G_h(z, w) \frac{1}{N} dd^c \log \|s_\zeta(w)\|_{h^N}^2 \\ &= \frac{1}{N} \log \|s_\zeta\|_{h^N}^2(z) - \frac{1}{N} \int_{\mathbb{C}\mathbb{P}^1} \log \|s_\zeta\|_{h^N}^2(z) \omega_h. \end{aligned}$$

□

4.3. Regularity of Green's functions. For use in the proof of the large deviation principle, we need the following regularity result on the Green's function. In what follows, $D = \{(z, z) : z \in \mathbb{C}\mathbb{P}^1\}$.

PROPOSITION 10. $G_h(z, w) \in C^\infty(\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus D)$, and in any local chart, near the diagonal it possesses the singularity expansion,

$$G_h(z, w) = 2 \log |z - w| + \rho(z) + O(|z - w|) \quad (68)$$

where $\rho(z)$ is a smooth function on \mathbb{CP}^1 known as the Robin constant. In particular, $G_h(z, \cdot) \in L^1(\mathbb{CP}^1, \omega_h)$ for any z , and there exists a constant $C_G < \infty$ so that

$$\sup_{(z, w) \in \mathbb{CP}^1 \times \mathbb{CP}^1} G(z, w) \leq C_G.$$

Proof. When ω is a Kähler metric, we may form its Laplacian Δ_ω and then the Green's function $G_\omega(z, w)$ is the kernel of Δ_ω^{-1} on the orthogonal complement of the constant functions. Thus, in the compact case, G_ω is defined by two conditions:

- (1) $\Delta_\omega G_\omega(z, w) = \delta_z(w) - \frac{1}{A}$, where $A = \int_{\mathbb{CP}^1} \omega$. That is, $G_\omega(z, w)$ is a (singular) ω -subharmonic function. In our case $A = 1$.
- (2) $\int_{\mathbb{CP}^1} G_\omega(z, w) \omega = 0$.

We denote by $\{\varphi_j\}_{j=0}^\infty$ an orthonormal basis of eigenfunctions of Δ_ω in $L^2(\mathbb{CP}^1, \omega)$, with $\varphi_0 = \frac{1}{\sqrt{A}}$ and with $\Delta \varphi_j = \lambda_j \varphi_j$ with $0 = \lambda_0 > \lambda_1 \geq \lambda_2 \downarrow -\infty$. Then G_ω has the eigenfunction expansion,

$$G_\omega(z, w) = \sum_{j=1}^{\infty} \frac{\varphi_j(z) \varphi_j(w)}{\lambda_j}. \quad (69)$$

The singularity expansion near the diagonal is then a standard fact which follows from the Hadamard-Riesz parametrix method (see [HoIII], Section 17.4).

We now consider general smooth $(1, 1)$ form ω_h . When ω_h fails to be Kähler, we introduce a Kähler metric ω in the same cohomology class as ω_h . Since $\int_{\mathbb{CP}^1} \omega = \int_{\mathbb{CP}^1} \omega_h$, the $\partial\bar{\partial}$ Lemma implies that there exists a relative Kähler potential φ_{hg} such that $\omega_h - \omega = dd^c \varphi_{hg}$. By definition (15), the relative potentials are given by

$$dd^c U_h^\omega = \omega - \omega_h, \quad dd^c U_\omega^{\omega_h} = \omega_h - \omega. \quad (70)$$

It follows that $U_h^\omega = -U_\omega^{\omega_h} + A_{gh}$ for a constant A_{gh} , and from $\int U_\omega^{\omega_h} \omega = 0$ we have

$$U_h^\omega = -U_\omega^{\omega_h} + \int U_\omega^{\omega_h} \omega_h. \quad (71)$$

By integrating both sides against ω_h we also have $\int U_h^\omega \omega = \int U_\omega^{\omega_h} \omega_h$.

We then claim that

$$G_h(z, w) - G_\omega(z, w) = U_h^\omega(z) + U_h^\omega(w) - \int U_h^\omega \omega. \quad (72)$$

Since the relative potential U_h^ω is a solution of the elliptic equation (70), and the left side is C^∞ , it follows that $U_h^\omega \in C^\infty$. Hence (72) implies the regularity result for any smooth h .

To conclude the proof, we need to prove the identity (72). We observe that the dd^c derivatives of both sides of (72) in either z or w agree, both equaling $\omega - \omega_h$. Hence, there exists a unique constant C_{gh} such that

$$G_h(z, w) - G_\omega(z, w) = U_h^\omega(z) + U_h^\omega(w) + C_{gh}. \quad (73)$$

To determine C_{gh} we integrate both sides of (73) against $\omega_h(z) \boxtimes \omega(w)$ and use that $\int G_\omega \omega = 0 = \int G_h \omega_h$. Hence,

$$C_{gh} = -\left(\int U_h^\omega \omega_h + \int U_h^\omega \omega\right) = -\int U_h^\omega \omega,$$

since $\int U_h^\omega \omega_h = 0$. This implies (72). \square

COROLLARY 11. *With C_G as in Proposition 10, for any $\mu \in \mathcal{M}(\mathbb{CP}^1)$, $\sup_z U_h^\mu \leq C_G$*

Proof. This follows from the fact that $U_h^\mu(z) = \int_{\mathbb{CP}^1} G_h(z, w) d\mu(w) \leq C_G$, as $\int d\mu = 1$. \square

4.4. Green's energy. From (16), we have for the Green's energy with respect to ω_h

$$\mathcal{E}_h(\mu) = \int_{\mathbb{CP}^1} U_h^\mu(z) d\mu(z) = \int_{\mathbb{CP}^1} U_h^\mu(z) (dd^c U_h^\mu + \omega_h) = \int_{\mathbb{CP}^1} U_h^\mu(z) dd^c U_h^\mu, \quad (74)$$

where we used (67) in the second equality and the fact that $\int U_h^\mu \omega_h = 0$ in the last equation.

In the next result, we outline a proof of the convexity of the energy functional for general smooth Hermitian metrics. It is used in the proof of the convexity of the rate function in Lemma 29. Convexity of the energy is well-known in weighted potential theory: It is proved in Lemma 1.8 of [ST] that $-\Sigma(\mu) \geq 0$ in the case where $\mu = \mu_1 - \mu_2$ is a signed Borel measure with compact support, where $\mu(\mathbb{C}) = 0$ and each of μ_1, μ_2 satisfies $-\Sigma(\mu_j) < \infty$. A different proof is given in [BG], Property 2.1(4) and another in Proposition 5.5 of [BB]. We give a somewhat different proof in our setting of \mathbb{CP}^1 .

We define the energy form on $\mathcal{M}(\mathbb{CP}^1)$ by

$$\langle \mu, \nu \rangle_\omega := \int_{\mathbb{CP}^1} G_\omega(z, w) d\mu(z) d\nu(w) = \int_{\mathbb{CP}^1} U_\omega^\mu d\nu = \int_{\mathbb{CP}^1} U_\omega^\nu d\mu. \quad (75)$$

As in [C], we denote the probability measures of finite energy $\|\mu\|_\omega^2 < \infty$ by $\mathcal{E}^+(\mathbb{CP}^1) \subset \mathcal{M}(\mathbb{CP}^1)$.

PROPOSITION 12. *For any smooth Hermitian metric on $\mathcal{O}(1)$, $-\mathcal{E}_h$ is a convex functional on $\mathcal{M}(\mathbb{CP}^1)$.*

Proof. We first prove convexity when ω is a Kähler metric.

LEMMA 13. *When ω is a Kähler metric, the energy form $\langle \mu, \nu \rangle_\omega$ is negative semi-definite on signed measures of finite energy. The unique measures of energy zero are multiples of ω .*

Proof. From the eigenfunction expansion (69), it follows that

$$\langle \mu, \nu \rangle_\omega = \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} G_\omega(z, w) d\mu(z) d\nu(w) = \sum_{j=1}^{\infty} \frac{\mu(\varphi_j) \nu(\varphi_j)}{\lambda_j}. \quad (76)$$

It is clear that for any signed measure μ , $\langle \mu, \mu \rangle_\omega \leq 0$ with equality if and only if $\mu(\varphi_j) = 0$ for all $j = 1, 2, \dots$. The constant term has been removed from the sum, so this case of equality is only possible if and only if $\mu = C\omega$ for some constant C . \square

We then let h be a general smooth metric. The following lemma follows immediately from the identity (72).

LEMMA 14. *Let h be any smooth Hermitian metric, and let ω be a Kähler form with $\int_{\mathbb{C}\mathbb{P}^1} \omega = \int_{\mathbb{C}\mathbb{P}^1} \omega_h$. Then, their energy forms are related by*

$$\langle \mu, \nu \rangle_{\omega_h} = \langle \mu, \nu \rangle_{\omega} + \nu(\mathbb{C}\mathbb{P}^1) \int U_{\omega}^{\omega_h} d\mu + \mu(\mathbb{C}\mathbb{P}^1) \int U_{\omega}^{\omega_h} d\nu.$$

It follows that on $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$, $-\mathcal{E}_h$ is a convex function. □

4.5. Green's function and L^2 norms.

LEMMA 15. *Let G_h be the Green's function relative to ω_h . Then,*

$$(i) \quad e^{\int_{\mathbb{C}\mathbb{P}^1} G_h(z,w) dd^c \log \|s(w)\|_{h^N}^2} = \|s\|_{h^N}^2(z) e^{-\int_{\mathbb{C}\mathbb{P}^1} \log \|s\|_{h^N}^2(z) \omega_h},$$

and

$$(ii) \quad \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \setminus D} G_h(z,w) dd^c \log \|s(z)\|_{h^N}^2 \boxtimes dd^c \log \|s(w)\|_{h^N}^2 = \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \setminus D} G_h(z,w) Z_s \boxtimes Z_s.$$

Proof. For the first formula, we take the logarithm of both sides, and integrate dd^c by parts in the integral $\int_{\mathbb{C}\mathbb{P}^1} G_h(z,w) dd^c \log \|s(w)\|_{h^N}^2$. It is possible since both factors are global currents on $\mathbb{C}\mathbb{P}^1$. The resulting integral equals $\log \|s(z)\|_{h^N}^2 - \int_{\mathbb{C}\mathbb{P}^1} \log \|s(z)\|_{h^N}^2 \omega_h$.

For (ii) we write $\frac{1}{N} dd^c \log \|s(z)\|_{h^N}^2 = Z_s - \omega_h$. Then

$$\begin{aligned} (ii) &= \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \setminus D} G_h(z,w) (Z_s - \omega_h) \boxtimes (Z_s - \omega_h) \\ &= \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \setminus D} G_h(z,w) Z_s \boxtimes Z_s \\ &\quad - 2 \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \setminus D} G_h(z,w) Z_s \boxtimes \omega_h + \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \setminus D} G_h(z,w) \omega_h \boxtimes \omega_h \\ &= \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \setminus D} G_h(z,w) Z_s \boxtimes Z_s, \end{aligned}$$

since $\int G_h(z,w) \omega_h = 0$ when integrating in either z or w . By Proposition 10, $G_h \in L^1(\mathbb{C}\mathbb{P}^1, \omega_h)$; the integral over $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \setminus D$ in the last terms is the same as over $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$. □

COROLLARY 16. *We have:*

$$e^{\int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \setminus D} G_h(z,w) dd^c \log \|s_{\zeta}(z)\|_{h^N}^2 \boxtimes dd^c \log \|s_{\zeta}(w)\|_{h^N}^2} = e^{\sum_{i \neq j} G_h(\zeta_i, \zeta_j)}$$

4.6. **Completion of proof of Proposition 3.** We now complete the proof of Proposition 3, which was started in §3.3. The purpose of this section is to convert the local expression (29) (see also (30)) for the joint probability current into a global invariant expression. We prove:

LEMMA 17. *Let $h = e^{-\varphi}$ be a smooth Hermitian metric on $\mathcal{O}(1)$, and let ω_h, G_h be as above. Let $s_{\zeta}(z) = \prod_{j=1}^N (z - \zeta_j) e^N$. Then, the joint probability current is given by:*

$$\begin{aligned} \frac{|\det \mathcal{A}_N(h, \nu)|^2 |\Delta(\zeta_1, \dots, \zeta_N)|^2 \prod_{j=1}^N d^2 \zeta_j}{\left(\int_{\mathbb{C}\mathbb{P}^1} \prod_{j=1}^N |(z - \zeta_j)|^2 e^{-N\varphi} d\nu(z) \right)^{N+1}} &= \frac{\exp\left(\frac{1}{2} \sum_{i \neq j} G_h(\zeta_i, \zeta_j)\right)}{\left(\int_{\mathbb{C}\mathbb{P}^1} e^{\int_{\mathbb{C}\mathbb{P}^1} G_h(z,w) d\mu_{\zeta}(w)} d\nu(z) \right)^{N+1}} \left(\prod_{j=1}^N e^{-2\varphi(\zeta_j)} d^2 \zeta_j \right) \\ &\times |\det \mathcal{A}_N(h, \nu)|^2 e^{(-\frac{1}{2} N(N-1) + N(N+1))E(h)}, \end{aligned}$$

where $E(h)$ is defined in (14) and \mathcal{A}_N is defined in (47). Moreover, $\prod_{j=1}^N e^{-2\varphi(\zeta_j)} d^2\zeta_j$ extends to a global smooth (N, N) form κ_N on $(\mathbb{C}\mathbb{P}^1)^N$.

Proof. We first claim that

$$|\Delta(\zeta)|^2 = \exp\left(\sum_{i<j} G_h(\zeta_i, \zeta_j)\right) \exp\left((N-1)\sum_j \varphi(\zeta_j) - \frac{1}{2}(N-1)NE(h)\right). \quad (77)$$

Indeed, by Lemma 8,

$$2\log|z-w| = G_h(z, w) + \varphi(z) + \varphi(w) - E(h).$$

We note that $\log|\Delta(\zeta)|^2 = 2\sum_{i<j} \log|\zeta_i - \zeta_j|$ and that

$$\begin{aligned} 2\sum_{i<j} \log|\zeta_i - \zeta_j| &= \sum_{i<j} G_h(\zeta_i, \zeta_j) + \sum_{i<j} (\varphi(\zeta_i) + \varphi(\zeta_j)) - E(h) \\ &= \sum_{i<j} G_h(\zeta_i, \zeta_j) + (N-1)\sum_j \varphi(\zeta_j) - \frac{1}{2}(N-1)NE(h) \\ &= \sum_{i<j} G_h(\zeta_i, \zeta_j) + N(N-1)\int \varphi d\mu_\zeta - \frac{1}{2}(N-1)NE(h). \end{aligned} \quad (78)$$

We then convert the denominator into the Green's function expression by the identities

$$\begin{aligned} \int_{\mathbb{C}\mathbb{P}^1} \prod_{j=1}^N |(z - \zeta_j)|^2 e^{-N\varphi} d\nu(z) &= \int_{\mathbb{C}\mathbb{P}^1} \|s_\zeta(z)\|_{h^N}^2 d\nu(z) \\ &= \left(\int_{\mathbb{C}\mathbb{P}^1} e^{\int_{\mathbb{C}\mathbb{P}^1} G_h(z, w) dd^c \log \|s_\zeta(w)\|_{h^N}^2} d\nu\right) e^{\int_{\mathbb{C}\mathbb{P}^1} \log \|s_\zeta\|_{h^N}^2(z) \omega_h} \\ &= \left(\int_{\mathbb{C}\mathbb{P}^1} e^{N\int_{\mathbb{C}\mathbb{P}^1} G_h(z, w) d\mu_\zeta(w)} d\nu\right) e^{\int_{\mathbb{C}\mathbb{P}^1} \log \|s_\zeta\|_{h^N}^2(z) \omega_h} \end{aligned} \quad (79)$$

by Lemmas 28 and 15 (i). Further, by Lemma 9,

$$\int \log \|s_\zeta(w)\|_{h^N}^2 \omega_h = N\left(\int \varphi d\mu_\zeta - E(h)\right).$$

We now raise the denominator (79) to the power $-(N+1)$ and multiply by (77) to obtain the Green's expression

$$\frac{\exp\left(\sum_{i<j} G_h(\zeta_i, \zeta_j)\right)}{\left(\int_{\mathbb{C}\mathbb{P}^1} e^{\int_{\mathbb{C}\mathbb{P}^1} G_h(z, w) d\mu_\zeta(w)} d\nu(z)\right)^{N+1}}$$

multiplied by the exponential of

$$(N-1)N\int \varphi d\mu_\zeta - \frac{1}{2}(N-1)NE(h) - N(N+1)\left(\int \varphi d\mu_\zeta - E(h)\right).$$

We note the cancellation in the N^2 term of $\int \varphi d\mu_\zeta$, leaving $-2N\int \varphi d\mu_\zeta = -2\sum_j \varphi(\zeta_j)$. This gives the stated result. The last statement follows from Lemma 5. \square

4.7. The approximate rate function I_N . Lemma 17 expresses the joint probability $\vec{K}_n^N(\zeta_1, \dots, \zeta_N)$ as a geometric $(N+1, N+1)$ form on configuration space. In order to extract a rate function, we further express it as a functional of the measures μ_ζ . We introduce the following functionals.

Definition: Let $\zeta \in (\mathbb{CP}^1)^{(N)}$ and let μ_ζ be as in (1). Let $D = \{(z, z) : z \in \mathbb{CP}^1\}$ be the diagonal. Put:

$$\begin{cases} \mathcal{E}_N^h(\mu_\zeta) = \int_{\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus D} G_h(z, w) d\mu_\zeta(z) d\mu_\zeta(w), \\ J_N^{h, \nu}(\mu_\zeta) = \log \|e^{U_h^{\mu_\zeta}}\|_{L^N(\nu)} \end{cases}$$

LEMMA 18. *We have*

$$\vec{K}_n^N(\zeta_1, \dots, \zeta_N) = \frac{1}{\hat{Z}_N(h)} e^{-N^2 \left(-\frac{1}{2} \mathcal{E}_N^h(\mu_\zeta) + \frac{N+1}{N} J_N^{h, \nu}(\mu_\zeta) \right)} \kappa_N$$

Proof. We are simply rewriting

$$\frac{\exp\left(\frac{1}{2} \sum_{i \neq j} G_h(\zeta_i, \zeta_j)\right)}{\left(\int_{\mathbb{CP}^1} e^{\int_{\mathbb{CP}^1} G_h(z, w) d\mu_\zeta(w)} d\nu(z)\right)^{N+1}} = e^{-N^2 I_N(\mu_\zeta)}, \quad (80)$$

on the right side of Lemma 17 and leaving the other factors as they are. Then,

$$\begin{aligned} I_N(\mu_\zeta) &= -\frac{1}{N^2} \sum_{i \neq j} \frac{1}{2} G_h(\zeta_i, \zeta_j) + \frac{N+1}{N^2} \log \left(\int_{\mathbb{CP}^1} e^{N \int_{\mathbb{CP}^1} G_h(z, w) d\mu_\zeta(w)} d\nu(z) \right) \\ &= -\frac{1}{N^2} \frac{1}{2} \int_{\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus D} G_h(z, w) d\mu_\zeta(z) d\mu_\zeta(w) + \frac{N+1}{N^2} \log \left(\int_{\mathbb{CP}^1} e^{N U_h^{\mu_\zeta}(z)} d\nu(z) \right) \\ &= -\frac{1}{N^2} \left(-\frac{1}{2} \mathcal{E}_N^h(\mu_\zeta) + \frac{N(N+1)}{N^2} J_N^{h, \nu}(\mu_\zeta) \right). \end{aligned}$$

□

5. WEIGHTED EQUILIBRIUM MEASURES

In this section, we define the notion of weighted equilibrium measure $\nu_{h, K}$ of a non-polar compact set K with respect to a Hermitian metric h and prove that it is unique. In fact, there are two characterizations of $\nu_{h, K}$:

- (i) $\nu_{h, K}$ is the minimizer of the Green's energy functional among measures supported on K .
- (ii) The potential of $\nu_{h, K}$ is the maximal ω_h -subharmonic function of K .

We will need both characterizations in order to prove that the unique minimizer of the function $I^{h, K}$ of (17) is $\nu_{h, K}$. The problem is that $I^{h, K}$ differs significantly from the Green's energy on K and it is not obvious that they have the same minimizer.

In the classical case of weighted potential theory on \mathbb{C} , the equivalence of the two definitions is proved in [ST], especially in the appendix by T. Bloom. Their framework of admissible weights on \mathbb{C} does not quite apply directly to the present setting of smooth Hermitian metrics on $\mathcal{O}(1)$ and potential theory on \mathbb{CP}^1 . The second definition (ii) is assumed in work on potential theory on Kähler manifolds, e.g. as in [GZ], Definition 4.1. Only recently in

[BB] have equilibrium measures been considered in terms of energy minimization. As a result, there is no simple reference for the facts we need, although their proofs are often small modifications of known proofs in the weighted case on \mathbb{C} . In that event, we only sketch the proof and refer the reader to the literature.

5.1. Equilibrium measures as energy minimizers. We now justify the first definition (i) by showing that there exists a unique energy minimizer (or maximizer, depending on the sign of the energy functional) among measures supported on a non-polar set K . We further prove that weighted equilibrium measures $\nu_{h,K}$ are unique and are supported on K (Proposition 19). We recall that K is a polar set if $\mathcal{E}_h(\mu) = -\infty$ for every finite non-zero Borel measure μ supported in K . In particular, a set satisfying (11) is non-polar.

Thus, we fix a compact non-polar subset $K \subset \mathbb{C}\mathbb{P}^1$ and consider the restriction of the energy functional $\mathcal{E}_\omega : \mathcal{M}(K) \rightarrow \mathbb{R}$ of probability measures supported on K .

PROPOSITION 19. *If $K \subset \mathbb{C}\mathbb{P}^1$ is non-polar, then \mathcal{E}_ω is bounded above on $\mathcal{M}(K)$. It has a unique maximum $\nu_{K,\omega} \in \mathcal{M}(K)$.*

We denote its potential, the *weighted equilibrium potential*, by

$$U_\omega^{\nu_{K,\omega}}(z) = \int G_\omega(z, w) d\nu_{K,\omega}(w). \quad (81)$$

Proof. We begin by sketching the proof in the case where $\omega = \omega_h$ is a Kähler metric. In this case, the proof follows the standard lines of [Ran] (Theorem 3.3.2 and Theorem 3.7.6) or [ST], Theorem 1.3 and particularly Theorem 5.10. Existence follows from the upper semi-continuity of \mathcal{E}_ω , which holds exactly as in the local weighted case.

Uniqueness by the method of [ST], Theorem I.1.3 or Theorem II.5.6 uses the non-positivity of the weighted logarithmic energy norm (Lemma I.1.8 of [ST]) or of the Green's energy norm (Theorem II.5.6). This argument applies directly to \mathcal{E}_ω when ω is Kähler : one assumes for purposes of contradiction that there exist two energy maximizers μ, ν of mass one. Then it follows by the argument of Theorem I. 1.3 (b) of [ST] that $\|\mu - \nu\|_\omega^2 = 0$; so $\mu - \nu = C\omega$. But integration over $\mathbb{C}\mathbb{P}^1$ shows that $C = 0$, proving that $\mu = \nu$.

We then consider a general smooth Hermitian metric h . Since \mathcal{E}_h is bounded above and $\mathcal{M}(K)$ is closed and hence compact, there exist measures in $\mathcal{M}(K)$ which maximize the energy \mathcal{E}_h . We now prove uniqueness:

LEMMA 20. *If $K \subset \mathbb{C}\mathbb{P}^1$ is non-polar, and h is any smooth metric, then \mathcal{E}_h has a unique maximizer $\nu_{K,h} \in \mathcal{M}(K)$.*

Proof. We put

$$V_{\omega_h}(K) = \max\{\mathcal{E}_h(\mu) : \mu \in \mathcal{M}(K)\} < \infty. \quad (82)$$

To prove uniqueness, we observe that, for any signed measure $\mu - \nu$ given by a difference of two elements of $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$, hence satisfying $\int_{\mathbb{C}\mathbb{P}^1} d(\mu - \nu) = 0$, we have

$$\|\mu - \nu\|_{\omega_h}^2 = \|\mu - \nu\|_\omega^2. \quad (83)$$

Indeed,

$$\begin{aligned}
\|\mu - \nu\|_{\omega_h}^2 &= \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1} G_h(z, w) d(\mu - \nu) \otimes d(\mu - \nu) \\
&= \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1} (G_\omega(z, w) + U_h^\omega(z) + U_h^\omega(w) - \int U_h^\omega \omega) d(\mu - \nu) \otimes d(\mu - \nu) \\
&= \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1} G_\omega(z, w) d(\mu - \nu) \otimes d(\mu - \nu) = \|\mu - \nu\|_\omega^2.
\end{aligned} \tag{84}$$

Hence, the energy form is negative semi-definite on the subspace of signed measures $\mu - \nu$ where μ, ν are positive and of the same mass.

Suppose that $\mu, \nu \in \mathcal{M}(K)$ and that both are maximizers of \mathcal{E}_h on $\mathcal{M}(K)$. Then $\int_{\mathbb{C}\mathbb{P}^1} d(\mu - \nu) = 0$ and hence $\|\mu - \nu\|_\omega^2 \leq 0$. Equality holds if and only if $\mu - \nu = C\omega$ for some C , and the fact that $\int d(\mu - \nu) = 0$ implies $C = 0$. But

$$\left\| \frac{1}{2}(\mu + \nu) \right\|_h^2 + \left\| \frac{1}{2}(\mu - \nu) \right\|_{\omega_h}^2 = \frac{1}{2}(\mathcal{E}_h(\mu) + \mathcal{E}_h(\nu)) = V_h(K). \tag{85}$$

Since $\mathcal{E}_h(\sigma) \leq V_{\omega_h}(K)$ for any $\sigma \in \mathcal{M}(\mathbb{C}\mathbb{P}^1)$, it follows that $\|\mu - \nu\|_\omega^2 = 0$ and hence $\mu = \nu$. This completes the proof of uniqueness. □

This completes the proof of Proposition 19. □

Definition: The weighted capacity of K with respect to ω is defined by

$$\text{Cap}_\omega(K) = e^{\sup\{\mathcal{E}_\omega(\mu) : \mu \in \mathcal{M}(K)\}} = e^{\mathcal{E}_\omega(\nu_{K,\omega})}. \tag{86}$$

5.2. Equilibrium measure and subharmonic envelopes. We now discuss the second characterization (ii) of equilibrium measures (see the beginning of §5) and prove that it is equivalent to the first.

Given a closed real $(1, 1)$ form ω (not necessarily a Kähler form), and a compact subset $K \subset \mathbb{C}\mathbb{P}^1$, define the global extremal function $V_{K,\omega}^*$ as the upper semi-continuous regularization of

$$V_{K,\omega}(z) := \sup\{u(z) : u \in SH(\mathbb{C}\mathbb{P}^1, \omega, K)\}, \tag{87}$$

where

$$SH(\mathbb{C}\mathbb{P}^1, \omega, K) := \{u \in SH(\mathbb{C}\mathbb{P}^1, \omega) : u \leq 0 \text{ on } K\}. \tag{88}$$

(See (63) for the definition of $SH(\mathbb{C}\mathbb{P}^1, \omega)$.)

The important properties of $V_{K,\omega}^*$ and $\nu_{K,\omega}$ are the following, a special case of Theorem 4.2 of [GZ]:

THEOREM 21. *Let $K \subset \mathbb{C}\mathbb{P}^1$ be a Borel set. If K is non-polar, then $V_{K,\omega}^* \in SH(\mathbb{C}\mathbb{P}^1, \omega)$ and satisfies:*

- (1) $\nu_{K,\omega} = 0$ on $\mathbb{C}\mathbb{P}^1 \setminus \overline{K}$.
- (2) $V_{K,\omega}^* = 0$ on $\text{supp } \nu_{K,\omega}$ and in the interior of K ;
- (3) $\int_{\overline{K}} \nu_{K,\omega} = \int \omega (= 1)$.

The following Proposition relates $V_{K,\omega}^*$ to the potential (81) of the equilibrium measure of Proposition 19.

PROPOSITION 22. *Let $K \subset \mathbb{C}\mathbb{P}^1$ be a non-polar compact subset and let ω be a smooth $(1, 1)$ form with $\int_{\mathbb{C}\mathbb{P}^1} \omega = 1$. Then,*

$$\nu_{K,\omega} = dd^c V_{K,\omega}^* + \omega. \quad (89)$$

Moreover,

$$U_{\omega}^{\nu_{K,\omega}} = V_{K,\omega}^* - \int_{\mathbb{C}\mathbb{P}^1} V_{K,\omega}^* \omega. \quad (90)$$

In particular, with $F_{K,\omega} = \int_{\mathbb{C}\mathbb{P}^1} V_{K,\omega}^* \omega$, we have that $U_{\omega}^{\nu_{K,\omega}} = -F_{K,\omega}$ on the support of $\nu_{K,\omega}$.

Proof. We only sketch the proof, which is barely different from the case of admissible potential theory on \mathbb{C} [ST].

The second statement (90) implies the first. It shows that both $U_{\omega}^{\nu_{K,\omega}}$ and $V_{K,\omega}^*$ belong to $SH(\mathbb{C}\mathbb{P}^1, \omega)$ and are potentials for $\nu_{K,\omega}$, i.e.

$$dd^c U_{\omega}^{\nu_{K,\omega}} = \nu_{K,\omega} - \omega = dd^c V_{K,\omega}^*. \quad (91)$$

The potentials must differ by a constant, which is determined by integrating with respect to ω and using that $\int U_{\omega}^{\mu} \omega = 0$ for any μ . The proof is essentially the same as in the classical unweighted case (see Lemma 2.4 of Appendix B.2 of [ST]).

It therefore suffices to prove (90). The proof in the case of weighted potential theory on \mathbb{C} is given in Theorem I.4.1 of [ST], as sharpened in Appendix B, Lemma 2.4 of [ST]. The main ingredients are the so-called principle of domination (see [ST], I.3), and the Frostman type theorem that $U_{\omega}^{\nu_{K,\omega}} \geq F_{\omega}$ q.e. on K and $U_{\omega}^{\nu_{K,\omega}} \leq F_{K,\omega}$ on $\text{supp } \nu_{K,\omega}$, hence $U_{\omega}^{\nu_{K,\omega}} = F_{K,\omega}$ on $\text{supp } \nu_{K,\omega}$ (see [ST], Theorem I.1.3 (d)-(f)). \square

We may view the energy function as a function on $SH(\mathbb{C}\mathbb{P}^1, \omega)$ rather than on $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$:

$$\mathcal{E}_{\omega}(\psi) := \int_{\mathbb{C}\mathbb{P}^1} \psi(dd^c \psi + \omega). \quad (92)$$

This definition is slightly more general than the preceding one since $\int U_{\omega}^{\mu} \omega = 0$ but ψ is not assumed to be so normalized.

Remark: We note that $V_{K,\omega}^* \in SH(\mathbb{C}\mathbb{P}^1, \omega, K)$ and that $U_{\omega}^{\nu_{K,\omega}} \in SH(\mathbb{C}\mathbb{P}^1, \omega, K)$. In classical weighted potential theory, both $U_{\omega}^{\nu_{K,\omega}}$ and $V_{K,\omega}^*$ are defined slightly differently, and their difference $F_{K,\omega}$ is known as the *Robin constant*.

COROLLARY 23. *We have*

- (1) $U_{\omega}^{\nu_{K,\omega}}$ maximizes \mathcal{E}_{ω} among all elements of $SH(\mathbb{C}\mathbb{P}^1, \omega, K)$.
- (2) $V_{K,\omega}^*$ maximizes \mathcal{E}_{ω} among all elements of $SH(\mathbb{C}\mathbb{P}^1, \omega, K)$.

5.3. Thin points, regular points and capacity. A set E is said to be *thin* at x_0 iff either of the following occur:

- x_0 is not a limit point of E ;
- there exists a disc $D_{\epsilon}(x_0)$ and $\epsilon > 0$ and a potential U^{μ} so that $U^{\mu}(x_0) > -\infty$, so that $U^{\mu}(x) \leq U^{\mu}(x_0) - \eta$ for all $x \in E \cap D_{\epsilon}(x_0) \setminus \{x_0\}$.

We refer to [Lan], Definition Ch. V §3 (5.3.1) or [D]. A point x_0 is called irregular for E if E is thin at x_0 . Thus, our assumption on K is that it is non-thin at all of its points. A subset of a thin set at x_0 is also thin at x_0 and the union of two thin sets at x_0 is thin there.

We further recall:

LEMMA 24. (see [ST], Corollary 6.11, or the Corollary to Theorem 3.7 of [Lan], Ch. III §2) *If $S \subset \mathbb{C}$ is compact and of positive capacity, then there exists a positive, finite measure ν , with support included in S , so that $U^\nu \in C(\mathbb{C}\mathbb{P}^1)$.*

The idea is that S contains a regular subset of positive capacity.

6. RATE FUNCTION AND EQUILIBRIUM MEASURE

We continue to fix a pair (h, ν) where h is a smooth Hermitian metric on $\mathcal{O}(1)$ and where ν is a measure satisfying (9). The purpose of this section is to prove that the rate function (18) of the LDP of Theorem 1 is a *good rate function* and also to prove that its unique minimizer is the equilibrium measure for (h, K) . That is, we prove:

PROPOSITION 25. *The function $I^{h,K}$ of (17) has the following properties:*

- (1) *It is a lower-semicontinuous functional.*
- (2) *It is convex.*
- (3) *Its unique minimizer is the equilibrium measure $\nu_{h,K}$.*
- (4) *Its minimum value equals $\frac{1}{2} \log \text{Cap}_{\omega_h}(K)$.*

We begin with the following elementary consequence of Proposition 10.

LEMMA 26. *For each $z \in \mathbb{C}\mathbb{P}^1$, the function $\mu \rightarrow U_h^\mu(z)$ is an upper semi-continuous function from $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$ to $\mathbb{R} \cup \{-\infty\}$. Further, so is the function $\mu \rightarrow \mathcal{E}_h(\mu)$.*

Proof. Fix $M \in \mathbb{R}$ and define $G_h^M(z, w) = G_h(z, w) \vee (-M)$. By Proposition 10, G_h^M is continuous on $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$. Set

$$U_h^{\mu, M}(z) = \int_{\mathbb{C}\mathbb{P}^1} G_h^M(z, w) d\mu(w), \quad \mathcal{E}_h^M(\mu) = \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1} G_h^M(z, w) d\mu(z) d\mu(w). \quad (93)$$

For fixed z , it follows that $\mu \rightarrow U_h^{\mu, M}(z)$ and $\mu \rightarrow \mathcal{E}_h^M(\mu)$ are continuous on $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$. Since $U_h^\mu(z) = \inf_M U_h^{\mu, M}(z)$ and $\mathcal{E}_h(\mu) = \inf_M \mathcal{E}_h^M(\mu)$, the claimed upper semi-continuity follows. \square

We next have the following.

LEMMA 27. (1) *The function $J^{h,K}(\mu) = \sup_{z \in K} U_h^\mu(z)$ is upper semi-continuous.*
 (2) *Assume that all points of K are regular. Then $J^{h,K}(\mu)$ is also lower semi-continuous.*

Proof. (i) Upper semi-continuity

We begin by proving the upper semi-continuity. Let $\mu_n \rightarrow \mu^*$ weakly in $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$. Fix $M \in \mathbb{R}$ and recall that $G_h^M(\cdot, \cdot)$ is continuous on $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$. Therefore, the map $(z, \mu) \rightarrow U_h^{\mu, M}(z)$ is continuous. Because K is compact, $\mu \mapsto \sup_{z \in K} U_h^{\mu, M}(z)$ is therefore continuous. Thus,

$$J^{h,K}(\mu_n) = \sup_{z \in K} U_h^{\mu_n}(z) \leq \sup_{z \in K} U_h^{\mu_n, M}(z) \rightarrow_{n \rightarrow \infty} \sup_{z \in K} U_h^{\mu^*, M}(z).$$

Since $U_h^{\mu^*,M}(z) \rightarrow_{M \rightarrow \infty} U_h^{\mu^*}(z)$ for any z by monotone convergence, we have

$$\sup_{z \in K} U_h^{\mu^*,M}(z) \rightarrow_{M \rightarrow \infty} \sup_{z \in K} U_h^{\mu^*}(z) = J^{h,K}(\mu).$$

Combining the last two displays completes the proof of (1).

(ii) Lower semi-continuity

Let K be a set all of whose points are regular. Suppose that $\mu_n \rightarrow \mu$. We claim that

$$\liminf_{n \rightarrow \infty} \sup_K U_h^{\mu_n} \geq \sup_{z \in K} U_h^\mu(z) := a.$$

(Recall that $a < \infty$.) For any $\epsilon > 0$ set

$$A_\epsilon = \{z \in \mathbb{C}\mathbb{P}^1 : U_h^\mu \geq a - \epsilon\}.$$

We claim the following

$$\forall \epsilon > 0, \quad \text{Cap}(A_\epsilon \cap K) > 0. \quad (94)$$

To prove the claim, let z^* be a point where U_h^μ attains its maximum on K (such a point exists by the upper semicontinuity of $z \mapsto U_h^\mu(z)$ and the compactness of K). By assumption, z^* is a regular point. Since

$$U_h^\mu < a - \epsilon, \quad \text{on } K \setminus A_\epsilon,$$

$K \setminus A_\epsilon$ is thin at z^* (see [Lan] (5.3.2), page 307). Suppose that there exists $\epsilon_0 > 0$ so that $\text{Cap}(A_{\epsilon_0} \cap K) = 0$. A set of capacity zero is thin at each of its points, see [Ch], Corollary on page 92. Since

$$K = K \setminus A_{\epsilon_0} \cup (K \cap A_{\epsilon_0}),$$

and since the union of two sets thin at z^* is thin at z^* , we see that K is thin at z^* . This contradicts the regularity of K at z^* and proves the claim.

We now complete the proof of the Proposition. Let $\mathbf{1}_{A_\epsilon \cap K}$ denote the characteristic function of $A_\epsilon \cap K$. Since U_h^μ is upper semi-continuous, $A_\epsilon \cap K$ is compact. By (94), it has positive capacity. It follows by Lemma 24 there exists a positive measure $\nu_{\epsilon,\mu,K}$ supported on $A_\epsilon \cap K$ whose potential $U^{\nu_{\epsilon,\mu,K}}$ is continuous.

We have,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{A_\epsilon} U_h^{\mu_n}(z) d\nu_{\epsilon,\mu,K}(z) &= \lim_{n \rightarrow \infty} \int U_h^{\nu_{\epsilon,\mu,K}}(z) d\mu_n(z) \\ &= \int U_h^{\nu_{\epsilon,\mu,K}} d\mu(z) \\ &= \int_{A_\epsilon} U_h^\mu(z) d\nu_{\epsilon,\mu,K}(z). \end{aligned}$$

Therefore,

$$\begin{aligned} \nu_{\epsilon,\mu,K}(A_\epsilon \cap K) \liminf_{n \rightarrow \infty} \sup_K U_h^{\mu_n} &\geq \liminf_{n \rightarrow \infty} \int_{A_\epsilon} U_h^{\mu_n}(z) d\nu_{\epsilon,\mu,K}(z) \\ &= \int_{A_\epsilon} U_h^\mu(z) d\nu_{\epsilon,\mu,K}(z) \geq (a - \epsilon) \nu_{\epsilon,\mu,K}(A_\epsilon \cap K). \end{aligned}$$

Since $\nu_{\epsilon,\mu,K}(A_\epsilon \cap K) > 0$ and since ϵ is arbitrary, this finishes the proof. \square

Remark: We note that if $d\nu$ is any measure on K whose potential U^ν is continuous, then $U^{\mathbf{1}_{A_\epsilon \cap K}^\nu}$ is automatically continuous. Indeed, $U^{\mathbf{1}_{A_\epsilon \cap K}^\nu}$ is upper semi-continuous, so we only need to prove that it is lower semi-continuous. But

$$U^{\mathbf{1}_{A_\epsilon \cap K}^\nu} = U^\nu - U^{\nu - \mathbf{1}_{A_\epsilon \cap K}^\nu},$$

and the first term on the right is continuous and the second, being the opposite of a potential, is lower semi-continuous.

A consequence of Lemma 27 is that $J^{h,K}(\cdot)$ is bounded on $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$, and thus $I^{h,K}(\cdot)$ is well-defined. Further, we have the following.

LEMMA 28. *The function $\tilde{I}^{h,K}(\cdot)$ on $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$ is a rate function.*

Proof. By Lemma 26 and Lemma 27, the function

$$-\frac{1}{2}\mathcal{E}_h(\cdot) + J^{h,K}(\cdot)$$

is well defined on $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$, bounded below, and lower semi-continuous. This implies the claim. \square

Next we prove convexity of the rate function. Convexity of the (unweighted) logarithmic energy is well-known [ST, BG, BB].

LEMMA 29. *$I^{h,K}$ is a strictly convex function on \mathcal{M} (with possible values $+\infty$)*

Proof. Convexity of the energy functional is proved in Proposition 12. To complete the proof, we note that the ‘potential term’ $J^{h,K}(\mu) = \sup_K U_h^\mu(z)$ is a maximum of affine functions of μ , hence is convex. \square

6.1. The global minimizer of $I^{h,K}$. Since $I^{h,K}$ is lower semi-continuous it has a minimum on $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$ and also on each closed ball $B(\sigma, \delta) \subset \mathcal{M}(\mathbb{C}\mathbb{P}^1)$. In this section we show that the global minimum is the equilibrium measure $\nu_{K,h}$ for the data (h, ν) defining $I^{h,K}$.

The equilibrium measure $\nu_{K,h}$ is the unique maximizer of $\mathcal{E}_h(\mu)$ on $\mathcal{M}(K)$. Our function differs from this constrained function in not being constrained to $\mathcal{M}(K)$ but rather possessing the term $\sup_K U_h^\mu$. We need to show that this term behaves like a ‘Lagrange multiplier’ enforcing the constraint. Unfortunately, it is not ‘smooth’ as a function of μ , so we cannot use calculus alone to demonstrate this.

LEMMA 30. *The global minimizer of $I^{h,K}$ is $\nu_{K,h}$. The global minimum is $\frac{1}{2} \log \text{Cap}_{\omega_h}(K)$.*

Proof.

$$\begin{aligned} 2I^{h,K}(\mu) &= -\mathcal{E}_h(\mu) + 2 \sup_K U_h^\mu \\ &= - \int_{\mathbb{C}\mathbb{P}^1} (U_h^\mu - \sup_K U_h^\mu)(dd^c U_h^\mu + \omega_h) + \sup_K U_h^\mu. \end{aligned} \tag{95}$$

We claim that $\int_{\mathbb{C}\mathbb{P}^1} (U_h^\mu - \sup_K U_h^\mu)(dd^c U_h^\mu + \omega_h) - \sup_K U_h^\mu$ is maximized by $V_{K,h}^*$. By definition, $U_h^\mu - \sup_{z \in K} U_h^\mu \leq 0$ on K . Hence, $U_h^\mu - \sup_{z \in K} U_h^\mu \leq V_{K,h}^*$. Since $dd^c U_h^\mu + \omega_h$ is

the positive measure $d\mu$,

$$\begin{aligned}
 -2I^{h,K}(\mu) &= \int_{\mathbb{C}} (U_h^\mu - \sup_K U_h^\mu) (dd^c U_h^\mu + \omega_h) - \sup_K U_h^\mu \\
 &\leq \int V_{K,h}^* (dd^c U_h^\mu + \omega_h) - \sup_K U_h^\mu \\
 &= \int (U_h^\mu - \sup_{z \in K} U_h^\mu) dd^c V_{K,h}^* - \sup_K U_h^\mu + \int_{\mathbb{C}\mathbb{P}^1} V_{K,h}^* \omega_h \\
 &= \int (U_h^\mu - \sup_{z \in K} U_h^\mu) (dd^c V_{K,h}^* + \omega_h) + \int_{\mathbb{C}\mathbb{P}^1} V_{K,h}^* \omega_h \\
 &\leq \int V_{K,h}^* (dd^c V_{K,h}^* + \omega_h) + \int_{\mathbb{C}\mathbb{P}^1} V_{K,h}^* \omega_h \\
 &= F_{K,\omega_h}.
 \end{aligned}$$

In the third line, we integrated dd^c by parts, and used that constants integrate to 0 against $dd^c V_{K,h}^*$. In the next to last line, we again use that $U_h^\mu - \sup_{z \in K} U_h^\mu \leq V_{K,h}^*$. In the last equality, we used Proposition 22.

Since

$$\int_{\mathbb{C}\mathbb{P}^1} (U_h^{\nu_{h,K}} - \sup_K U_h^{\nu_{h,K}}) (dd^c U_h^{\nu_{h,K}} + \omega_h) - \sup_K U_h^{\nu_{h,K}} = \int V_{K,h}^* (dd^c V_{K,h}^* + \omega_h) + F_{K,\omega_h},$$

we see that $I^{h,K}$ is minimized by $\nu_{h,K}$.

One easily checks that all of the inequalities are equalities for $\nu_{K,h}$. When K is regular, $V_{K,h}^* = (U_h^{\nu_{K,h}} - \sup_K U_h^{\nu_{K,h}})$ is continuous. We can determine the sup by integrating both sides against ω_h as in Proposition 22:

$$\int V_{K,h}^* \omega_h = - \sup_K U_h^{\nu_{K,h}}.$$

We have,

$$\begin{aligned}
 -2I^{h,K}(\nu_{K,h}) &= \int_{\mathbb{C}\mathbb{P}^1} (U_h^{\nu_{K,h}} - \sup_K U_h^{\nu_{K,h}}) (dd^c U_h^{\nu_{K,h}} + \omega_h) - \sup_K U_h^{\nu_{K,h}} \\
 &= \int V_{K,h}^* (dd^c V_{K,h}^* + \omega_h) + \int_{\mathbb{C}\mathbb{P}^1} V_{K,h}^* \omega_h = F_{K,\omega_h}.
 \end{aligned}$$

On the other hand, since $U_{\omega_h}^{\nu_{K,h}} = -F_{K,\omega_h}$ on K , we have that

$$\log \text{Cap}_{\omega_h}(K) = \mathcal{E}_{\omega_h}(\nu_{K,h}) = \int U_{\omega_h}^{\nu_{K,h}} d\nu_{K,h} = -F_{K,\omega_h}.$$

This completes the proof. □

7. LARGE DEVIATIONS THEOREMS IN GENUS ZERO: PROOF OF THEOREM 1

In this section, we prove Theorem 1. We already know that $\tilde{I}^{h,K}$ is a good rate function. We still need to prove that it actually is the rate function of the large deviations principle.

As in [BG] (Section 3), it is equivalent to prove that

$$-I(\sigma) := \limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) = \liminf_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)).$$

See Theorem 4.1.11 of [DZ].

The proof follows the approach in [BG, BZ] of large deviations principles for empirical measures of eigenvalues of certain random matrices. However, we take full advantage of the compactness of $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$, and of the properties of the Green function G , see Lemma 26 and Proposition 27.

We first prove the result without taking the normalizing constants $Z_N(h)$ of Proposition 3 into account. Then in §7.4, we determine the logarithmic asymptotics of the normalizing constants.

7.1. Heuristic derivation of $I^{h,K}$. Since the proof of the large deviations principle is technical, we first give a formal or heuristic derivation of the rate function (20) from the expression for the joint probability distribution of zeros in Lemma 18 in the spirit of the discussion in §1.5. We then fill in the technical gaps to give a rigorous proof.

7.1.1. Heuristic derivation. Lemma 18 expresses $\vec{K}_n^N(\zeta_1, \dots, \zeta_N)$ as a product of three factors: the normalizing constant $\frac{1}{Z_N(h)}$, the factor $e^{-N^2(-\frac{1}{2}\mathcal{E}_N^h(\mu_\zeta) + \frac{N+1}{N}J_N^{h,\nu}(\mu_\zeta))}$ and the integration measure $\prod_{j=1}^N e^{-2N\varphi(\zeta_j)} d^2\zeta_j$. The normalizing constant will be worked out asymptotically in §7.4 using the fact that $\vec{K}_n^N(\zeta_1, \dots, \zeta_N)$ is a probability measure. The integration measure $\prod_{j=1}^N e^{-2N\varphi(\zeta_j)} d^2\zeta_j$ is an invariantly defined smooth (N, N) on $(\mathbb{C}\mathbb{P}^1)^N$ of finite mass independent of N , and thus does not contribute to the logarithmic asymptotics. Hence, only the factor $\left(-\frac{1}{2}\mathcal{E}_N^h(\mu_\zeta) + \frac{N+1}{N}J_N^{h,\nu}(\mu_\zeta)\right)$ contributes to the rate function.

The term $\mathcal{E}_N^h(\mu_\zeta)$ closely resembles the energy except that the diagonal D has been punctured out of the domain of integration, as it must since μ_ζ has infinite energy. It must be shown that the true energy is the correct limiting form when measuring log probabilities of balls of measures.

The second term satisfies,

$$\lim_{N \rightarrow \infty} J_N^{h,\nu}(\mu_\zeta) = \log \|e^{U_h^\mu}\|_{L^N(\nu)} \uparrow \log \|e^{U_h^\mu}\|_{L^\infty(\nu)} = \sup_K U_h^\mu \quad (96)$$

monotonically as $N \rightarrow \infty$. Thus, it is natural to conjecture that the rate function for large deviations of empirical measures is given by (20).

We now turn to the rigorous proof.

7.2. Proof of the upper bound. In this section, we prove the upper bound part of the large deviation principle, that is we prove that

$$\lim_{\delta \downarrow 0} \limsup_N \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) \leq -\tilde{I}^{h,K}(\sigma). \quad (97)$$

The first step is:

LEMMA 31. Fix $\epsilon > 0$. If ν satisfies the Bernstein-Markov condition (9), then there exists a $N_0 = N_0(\epsilon)$ such that for all $N > N_0$ and all $\mu_\zeta \in \mathcal{M}(\mathbb{C}\mathbb{P}^1)$,

$$\log \|e^{U_h^{\mu_\zeta}}\|_{L^N(\nu)} \geq \sup_{z \in K} U_h^{\mu_\zeta} - \epsilon.$$

Proof. We are assuming that, for all $s \in H^0(C, L^N)$,

$$\sup_{z \in K} |s(z)|_{h^N} \leq C_\epsilon e^{\epsilon N} \left(\int_K |s(z)|_{h^N}^2 d\nu(z) \right)^{1/2}. \quad (98)$$

By Lemma 9 we may write

$$|s_\zeta(z)|_{h^N}^2 = e^{NU_h^{\mu_\zeta}(z)} e^{N(\int \varphi d\mu_\zeta - E(h))}$$

Hence,

$$\begin{aligned} \|e^{U_h^{\mu_\zeta}}\|_{L^N e^{(\int \varphi d\mu_\zeta - E(h))}} &= \left(\int_K |s_\zeta(z)|_{h^N}^2 d\nu(z) \right)^{1/N} \\ &\geq \left(C_\epsilon^{-1} e^{-N\epsilon} \sup_{z \in K} |s_\zeta(z)|_{h^N}^2 \right)^{\frac{1}{N}} \\ &\implies \log \|e^{U_h^{\mu_\zeta}}\|_{L^N} \geq \sup_{z \in K} U_h^{\mu_\zeta} - \epsilon + \frac{1}{N} \log C_\epsilon, \end{aligned} \quad (99)$$

for all $\epsilon > 0$. □

Write

$$\Theta_N = -\frac{1}{N^2} \log \hat{Z}_N(h).$$

As we will see in the course of the proof of Lemma 4, $\Theta_N \rightarrow_{N \rightarrow \infty} \log \text{Cap}_{\omega_h}(K)$.

By Lemma 17 and Lemma 18,

$$\frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) = \frac{1}{N^2} \log \int_{\zeta \in (\mathbb{C}\mathbb{P}^1)^N; \mu_\zeta \in B(\sigma, \delta)} e^{-N^2 I_N(\mu_\zeta)} \kappa_N + \Theta_N. \quad (100)$$

Fix $M \in \mathbb{R}$ and let $G_h^M = G_h \vee (-M)$ be the truncated Green function. By Lemma 26, G_h^M is continuous on $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$. Further, with notation as in (93),

$$\begin{aligned} -\frac{1}{N^2} \sum_{i < j} G_h(\zeta_i, \zeta_j) &\geq -\frac{1}{N^2} \sum_{i < j} G_h^M(\zeta_i, \zeta_j) \\ &\geq -\frac{1}{2} \int \int_{\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1} G_h^M(z, w) d\mu_\xi(z) d\mu_\xi(w) - \frac{C(M)}{N} = \mathcal{E}_h^M(\mu_\xi) - \frac{C(M)}{N}, \end{aligned}$$

where the constant $C(M)$ does not depend on ξ . Using Lemma 31 (and also Corollary 11), we then have that for any $\epsilon > 0$ and all $N > N_0(\epsilon)$,

$$\begin{aligned} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) &\leq \frac{1}{N^2} \log \int_{\xi \in (\mathbb{C}\mathbb{P}^1)^N; \mu_\xi \in B(\sigma, \delta)} e^{\frac{N^2}{2} \mathcal{E}_h^M(\mu_\xi) - N^2 J_h^K(\mu_\xi)} \kappa_N \\ &\quad + \left(\Theta_N + \frac{C'(M)}{N} + \epsilon \right), \end{aligned} \quad (101)$$

for some constant $C'(M)$. It follows that

$$\limsup_N \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) \leq \Theta_N + \limsup_{\delta \downarrow 0} \sup_{\mu \in B(\sigma, \delta)} - \left(-\frac{1}{2} \mathcal{E}_h^M(\sigma) + J_h^K(\sigma) \right).$$

Here, we use that

$$\frac{1}{N^2} \log \int_{(\mathbb{C}\mathbb{P}^1)^N} \kappa_N = O\left(\frac{\log N}{N}\right), \quad (102)$$

which follows from Lemma 5.

It then follows from the continuity of $\mathcal{E}_h^M(\sigma)$ and the lower semi-continuity of $J_h^K(\sigma)$ (see Lemma 27) that

$$\lim_{\delta \downarrow 0} \limsup_N \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) \leq \lim_{N \rightarrow \infty} \Theta_N + \frac{1}{2} \mathcal{E}_h^M(\sigma) - J_h^K(\sigma) + \epsilon.$$

Since $\mathcal{E}_h^M(\sigma) \rightarrow \mathcal{E}_h(\sigma)$ as $M \rightarrow \infty$ by monotone convergence, and since ϵ is arbitrary, we obtain (97). \square

7.3. Proof of the lower bound. In this section, we prove that

$$\lim_{\delta \downarrow 0} \liminf_N \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma, \delta)) = -I(\sigma). \quad (103)$$

The strategy is again similar to [BG] and [BZ]. Note first that to prove (103), it is enough to find, for any σ with $I(\sigma) < \infty$, a sequence $\sigma_\epsilon \rightarrow_{\epsilon \rightarrow 0} \sigma$ weakly in $\mathcal{M}(\mathbb{C}\mathbb{P}^1)$ such that $I(\sigma_\epsilon) \rightarrow I(\sigma)$ and (103) holds for σ_ϵ .

So, define $\sigma_\epsilon = e^{\epsilon \Delta} \sigma$ as in Lemma 34 of the appendix. By that lemma, the property $I(\sigma_\epsilon) \rightarrow_{\epsilon \rightarrow 0} I(\sigma)$ holds. It thus only remains to prove (103) when σ is replaced by σ_ϵ . Thus, the large deviations lower bound is a consequence of the following lemma.

LEMMA 32. *Let $\sigma = f\omega \in \mathcal{M}(\mathbb{C}\mathbb{P}^1)$ with f a strictly positive and continuous function on $\mathbb{C}\mathbb{P}^1$. Then, (103) holds.*

Proof. We follow [BZ], Lemma 2.5. It will be convenient to consider three charts on $\mathbb{C}\mathbb{P}^1$, denoted W_0, W_1, W_2 , with distance d_{W_i} on W_i , that cover $\mathbb{C}\mathbb{P}^1$, and a constant R , in such a way that for any two points $z, w \in \mathbb{C}\mathbb{P}^1$ there exists a chart $W_{z,w}$ with distance $d_{W_{z,w}}$ so that in local coordinates, $d(z, w) := d_{W_{z,w}}(z, w) \leq R$ (If more than one such chart exists for a given pair z, w , fix one arbitrarily as $W_{z,w}$; The charts W_0 can be taken as the standard chart \mathbb{C} , with W_1 and W_2 its translation to two fixed distinct points in $\mathbb{C}\mathbb{P}^1$.)

Construct a sequence of discrete probability measures

$$d\sigma_N = \frac{1}{N} \sum_{j=1}^N \delta_{Z_j} \in B(\sigma, \delta)$$

with the following properties:

- (1) $\sigma_N \in B(\sigma, \delta/2)$ for all N large;
- (2) $d(Z_i, Z_j) \geq \frac{C(\sigma, \delta)}{\sqrt{N}}$.

(Since in a local chart, σ possesses a bounded density with respect to Lebesgue's measure, such a sequence can be constructed by adapting to the local charts W_i the construction in the proof of Lemma 5 in [BZ].) Define

$$D_I^\eta = \{\zeta \in (\mathbb{C}\mathbb{P}^1)^N : d(\zeta_j, Z_j) \leq \frac{\eta}{N}, \quad j = 1, \dots, N\}.$$

Then, for η small enough and all N large, all $\zeta \in D_I^\eta$ satisfy that $\mu_\zeta \in B(\sigma, \delta)$. Since $D_I^\eta \subset B(\sigma, \delta)$,

$$\mathbf{Prob}_N(B(\sigma, \delta)) \geq \int_{D_I^\eta} e^{-N^2 I_N(\mu_\zeta)} \kappa_N + \Theta_N. \quad (104)$$

(Recall Lemma 17 for the definition of the (N, N) form κ_N .) By Proposition 10 and our construction, there exists a constant $C_1 = C_1(\eta, \sigma)$ with $C_1 \rightarrow_{\eta \rightarrow 0} 0$ such that for any $\xi \in D_I^\eta$ and $i, j \in \{1, \dots, N\}$, $i \neq j$,

$$G_h(\xi_i, \xi_j) \geq G_h(Z_i, Z_j) - C_1(\eta).$$

For $\epsilon > 0$, set

$$\mathcal{E}_h^\epsilon(\sigma) := \int_{\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus D_\delta} G_h(z, w) d\sigma(z) d\sigma(w),$$

where $D_\epsilon = \{(z, w) \in (\mathbb{CP}^1 \times \mathbb{CP}^1) : d(z, w) < \epsilon\}$. We have by monotone convergence that

$$\mathcal{E}_h(\sigma) = \lim_{\epsilon \rightarrow 0} \mathcal{E}_h^\epsilon(\sigma).$$

Because G_h is continuous on $\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus D_\epsilon$, we have that

$$N^{-2} \sum_{i \neq j, d(Z_i, Z_j) \geq \epsilon} G_h(Z_i, Z_j) \geq \mathcal{E}_h(\sigma) - C_2(\epsilon, \delta),$$

where for fixed ϵ , $C_2(\epsilon, \delta) \rightarrow_{\delta \rightarrow 0} 0$. On the other hand, let $J_i(r) = \#\{j \in \{1, \dots, N\} : j \neq i, d(Z_i, Z_j) \in [r/\sqrt{N}, (r+1)/\sqrt{N}]\}$. From our construction, there exists a constant $C_3 = C_3(\sigma, \delta)$ such that $J_i(r) \leq C_3 r$ for any $i \in \{1, \dots, N\}$ and all $r < \epsilon\sqrt{N}$, if ϵ is smaller than some ϵ_0 independent of δ . Thus, applying Proposition 10, we have that

$$N^{-2} \sum_{i \neq j, d(Z_i, Z_j) \leq \epsilon} |G_h(Z_i, Z_j)| \leq \frac{C_3(\epsilon, \delta, \sigma)}{N^2} \sum_{i=1}^N \sum_{k=1}^{\epsilon\sqrt{N}} k \log(k/\sqrt{N}) = O\left(\frac{\log N}{\sqrt{N}}\right).$$

For $\epsilon' > 0$ given, fix $\epsilon > 0$ so that

$$|\mathcal{E}_h(\sigma) - \mathcal{E}_h^\epsilon(\sigma)| < \epsilon'.$$

Then, taking $N \rightarrow \infty$ we conclude that

$$\limsup_{N \rightarrow \infty} |N^{-2} \sum_{i \neq j} G_h(Z_i, Z_j) - \mathcal{E}_h(\sigma)| \leq C_2(\epsilon, \delta) + \epsilon'.$$

In particular, taking $\delta = \delta(\epsilon')$ small enough gives

$$\limsup_{N \rightarrow \infty} |N^{-2} \sum_{i \neq j} G_h(Z_i, Z_j) - \mathcal{E}_h(\sigma)| \leq 2\epsilon'.$$

By Proposition 27, reducing δ further if necessary, we also have $|J^{h,K}(\sigma_N) - J^{h,K}(\sigma)| \leq \epsilon'$. Combining these estimates and substituting in (104), one gets that for any $\epsilon' > 0$ and all N large enough,

$$\mathbf{Prob}_N(B(\sigma, \delta)) \geq e^{-N^2 I(\sigma) - 3\epsilon' N^2} \int_{D_I^\eta} \kappa_N \quad (105)$$

To complete the proof, we again use (102). Indeed, as above (see Proposition 5), κ is a smooth positive $(1, 1)$ form on $\mathbb{C}\mathbb{P}^1$. Now, $D_I^\eta \subset (\mathbb{C}\mathbb{P}^1)^N$ is a product of the one-complex dimensional sets $D_j^{\eta;N} := \{\zeta_j : d(\zeta_j, Z_j) \leq \frac{\eta}{N}\} \subset \mathbb{C}\mathbb{P}^1$. Hence,

$$\int_{D_I^\eta} \kappa_N = \left(\int_{D_1^{\eta;N}} \kappa \right)^N.$$

Since κ is a smooth area form, $\int_{D_1^{\eta;N}} \kappa \sim CN^{-2}$. It follows that

$$\frac{1}{N^2} \log \int_{D_I^\eta} \kappa_N = O\left(\frac{\log N}{N}\right).$$

Combined with (105) and the fact that ϵ' was arbitrary, this completes the proof. \square

7.4. The normalizing constant: Proof of Lemma 4. Finally, we consider the normalizing constants of Proposition 3, in particular the determinant $\det \mathcal{A}_N(h, \nu) = \det(\langle z^k, \psi_\ell \rangle)$ of the change of basis matrix (47). Here, $\langle \cdot, \cdot \rangle$ is the inner product $G_N(h, \nu)$. The same asymptotics have been studied before in the theory of orthogonal polynomials (see e.g. [B]) and in the setting of line bundles in [Don2, BB]. The following gives an alternative to the proof in [BB] in the special case at hand.

We claim that

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \hat{Z}_N(h) = \frac{1}{2} \log \text{Cap}_{\omega_h}(K).$$

Proof. We prove this by combining the large deviations result for the un-normalized probability measure with the fact that \mathbf{Prob}_N is a probability measure. By Lemma 30 and proof of the large deviations upper bound,

$$\begin{aligned} 0 &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbf{Prob}_N(\mathcal{M}(\mathbb{C}\mathbb{P}^1)) \\ &\leq \limsup_{N \rightarrow \infty} \frac{-1}{N^2} \log \hat{Z}_N(h) - \inf_{\mu \in \mathcal{M}(\mathbb{C}\mathbb{P}^1)} I^{h,K}(\mu) \\ &= \limsup_{N \rightarrow \infty} \frac{-1}{N^2} \log \hat{Z}_N(h) - I^{h,K}(\nu_{h,K}) \\ &= \limsup_{N \rightarrow \infty} \frac{-1}{N^2} \log \hat{Z}_N(h) - \frac{1}{2} \log \text{Cap}_{\omega_h}(K). \end{aligned} \tag{106}$$

A similar argument using the large deviations lower bound shows the reverse inequality for $\liminf_{N \rightarrow \infty} \frac{-1}{N^2} \log \hat{Z}_N$. \square

COROLLARY 33.

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log |\det \mathcal{A}_N(h)|^{-2} = -\frac{1}{2} E(h) + \frac{1}{2} \log \text{Cap}_{\omega_h}(K).$$

Proof. By Lemma 17,

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \hat{Z}_N(h) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log |\det \mathcal{A}_N(h)|^{-2} + \frac{1}{2} E(h).$$

\square

8. APPENDIX

This Appendix contains proofs of some technicalities used in the proofs of Theorem 1.

8.1. Regularization of measures. In the large deviations lower bound, we will need to prove that any $\mu \in \mathcal{M}(\mathbb{CP}^1)$ may be weakly approximated by measures with continuous densities. In [BZ], this was proved using convolution with Gaussians; since we are working on \mathbb{CP}^1 , we need a suitable replacement.

We once again use the auxiliary Kähler metric and its Laplacian Δ_ω . It generates the heat operator $e^{t\Delta_\omega}$. We denote its heat kernel by

$$K_\omega(t, z, w) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \varphi_j(z) \varphi_j(w), \quad (107)$$

and write

$$e^{t\Delta_\omega} \mu(z) = \int_{\mathbb{CP}^1} K_\omega(t, z, w) d\mu(w). \quad (108)$$

It is well-known and easy to see that $e^{t\Delta_\omega} \mu(z) \in C^\infty(\mathbb{CP}^1)$ for any $\mu \in \mathcal{M}(\mathbb{CP}^1)$. The following simple Lemma is sufficient for our purposes:

LEMMA 34. *If $I^{h,K}(\mu) < \infty$, then $e^{t\Delta_\omega} \mu \rightarrow \mu$ in \mathcal{E}_+ as $t \rightarrow 0^+$. In particular, $I^{h,K}(e^{t\Delta_\omega} \mu) \rightarrow I^{h,K}(\mu)$. Moreover, $e^{t\Delta_\omega} \mu \in \mathcal{M}(\mathbb{CP}^1)$.*

Proof. It is well known (and follows by the maximum principle for the heat equation) that $K_\omega(t, z, w) > 0$. Hence, $\mu_t := (e^{t\Delta_\omega} \mu) \omega$ is a positive measure. Further, $\int_{\mathbb{CP}^1} \mu_t = 1$ since $\int_{\mathbb{CP}^1} K(t, z, w) \omega = 1$.

As observed above, it is equivalent to say that $e^{t\Delta_\omega} \mu \rightarrow \mu$ in $H^{-1}(\mathbb{CP}^1)$. It suffices to observe that by monotone convergence,

$$\lim_{t \rightarrow 0^+} \sum_{j=1}^{\infty} \frac{1 - e^{-t\lambda_j}}{\lambda_j} |\mu(\varphi_j)|^2 = 0.$$

□

8.2. Residue at infinity of certain integrals. This section is rather technical. We go over the ‘residue at infinity’ of a number of integrals that arise in the calculations of the LDP.

Integrals of the form $\int_{\mathbb{C}} v dd^c u$ often arise when u, v are potentials for probability measures. The subtlety is the ‘boundary term’ at infinity. To illustrate, we note that although $dd^c \log(1 + |z|^2)^{1/2}$ is exact, the integral (e.g.) $\int_{\mathbb{C}} dd^c \log(1 + |z|^2)^{1/2} = 1$ and not zero. The integral has been studied in all dimensions in [BT], and although we are only working in dimension one we will follow their presentation.

The formula for $\int_{\mathbb{C}} v dd^c u$ depends on the class of functions that u, v belong to. Following [BT], one defines the Robin function ρ_u of a subharmonic function u by

$$\rho_u(z) = \limsup_{\lambda \rightarrow \infty} (u(\lambda z) - \log^+ |\lambda z|). \quad (109)$$

Here, u^* denotes the upper semi-continuous regularization of u . We denote by \mathcal{L} the Lelong class of subharmonic functions satisfying $u(z) \leq \log^+ |z| + C$, and put $\mathcal{L}_\rho := \{u \in \mathcal{L}(\mathbb{C}) : \rho_u \neq -\infty\}$. Note that $\rho_1 \equiv -\infty$.

8.2.1. *Case (i):* $u, v \in \mathcal{L}_\rho$. It is proved in [BT] that if $u, v \in \mathcal{L}_\rho$, then

$$\int_{\mathbb{C}} u dd^c v - v dd^c u = 2\pi(\rho_u^*(\infty) - \rho_v^*(\infty)), \quad (110)$$

The formula applies in particular to:

- (1) $\int_{\mathbb{C}} \psi dd^c \psi$ where $\omega = dd^c \psi$ on \mathbb{C} ; here ω is the curvature $(1, 1)$ form of any Hermitian metric on $\mathcal{O}(1)$. For any such metric, $\psi \in \mathcal{L}_\rho$.
- (2) $\int_{\mathbb{C}} \log |z - w| dd^c \psi$ where ψ is as above.
- (3) $\int_{\mathbb{C}\mathbb{P}^1} \log \|s_\zeta\|_{h^N}^2 \omega_h$, where as usual $s_\zeta = \prod_{j=1}^N (z - \zeta_j) e^N$.

In the case (1), the integration by parts formula just reproduces the same expression, but we include it for future reference.

In case (2), we first consider the Fubini-Study case, where the Hermitian metric h_{FS} is locally given in the standard frame by $\psi(w) = \log(1 + |w|^2)^{1/2}$. It is simple to see that

$$\int_{\mathbb{C}} \log |z - w| dd^c \log(1 + |w|^2) = \frac{1}{2} \log(1 + |z|^2). \quad (111)$$

This follows from (35) and the fact that $\rho_{\log(1+|w|^2)}(\infty) = \rho_{\log|z-w|}(\infty) = 0$, since for sufficiently large $|w|$,

$$\log |z - \lambda w| - \log |\lambda w| = \log \left| 1 - \frac{z}{\lambda w} \right| = \Re \log \left(1 - \frac{z}{\lambda w} \right) = -\Re \frac{z}{\lambda w} + \dots = o(1),$$

$$\log(1 + |\lambda w|^2)^{1/2} - \log |\lambda w| = \log(1 + |\lambda w|^{-1})^{1/2} = o(1).$$

In the case of a general Hermitian metric on $\mathcal{O}(1)$, we may write $h = h_{FS} e^{-\Phi}$ where $\Phi \in C^\infty(\mathbb{C}\mathbb{P}^1)$. Then on \mathbb{C} , the weight has the form $\log(1 + |w|^2)^{1/2} + \Phi \in \mathcal{L}_\rho$. Now $\rho_\psi(\infty) = \Phi(\infty)$. Hence,

$$\int_{\mathbb{C}} \log |z - w| dd^c \psi = \left(\frac{1}{2} \psi(z) - 2\pi \rho_\psi(\infty) \right) = \frac{1}{2} \psi(z) - 2\pi \Phi(\infty). \quad (112)$$

In case (3), we again express the Hermitian metric as $h = e^{-\Phi} h_{FS}$, and then

$$\begin{aligned} \int_{\mathbb{C}\mathbb{P}^1} \log \|s_\zeta\|_{h^N}^2 \omega_h &= 2 \sum_{j=1}^N \int_{\mathbb{C}} \log |z - \zeta_j| dd^c (\log(1 + |z|^2) + \Phi(z)) \\ &= \sum_j \log(1 + |\zeta_j|^2) + 2\pi(\Phi(\zeta_j) - \Phi(\infty)) \\ &= N \int (\log(1 + |w|^2) + 2\pi(\Phi(w) - \Phi(\infty))) d\mu_\zeta(w). \end{aligned} \quad (113)$$

This integral is also evaluated in Lemma 9.

8.2.2. *Case (ii)* $u, v \in C^2(\mathbb{C}\mathbb{P}^1)$. Obviously, in this case we can simply integrate by parts.

8.2.3. *Case (iii)* $u \in \mathcal{L}_\rho, v \in C^2(\mathbb{C}\mathbb{P}^1)$. We are interested in three cases:

- (1) $\int_{\mathbb{C}} \log |z - w| dd^c v$ where $v \in C^2(\mathbb{C}\mathbb{P}^1)$.
- (2) $\int_{\mathbb{C}} \log |z - w| dd^c \varphi$ where $e^{-\varphi}$ is the local expression on \mathbb{C} of a global smooth Hermitian metric h_φ on $\mathcal{O}(1)$. Equivalently, there exists a global continuous $(1, 1)$ form ω such that $\int_{\mathbb{C}\mathbb{P}^1} \omega = 1$ and such that $\omega - \delta_\infty = dd^c u$.
- (3) The integral $\int_{\mathbb{C}} u dd^c v$, where u is the potential on \mathbb{C} of a global smooth hermitian metric on $\mathcal{O}(1)$.
- (4) $v \equiv 1$, i.e. $\int_{\mathbb{C}} dd^c \varphi = \int_{\mathbb{C}\mathbb{P}^1} \omega = 1$.

In case (1), the logarithmic factor lies in \mathcal{L}_ρ but $v \notin \mathcal{L}_\rho$. When $v \in C^2(\mathbb{C}\mathbb{P}^1)$, we claim that

$$\int_{\mathbb{C}} \log |z - w| dd^c v = \frac{1}{2}v(z) - 2\pi v(\infty). \quad (114)$$

Indeed, we form the metric $h_{FS}e^{-v}$, apply (112) with $\psi = \log(1 + |w|^2)^{1/2} + v$ and then subtract the integral for $\log(1 + |w|^2)^{1/2}$.

Similarly, in case (2) we write the Hermitian metric $h_\varphi = e^{-\varphi}$ in the form $h_\varphi = e^{-\psi}h_{FS}$ with $\psi \in C^\infty(\mathbb{C}\mathbb{P}^1)$. Then by (112) and (114),

$$\begin{aligned} \int_{\mathbb{C}} \log |z - w| dd^c \varphi &= \int_{\mathbb{C}} \log |z - w| dd^c \log(1 + |w|^2)^{1/2} + \int_{\mathbb{C}} \log |z - w| dd^c \psi \\ &= \frac{1}{2} \log(1 + |z|^2)^{1/2} + \frac{1}{2} \psi(z) - 2\pi \psi(\infty) \\ &= \varphi(z)/2 - 2\pi \rho_\varphi(\infty). \end{aligned} \quad (115)$$

By a similar calculation, in case (3) we have,

$$\int_{\mathbb{C}} u dd^c v = \int_{\mathbb{C}} v dd^c u + 2\pi(\rho_u(\infty) - v(\infty)), \quad (116)$$

while in case (4) we have

$$\int_{\mathbb{C}} dd^c \varphi = -2\pi \rho_\varphi(\infty) = -2\pi. \quad (117)$$

REFERENCES

- [ABMNV] L. Alvarez-Gaumé, J-B. Bost, G. Moore, P. Nelson, and C. Vafa, Bosonization on higher genus Riemann surfaces. *Comm. Math. Phys.* 112 (1987), no. 3, 503–552.
- [BT] W. Bedford and B. A. Taylor, Plurisubharmonic functions with logarithmic singularities. *Ann. Inst. Fourier (Grenoble)* 38 (1988), no. 4, 133–171.
- [BG] G. Ben Arous and A. Guionnet, Large deviations for Wigner’s law and Voiculescu’s non-commutative entropy. *Probab. Theory Related Fields* 108 (1997), no. 4, 517–542.
- [BZ] G. Ben Arous and O. Zeitouni, Large deviations from the circular law. *ESAIM Probab. Statist.* 2 (1998), 123–134.
- [Ber1] R. Berman, Bergman kernels and equilibrium measures for ample line bundles (arXiv:0704.1640)
- [Ber2] R. Berman, Bergman kernels and weighted equilibrium measures of \mathbb{C}^n (arXiv:math/0702357).
- [BB] R. Berman and S. Boucksom, Capacities and weighted volumes of line bundles, arXiv:0803.1950.
- [BSZ] P. Bleher, B. Shiffman, and S. Zelditch, Poincaré-Lelong approach to universality and scaling of correlations between zeros. *Comm. Math. Phys.* 208 (2000), no. 3, 771–785.
- [B] T. Bloom, Random polynomials and Green functions. *Int. Math. Res. Not.* 2005, no. 28, 1689–1708.
- [B2] T. Bloom, Weighted polynomials and weighted pluripotential theory (arXiv:math/0610330).
- [BS] T. Bloom and B. Shiffman, Zeros of random polynomials on \mathbb{C}^m , *Math. Res. Lett.* 14 (2007), no. 3, 469–479.
- [BBL1] E. Bogomolny, O. Bohigas, and P. Leboeuf, Quantum chaotic dynamics and random polynomials. *J. Statist. Phys.* 85 (1996), no. 5-6, 639–679.
- [BBL2] E. Bogomolny, O. Bohigas, and P. Leboeuf, Distribution of roots of random polynomials. *Phys. Rev. Lett.* 68 (1992), no. 18, 2726–2729.
- [C] H. Cartan, Théorie du potentiel newtonien: énergie, capacité, suites de potentiels. *Bull. Soc. Math. France* 73, (1945). 74–106.
- [Ch] G. Choquet, Sur les points d’effilement d’un ensemble. Application à l’étude de la capacité. *Ann. Inst. Fourier. Grenoble* 9 1959 91–101
- [Dem] J. P. Demailly, Complex Analytic and Differential Geometry, www.fourier.ujf-grenoble.fr/~demailly.

- [DZ] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*. Second edition. Applications of Mathematics (New York), 38. Springer-Verlag, New York, 1998.
- [D] J. Deny, Les potentiels d'énergie finie. *Acta Math.* 82, (1950). 107–183.
- [Don2] S.K. Donaldson, Scalar curvature and projective embeddings. II. *Q. J. Math.* 56 (2005), no. 3, 345–356.
- [Fal] G. Faltings, Calculus on arithmetic surfaces. *Ann. of Math. (2)* 119 (1984), no. 2, 387–424
- [GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.
- [GrS] U. Grenander and G. Szego, *Toeplitz forms and their applications*. Second edition. Chelsea Publishing Co., New York, 1984
- [GZ] V. Guedj and A. Zeriahi, Intrinsic capacities on compact Kähler manifolds. *J. Geom. Anal.* 15 (2005), no. 4, 607–639.
- [H] J. H. Hannay, Chaotic analytic zero points: exact statistics for those of a random spin state. *J. Phys. A* 29 (1996), no. 5, L101–L105.
- [He] G. Hein, Computing Green currents via the heat kernel. *J. Reine Angew. Math.* 540 (2001), 87–104.
- [HP] F. Hiai and D. Petz. *The semicircle law, free random variables and entropy*, volume 77 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.
- [Ho] L. Hörmander, *The Analysis of Linear Partial Differential Operators, I*, *Grund. Math. Wiss.* 256, Springer-Verlag, New York, 1983.
- [HoII] L. Hörmander, *The analysis of linear partial differential operators. II. Differential operators with constant coefficients*. *Classics in Mathematics*. Springer-Verlag, Berlin, 2005.
- [HoIII] L. Hörmander, *The analysis of linear partial differential operators. III*. *Classics in Mathematics*. Springer-Verlag, Berlin, 2005.
- [Lan] N.S. Landkof, *Foundations of modern potential theory*. *Die Grundlehren der mathematischen Wissenschaften*, Band 180. Springer-Verlag, New York-Heidelberg, 1972.
- [LP] A. Lascoux and P. Pragacz, Jacobians of symmetric polynomials. *Ann. Comb.* 6 (2002), no. 2, 169–172.
- [Ran] T. Ransford, *Potential theory in the complex plane*. *London Mathematical Society Student Texts*, 28. Cambridge University Press, Cambridge, 1995.
- [ST] E.B. Saff and V. Totik, *Logarithmic potentials with external fields*. Appendix B by Thomas Bloom. *Grundlehren der Mathematischen Wissenschaften* 316. Springer-Verlag, Berlin, 1997.
- [SZ] B. Shiffman and S. Zelditch, Equilibrium distribution of zeros of random polynomials. *Int. Math. Res. Not.* 2003, no. 1, 25–49.
- [SZ2] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.* 200 (1999), no. 3, 661–683.
- [SZ3] B. Shiffman and S. Zelditch, Number variance of random zeros, *Geom. Funct. Anal.* 18 (2008), no. 4, 1422–1475 (math.CV/0512652).
- [SZZr] B. Shiffman, S. Zelditch, and S. Zrebiec, Overcrowding and hole probabilities for random zeros on complex manifolds. *Indiana Univ. Math. J.* 57 (2008), no. 5, 1977–1997.
- [SoTs] M. Sodin and B. Tsirelson, Random complex zeroes. III. Decay of the hole probability. *Israel J. Math.* 147 (2005), 371–379.
- [Z] S. Zelditch, Large deviations of empirical zero point measures on Riemann surfaces, II: $g \geq 1$ (preprint, 2008).

FACULTY OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL AND
SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS 55455, USA

E-mail address: zeitouni@math.umn.edu

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA

E-mail address: szelditch@jhu.edu