COUNTING STRING/M VACUA

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ABSTRACT. We report on some recent work with M. R. Douglas and B. Shiffman on vacuum statistics for flux compactifications in string/M theory.

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1. INTRODUCTION

According to string/M theory, the vacuum state of our universe is a 10 dimensional space-time of the form $M^{3,1} \times X$, where $M^{3,1}$ is Minkowski space and X is a small 3-complex dimensional Calabi-Yau manifold X known as the 'small' or 'extra' dimensions [CHSW, St]. The *vacuum selection problem* is that there are many candidate vacua for the Calabi-Yau 3-fold X. Here, we report on recent joint work with B. Shiffman and M. R. Douglas devoted to counting the number of supersymmetric vacua of type IIb flux compactifications [DSZ1, DSZ2, DSZ3]. We also describe closely related the physics articles of Ashok-Douglas and Denef-Douglas [D, AD, DD] on the same problem.

At the time of writing of this article, vacuum statistics is being intensively investigated by many string theorists (see for instance [DGKT, CQ, GKT, S, Ar] in addition to the articles cited above). One often hears that the number of possible vacua is of order 10^{500} (see e.g. [BP]). This large figure is sometimes decried (at this time) as a blow to predictivity of string/M theory or extolled as giving string theory a rich enough 'landscape' to contain vacua that match the physical parameters (e.g. the cosmological constant) of our universe. However, it is very difficult to obtain sufficiently accurate results on vacuum counting to justify the claims of 10^{500} total vacua, or even the existence of one vacuum which is consistent

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with known physical parameters. The purpose of our work is to develop methods and results relevant to accurate vacuum counting.

From a mathematical viewpoint, supersymmetric vacua are critical points

$$\nabla W(Z) = 0 \tag{1}$$

of certain holomorphic sections W_G called *flux superpotentials* of a line bundle $\mathcal{L} \to \mathcal{C}$ over the moduli space $\mathcal{C}$ of complex structures on $X \times T^2$ where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Flux superpotentials depend on a choice of flux $G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$. There is a constraint on G called the 'tadpole constraint', so that G is a lattice point lying in a certain hyperbolic shell $0 \leq Q[G] \leq L$ in $H^3(X, \mathbb{C})$ (16). Our goal is to count all critical points of all flux superpotentials W_G in a given compact set of $\mathcal{C}$ as G ranges over such lattice points. Thus, counting vacua in $K \subset \mathcal{C}$ is a combination of an equidistribution problem for projections of lattice points and an equidistribution problem for critical points of random holomorphic sections.

The work we report on gives a rigorous foundation for the program initiated by M. R. Douglas [D] to count vacua by making an approximation to the Gaussian ensembles the other two authors were using to study statistics of zeros of random holomorphic sections (cf. [SZ, BSZ]). The results we describe here are the first rigorous results on counting vacua in a reasonably general class of models (type IIb flux compaticifications). They are admittedly still in a rudimentary stage, in particular because they are asymptotic rather than effective. We will discuss the difficulties in making them effective below.

This report is a written version of our talk at the QMath9 conference in Giens in October, 2004. A more detailed expository article with background on statistical algebraic geometry as well as string theory is given in [Z], which was based on the author's AMS address in Atlanta, January 2005.

2. Type IIb flux compactifications of string/M theory

The string/M theories we consider are type IIb string theories compactified on a complex 3-dimensional Calabi-Yau manifold X with flux [GKP, GVW, GKTT, GKT, AD]. We recall that a Calabi-Yau 3-fold is a compact complex manifold X of dimension 3 with trivial canonical bundle K_X, i.e. $c_1(X) = 0$ [Gr, GHJ]. Such X possesses a unique Ricci flat Kähler metric in each Kähler class. In what follows, we fix the Kähler class, and then the CY metrics correspond to the complex structures on X. We denote the moduli space of complex structures on X by $\mathcal{M}_{\mathbb{C}}$. In addition to the complex structure moduli on X there is an extra parameter τ called the dilaton axion, which ranges over complex structure moduli on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Hence, the full configuration space $\mathcal{C}$ of the model is the product

$$\mathcal{C} = \mathcal{M}_{\mathbb{C}} \times \mathcal{E}, \quad (Z = (z, \tau); \;\; z \in \mathcal{M}_{\mathbb{C}}, \tau \in \mathcal{E}) \tag{2}$$

where $\mathcal{E} = \mathcal{H}/SL(2, \mathbb{Z})$ is the moduli space of complex 1-tori (elliptic curves). One can think of $\mathcal{C}$ as a moduli space of complex structures on the CY 4-fold $X \times T^2$.

By 'flux' is meant a complex integral 3-form

$$G = F + iH \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}). \tag{3}$$

The *flux superpotential* $W_G(Z)$ corresponding to G is defined as follows: On a Calabi-Yau 3-fold, the space $H^{3,0}_z(X)$ of holomorphic $(3,0)$-forms for each complex structure z on X has dimension 1, and we denote a holomorphically varying family by $\Omega_z \in H^{3,0}_z(X)$. Given G as in (3) and $\tau \in \mathcal{H}$, physicists define the superpotential corresponding to G, τ by:

$$W_G(z,\tau) = \int_X (F - \tau H) \wedge \Omega_z. \tag{4}$$

This is not well-defined as a function on $\mathcal{C}$, since Ω_z is not unique and τ corresponds to the holomorphically varying form $\omega_\tau = dx + \tau dy \in H^{1,0}_\tau(T^2)$ which is not unique either. To be more precise, we define W_G to be a holomorphic section of a line bundle $\mathcal{L} \to \mathcal{C}$, namely the dual line bundle to the Hodge line bundle $H^{4,0}_{z,\tau} = H^{3,0}_z(X) \otimes H^{1,0}(T^2) \to \mathcal{C}$. We form the 4-form on $X \times T^2$

$$\tilde{G} = F \wedge dy + H \wedge dx$$

and define a linear functional on $H^{3,0}_z(X) \otimes H^{1,0}_\tau(T^2)$ by

$$\langle W_G(z,\tau), \Omega_z \wedge \omega_\tau \rangle = \int_{X \times T^2} \tilde{G} \wedge \Omega_z \wedge \omega_\tau. \tag{5}$$

When $\omega_\tau = dx + \tau dy$ we obtain the original formula. As $Z = (z, \tau) \in \mathcal{C}$ varies, (5) defines a holomorphic section of the line bundle $\mathcal{L}$ dual to $H^{3,0}_z \otimes H^{1,0}_\tau \to \mathcal{C}$.

The Hodge bundle carries a natural Hermitian metric

$$h_{WP}(\Omega_z \wedge \omega_\tau, \Omega_z \wedge \omega_\tau) = \int_{X \times T^2} \Omega_z \wedge \omega_\tau \wedge \overline{\Omega_z \wedge \omega_\tau}$$

known as the Weil-Petersson metric, and an associated metric (Chern) connection by ∇_{WP}. The Kähler potential of the Weil-Petersson metric on $\mathcal{M}_{\mathbb{C}}$ is defined by

$$\mathcal{K} = -\ln\langle \Omega, \Omega \rangle = \ln \int_X \Omega \wedge \overline{\Omega}. \tag{6}$$

There is a similar definition on $\mathcal{E}$ and we take the direct sum to obtain a Kähler metric on $\mathcal{C}$. We endow $\mathcal{L}$ with the dual Weil-Petersson metric and connection. The hermitian line bundle $(H^{4,0}, h_{WP}) \to \mathcal{M}_{\mathbb{C}}$ is a positive line bundle, and it follows that $\mathcal{L}$ is a negative line bundle.

The vacua we wish to count are the classical vacua of the effective supergravity Lagrangian of the string/M model, which is derived by 'integrating out' the massive modes (cf. [St]). The only term relevant of the Lagrangian to our counting problem is the scalar potential [WB]

$$V_G(Z) = |\nabla W_G(Z)|^2 - 3|W(Z)|^2, \tag{7}$$

where the connection and hermitian metric are the Weil-Petersson ones. We only consider the supersymmetric vacua here, which are the special critical points Z of V_G satisfying (1).

3. Critical points and Hessians of holomorphic sections

We see that type IIb flux compactifications involve holomorphic sections of hermitian holomorphic line bundles over complex manifolds. Thus, counting flux vacua is a problem in complex geometry. In this section, we provide a short review from [DSZ1, DSZ2].

Let $L \to M$ denote a holomorphic line bundle over a complex manifold, and endow L with a hermitian metric h. In a local frame e_L over an open set $U \subset M$, one defines the Kähler potential K of h by

$$|e_L(Z)|^2_h = e^{-K(Z)}\ . \tag{8}$$

We write a section $s \in H^0(M, L)$ locally as $s = fe_L$ with $f \in \mathcal{O}(U)$. We further choose local coordinates z. In this frame and local coordinates, the covariant derivative of a section s takes the local form

$$\nabla s = \sum_{j=1}^m \left(\frac{\partial f}{\partial Z_j} - f \frac{\partial K}{\partial Z_j} \right) dZ_j \otimes e_{\mathcal{L}} = \sum_{j=1}^m e^K \frac{\partial}{\partial Z_j} \left(e^{-K} f \right) dZ_j \otimes e_{\mathcal{L}} \,. \tag{9}$$

The critical point equation $\nabla s(Z) = 0$ thus reads,

$$\frac{\partial f}{\partial Z_j} - f \frac{\partial K}{\partial Z_j} = 0.$$

It is important to observe that although s is holomorphic, ∇s is not, and the critical point equation is only C^∞ and not holomorphic. This is due to the factor $\frac{\partial K}{\partial Z_j}$, which is only smooth. Connection critical points of s are the same as ordinary critical points of $\log |s(Z)|_h$. Thus, the critical point equation is a system of real equations and the number of critical points varies with the holomorphic section. It is not a topological invariant, as would be the number of critical points of m sections in dimension m, even on a compact complex manifold. This is one reason why counting critical points, hence vacua, is so complicated.

We now consider the Hessian of a section at a critical point. The Hessian of a holomorphic section s of a general Hermitian holomorphic line bundle $(L, h) \to M$ at a critical point Z is the tensor

$$D\nabla W(Z) \in T^* \otimes T^* \otimes L$$

where D is a connection on $T^* \otimes L$. At a critical point Z, $D\nabla s(Z)$ is independent of the choice of connection on T^*. The Hessian $D\nabla W(Z)$ at a critical point determines the complex symmetric matrix H^c (which we call the 'complex Hessian'). In an adapted local frame (i.e. holomorphic derivatives vanish at Z_0) and in Kähler normal coordinates, it takes the form

$$H^c := \begin{pmatrix} H' & H'' \\ \overline{H''} & \overline{H'} \end{pmatrix} = \begin{pmatrix} H' & -f(Z_0)\Theta \\ -\overline{f(z_0)\Theta} & \overline{H'} \end{pmatrix} \,, \tag{10}$$

whose components are given by

$$H'_{jq} = \left(\frac{\partial}{\partial Z_j} - \frac{\partial K}{\partial Z_j}\right)\left(\frac{\partial}{\partial Z_q} - \frac{\partial K}{\partial Z_q}\right) f(Z_0) \,, \tag{11}$$

$$H''_{jq} = - f \frac{\partial^2 K}{\partial Z_j \partial \bar{Z}_q}\Big|_{Z_0} = -f(Z_0)\Theta_{jq} \,, \quad \Theta_h(z_0) = \sum_{j,q} \Theta_{jq} dZ_j \wedge d\bar{Z}_q \,. \tag{12}$$

Here, $\Theta_h(z_0) = \sum_{j,q} \Theta_{jq} dZ_j \wedge d\bar{Z}_q$ is the curvature.

4. The critical point problem

We can now define the critical point equation (1) precisely. We define a supersymmetric vacuum of the flux superpotential W_G corresponding to the flux G of (3) to be a critical point $\nabla_{WP} W_G(Z) = 0$ of W_G relative to the Weil-Petersson connection on $\mathcal{L}$.

We obtain a local formula by writing $W_G(Z) = f_G(Z) e_Z$ where e_Z is local frame for $\mathcal{L} \to \mathcal{C}$. We choose the local frame e_Z to be dual to $\Omega_z \otimes \omega_\tau$, and then $f_G(z, \tau)$ is given by the formula

(4). The $\mathcal{E}$ component of ∇_{WP} is $\frac{\partial}{\partial \tau} - \frac{1}{\tau - \bar{\tau}}$. The critical point equation is the system:

$$\begin{cases} \int_X (F - \tau H) \wedge \{\frac{\partial \Omega_z}{\partial z_j} + \frac{\partial K}{\partial z_j} \Omega_z\} = 0, \\ \\ \int_X (F - \bar{\tau} H) \wedge \Omega_z = 0, \end{cases} \tag{13}$$

where K is from (6).

Using the *special geometry* of $\mathcal{C}$ ([St3, Can1]), one finds that the critical point equation is equivalent to the following restriction on the Hodge decomposition of $H^3(X, \mathbb{C})$ at z:

$$\nabla_{WP} W_G(z, \tau) = 0 \iff F - \tau H \in H_z^{2,1} \oplus H_z^{0,3}. \tag{14}$$

Here, we recall that each complex structure $z \in \mathcal{M}_{\mathbb{C}}$ gives rise to a Hodge decomposition

$$H^3(X, \mathbb{C}) = H_z^{3,0}(X) \oplus H_z^{2,1}(X) \oplus H_z^{1,2}(X) \oplus H_z^{0,3}(X) \tag{15}$$

into forms of type (p, q). In the case of a CY 3-fold, $h^{3,0} = h^{0,3} = 1$, $h^{1,2} = h^{2,1}$ and $b_3 = 2 + 2h^{2,1}$.

Next, we specify the tadpole constraint. We define the real symmetric bilinear form on $H^3(X, \mathbb{C})$ by

$$Q(\psi, \varphi) = i^3 \int_X \psi \wedge \bar{\varphi}. \tag{16}$$

The Hodge-Riemann bilinear relations for a 3-fold say that the form Q is definite in each $H_z^{p,q}(X)$ for $p + q = 3$ with sign alternating $+ - + -$ as one moves left to right in (15). The tadpole constraint is that

$$Q[G] = i^3 \int_X G \wedge \bar{G} \leq L. \tag{17}$$

Here, L is determined by X in a complicated way (it equals $\chi(Z)/24$ where Z is CY 4-fold which is an elliptic fibration over X/g, where $\chi(Z)$ is the Euler characteristic and where g is an involution of X). Although Q is an indefinite symmetric bilinear form, we see that $Q >> 0$ on $H_z^{2,1}(X) \oplus H_z^{0,3}$ for any complex structure z.

We now explain the sense in which we are dealing with a lattice point problem. The definition of W_G makes sense for any $G \in H^3(X, \mathbb{C})$, so we obtain a real (but not complex) linear embedding $H^3(X, \mathbb{C}) \subset H^0(\mathcal{C}, \mathcal{L})$. Let us denote the image by $\mathcal{F}$ and call it the space of complex-valued flux superpotentials with dilaton-axion. The set of W_G with $G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$ is then a lattice $\mathcal{F}_{\mathbb{Z}} \subset \mathcal{F}$, which we will call the lattice of quantized (or integral) flux superpotentials.

Each integral flux superpotential W_G thus gives rise to a finite set of critical points $Crit(W_G) \subset \mathcal{C}$, any of which could be the vacuum state of the universe. Moreover, the flux G can be any element of $H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$ satisfying the tadpole constraint (17). Thus, the set of possible vacua is the union

$$\text{Vacua}_L = \bigcup_{G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1} Z),\ 0 \leq Q[G] \leq L} \text{Crit}(W_G). \tag{18}$$

Our purpose is to count the number of vacua $\#\text{Vacua}_L \cap K$ in any given compact subset $K \subset \mathcal{C}$.

More generally, we wish to consider the sums

$$N_\psi(L) = \sum_{N \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}): Q[N] \leq L} \langle C_N, \psi \rangle, \tag{19}$$

where

$$\langle C_N, \psi \rangle = \sum_{(z,\tau): \nabla N(z,\tau) = 0} \psi(N, z, \tau), \tag{20}$$

and where ψ is a reasonable function on the incidence relation

$$\mathcal{I} = \{(W; z, \tau) \in \mathcal{F} \times \mathcal{C} : \nabla W(z, \tau) = 0\}. \tag{21}$$

We often write $Z = (z, \tau) \in \mathcal{C}$. Points (W, Z) such that Z is a degenerate critical point of W cause problems. They belong to the *discriminant variety* $\widetilde{\mathcal{D}} \subset \mathcal{I}$ of singular points of the projection $\pi : \mathcal{I} \to \mathcal{F}$. We note that $\pi^{-1}(W) = \{(W, Z) : Z \in Crit(W)\}$. This number is constant on each component of $\mathcal{F} \setminus \mathcal{D}$ where $\mathcal{D} = \pi(\widetilde{\mathcal{D}})$ but jumps as we cross over $\mathcal{D}$.

To count critical points in a compact subset $\mathcal{K} \subset \mathcal{C}$ of moduli space, we would put $\psi = \chi_K(z, \tau)$. We often want to exclude degenerate critical points and then use test functions $\psi(W, Z)$ which are homogeneous of degree 0 in W and vanish on $\widetilde{\mathcal{D}}$ Another important example is the cosmological constant $\psi(N, z, \tau) = V_N(z, \tau)$, i.e. the value of the potential at the vacuum, which is homogeneous of degree 2 in W.

5. Statement of results

We first state an initial estimate which is regarded as 'trivial' in lattice counting problems. In pure lattice point problems it is sharp, but we doubt that it is sharp in the vacuum counting problem because of the 'tilting' of the projection $\mathcal{I} \to \mathcal{C}$. We denote by χ_Q the characteristic function of the hyperbolic shell $0 < Q_Z[W] < 1 \subset \mathcal{F}$ and by χ_{Q_Z} the characteristic function of the elliptic shell $0 < Q_Z[W] < 1 \subset \mathcal{F}_Z$.

Proposition 5.1. *Suppose that $\psi(W, Z) = \chi_K$ where $K \subset \mathcal{I}$ is an open set with smooth boundary. Then:*

$$\mathcal{N}_\psi(L) = L^{b_3} \left[\int_{\mathcal{C}} \int_{\mathcal{F}_Z} \psi(W, Z) |\det H^c W(Z)| \chi_{Q_Z} dW + R_K(L) \right],$$

where

1. *If $\overline{K}$ is disjoint from the $\widetilde{\mathcal{D}}$, then $R_K(L) = O\left(L^{-1/2}\right)$.*
2. *If $\overline{K}$ is a general compact set (possibly intersecting the discriminant locus), then $R_K(L) = O\left(L^{-1/4}\right)$*

Here, $H^c W(Z)$ is the complex Hessian of W at the critical point Z in the sense of (10). We note that the integral converges since $\{Q_Z \leq 1\}$ is an ellipsoid of finite volume. This is an asymptotic formula which is a good estimate on the number of vacua when L is large (recall that L is a topological invariant determined by X).

The reason for assumption (1) is that number of critical points and the summand $\langle C_W, \psi \rangle$ jump across $\mathcal{D}$, so in $N_\psi(L)$ we are summing a discontinuous function. This discontinuity could cause a relatively large error term in the asymptotic counting. However, superpotentials of physical interest have non-degenerate supersymmetric critical points. Their Hessians at the critical points are 'fermionic mass matrices', which in physics have only non-zero eigenvalues (masses), so it is reasonable assume that $supp\psi$ is disjoint from $\mathcal{D}$.

Now we state the main result.

THEOREM **5.2.** *Suppose $\psi(W, z, \tau) \in C_b^\infty(\mathcal{F} \times \mathcal{C})$ is homogeneous of degree 0 in W , with $\psi(W, z, \tau) = 0$ for $W \in \mathcal{D}$. Then*

$$\mathcal{N}_\psi(L) = L^{b_3} \left[\int_{\mathcal{C}} \int_{\mathcal{F}_{z,\tau}} \psi(W, z, \tau) |\det H^c W(z, \tau)| \chi_{Q_{z,\tau}}(W) dW dV_{WP}(z, \tau) + O\left(L^{-\frac{2b_3}{2b_3+1}}\right) \right].$$

Here, $b_3 = \dim H_3(X, \mathbb{R}), Q_{z,\tau} = Q|_{\mathcal{F}_{z,\tau}}$, and $\chi_{Q_{z,\tau}}(W)$ is the characteristic function of $\{Q_{z,\tau} \leq 1\} \subset \mathcal{F}_{z,\tau}$. Also C_b^∞ denotes bounded smooth functions.

There is a simple generalization to homogeneous functions of any degree such as the cosmological constant. The formula is only the starting point of a number of further versions which will be presented in §8 in which we 'push-forward' the dW integral under the Hessian map, and then perform an Itzykson-Zuber-Harish-Chandra transformation on the integral. The latter version gets rid of the absolute value and seems to most useful for numerical studies. Further, one can use the special geometry of moduli space to simplify the resulting integral. Before discussing them, we pause to compare our results to the expectations in the string theory literature.

6. Comparison to the physics literature

The reader following the developments in string theory may have encountered discussions of the 'string theory landscape' (see e.g. [S, BP]). The multitude of superpotentials and vacua is a problem for the predictivity of string theory. It is possible that a unique vacuum will distinguish itself in the future, but until then all critical points are candidates for the small dimensions of the universe, and several groups of physicists are counting or enumerating them in various models (see e.g. [DD, CQ, DGKT]).

The graph of the scalar potential energy may be visualized as a landscape [S] whose local minima are the possible vacua. It is common to hear that there are roughly 10^{500} possible vacua. This heuristic figure appears to originate in the following reasoning: assuming $b_3 \sim 250$, the potential energy $V_G(Z)$ is a function roughly 500 variables (including fluxes G). The critical point equation for a function of m variables is a system of m equations. Naively, the number of solutions should grow like d^m where d is the number of solutions of the jth equation with the other variables held fixed. This would follow from Bézout's formula if the function was a polynomial and if we were counting complex zeros. Thus, if the 'degree' of V_G were a modest figure of 10 we would obtain the heuristic figure.

Such an exponential growth rate of critical points in the number of variables also arises in estimates of the number of metastable states (local minima of the Hamiltonian) in the theory of spin glasses. In fact, an integral similar to that in Theorem 5.2 arises in the formula for the expected number of local minima of a random spin glass Hamiltonian. Both heuristic and rigorous calculations lead to an exponential growth rate of the number of local minima as the number of variables tends to infinity (see e.g. [F] for a mathematical discussion and references to the literature). The mathematical similarity of the problems at least raises the question whether the number of string/M vacua should grow exponentially in the number $2b_3$ of variables (G, Z), i.e. in the 'topological complexity' of the Calabi-Yau manifold X.

Our results do not settle this problem, and indeed it seems to be a difficult question. Here are some of the difficulties: First, in regard to the Bézout estimate, the naive argument ignores the fact that the critical point equation is a real C^∞ equation, not a holomorphic one and so the Bézout estimate could be quite inaccurate. Moreover, a flux superpotential is not a polynomial and it is not clear what 'degree' it has, as measured by its number critical points. In simple examples (see e.g. [AD, DD, DGKT], the superpotentials do not have many critical points and it is rather the large number of fluxes satisfying the tadpole constraint which produces the leading term L^{b_3}. This is why the flux G has to be regarded as one of the variables if one wants to rescue the naive counting argument. In addition, the tadpole constraint has a complicated dimensional dependence. It induces a constraint on the inner integral in Theorem 5.2 to an ellipse in b_3 dimensions, and the volume of such a domain shrinks at the rate $1/(b_3)!$. Further, the volume of the Calabi-Yau moduli space is not known, and could be very small. Thus, there are a variety of competing influences on the growth rate of the number of vacua in b_3 which all have a factorial dependence on the dimension.

To gain a better perspective on these issues, it is important to estimate the integral giving the leading coefficient and the remainder in Theorem 5.2. The inner integral is essentially an integral of a homogeneous function of degree b_3 over an ellipsoid in b_3 dimensions, and is therefore very sensitive to the size of b_3. The full integral over moduli space carries the additional problem of estimating its volume. Further, one needs to estimate how large L is for a given X. Without such effective bounds on L, it is not even possible to say whether any vacua exist which are consistent with known physical quantities such as the cosmological constant.

7. Sketch of proofs

The proof of Theorem 5.2 is in part an application of a lattice point result to the lattice of flux superpotentials. In addition, it uses the formalism on the density of critical points of Gaussian random holomorphic sections in [DSZ1]. The lattice point problem is to study the distribution of radial projections of lattice points in the shell $0 \leq Q[G] \leq L$ on the surface $Q[G] = 1$. Radial projections arise because the critical point equation $\nabla W_G = 0$ is homogeneous in G.

Thus, we consider the model problem: Let $\mathbf{Q} \subset \mathbb{R}^n$ $(n \geq 2)$ be a smooth, star-shaped set with $0 \in \mathbf{Q}^\circ$ and whose boundary has a non-degenerate second fundamental form. Let $|X|_{\mathbf{Q}}$ denote the norm of $X \in \mathbb{R}^n$ defined by $\mathbf{Q} = \{X \in \mathbb{R}^n : |X|_{\mathbf{Q}} < 1\}$. In the following, we denote the large parameter by $\sqrt{L}$ to maintain consistency with Theorem 5.2.

Theorem 7.1. [DSZ3] *If f is homogeneous of degree* 0 *and $f|_{\partial Q} \in C_0^\infty(\partial Q)$, then*

$$S_f(L) := \sum_{k \in \mathbb{Z}^n \cap \sqrt{L}\mathbf{Q} \setminus \{0\}} f(k) = L^{\frac{n}{2}} \int_{\mathbf{Q}} f \, dX + O(L^{\frac{n}{2} - \frac{n}{n+1}}), \quad L \to \infty.$$

Although we have only stated it for smooth f, the method can be generalized to $f|_{\partial Q} = \chi_K$ where K is a smooth domain in ∂Q [Z2]. However, the remainder then depends on K and reflects the extent to which projections of lattice points concentrate on $\partial K \subset \partial Q$. The asymptotics are reminiscent of the the result of van der Corput, Hlawka, Herz and Randol on the number of lattice points in dilates of a convex set, but as of this time of writing we have not located any prior studies of the radial projection problem. Number theorists have however studied the distribution of lattice points lying exactly on spheres (Linnik, Pommerenke). We also refer the interested reader to [DO] for a recent article counting lattice points in certain rational cones using methods of automorphic forms, in particular L-functions. We thank B. Randol for some discussions of this problem; he has informed the author that the result can also be extended to more general kinds of surfaces with degenerate second fundamental forms.

Applying Theorem 7.1 to the string/M problem gives that

$$\mathcal{N}_\psi(L) = L^{b_3}\left[\int_{\{Q[W]\leq 1\}} \langle C_W, \psi\rangle \, dW + O\left(L^{-\frac{2b_3}{2b_3+1}}\right)\right]. \tag{22}$$

We then write (22) as an integral over the incidence relation (21) and change the order of integration to obtain the leading coefficient

$$\int_{\{Q[W]\leq 1\}} \langle C_W, \psi\rangle \, dW = \int_{\mathcal{C}}\int_{\mathcal{F}_{z,\tau}} \psi(W,z,\tau)|\det H^c W(z,\tau)|\chi_{Q_{z,\tau}} dW dV_{WP}(z,\tau) \tag{23}$$

in Theorem 5.2. Heuristically, the integral on the left side is given by

$$\int_{\mathcal{F}}\int_{\mathcal{C}} \psi(W,Z)|\det H^c W(Z)|\delta(\nabla W(z))\chi_Q(Z) dW dV(Z). \tag{24}$$

The factor $|\det H^c W(Z)|$ arises in the pullback of δ under $\nabla W(Z)$ for fixed W, since it weights each term of (20) by $\frac{1}{|\det H^c W(Z)|}$. We obtain the stated form of the integral in (23) by integrating first in W and using the formula for the pull-back of a δ function under a linear submersion. That formula also contains another factor $\frac{1}{\det A(Z)}$ where $A(Z) = \nabla_{Z'_j}\nabla_{Z''_k}\Pi_Z(Z',Z'')|_{Z'=Z''=Z}$, where Π_Z is the Szegö kernel of $\mathcal{F}_Z$, i.e. the orthogonal projection onto that subspace. Using special geometry, the matrix turns out to be just I and hence the determinant is one.

8. Other formulae for the critical point density

In view of the difficulty of estimating the leading term in Theorem 5.2, it is useful to have alternative expressions. We now state two of them.

The first method is to change variables to the Hessian $H^c W(Z)$ under the Hessian map

$$H_Z : \mathcal{S}_Z \to \mathrm{Sym}(m,\mathbb{C}) \oplus \mathbb{C}, \quad H_Z(W) = H^c W(Z), \tag{25}$$

where $m = \dim \mathcal{C} = h^{2,1}+1$. It turns out that Hessian map is an isomorphism to a real b_3-dimensional space $\mathcal{H}_Z \oplus \mathbb{C}$, where

$$\mathcal{H}_Z = \mathrm{span}_{\mathbb{R}}\left\{\begin{pmatrix} 0 & e_j \\ e_j^t & \mathcal{F}^j(z)\end{pmatrix}, \begin{pmatrix} 0 & ie_j \\ ie_j^t & -i\mathcal{F}^j(z)\end{pmatrix}\right\}_{j=1,\dots,h^{2,1}}. \tag{26}$$

Here, e_j is the j-th standard basis element of $\mathbb{C}^{h^{2,1}}$ and $\mathcal{F}^j(z) \in \mathrm{Sym}(h^{2,1},\mathbb{C})$ is the matrix $\left(\mathcal{F}^j_{ik}(z)\right)$ whose entries define the so-called 'Yukawa couplings' (see [St3, Can1] for the definition). We define the positive definite operator $C_Z : \mathcal{H}_Z \to \mathcal{H}_Z$ by:

$$(C_Z^{-1} H_Z W, H_Z W) = Q_Z(W, \overline{W}). \tag{27}$$

The entries in C_Z are quadratic expressions in the $\mathcal{F}^j_{ik}$ (see [DSZ3]).

Theorem 8.1. *We have:*

$$\begin{aligned}\mathcal{K}^{\rm crit}(Z) &= \tfrac{1}{b_3!\det C'_Z}\int_{\mathcal{H}_Z\oplus\mathbb{C}}|\det H^*H-|x|^2I|\ \ e^{-(C_Z^{-1}H,H)+|x|^2)}\,dH\,dx\ ,\\ &= \tfrac{1}{\det C'_Z}\int_{\mathcal{H}_Z\oplus\mathbb{C}}|\det H^*H-|x|^2I|\,\chi_{C_Z}(H,x)dHdx,\end{aligned}$$

where χ_{C_Z} is the characteristic function of the ellipsoid $\{(C_ZH,H)+|x|^2)\leq 1\}$.

Finally, we give formula of Itzykson-Zuber type as in [DSZ2, Lemma 3.1], which is useful in that it has a fixed domain of integration.

Theorem 8.2. *Let $\Lambda_Z=C_Z\oplus I$ on $\mathcal{H}_Z\oplus\mathbb{C}$ and let P_Z denote the orthogonal projection from $\mathrm{Sym}(m,\mathbb{C})$ onto $\mathcal{H}_Z$. Then:*

$$\mathcal{K}^{\rm crit}(Z)=c_m\lim_{\varepsilon'\to 0^+}\int_{\mathbb{R}^m}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^m}\int_{\mathrm{U}(m)}\frac{\Delta(\xi)\,\Delta(\lambda)\,|\prod_j\lambda_j|\,e^{i\langle\xi,\lambda\rangle}e^{-\varepsilon|\xi|^2-\varepsilon'|\lambda|^2}}{\sqrt{\det\left[i\Lambda_ZP_Z\rho(g)^*\widehat{D}(\xi)\rho(g)+I\right]}}\,dg\,d\xi\,d\lambda,$$

where:

- $m=h^{2,1}+1,\quad c_m=\frac{(-i)^{m(m-1)/2}}{2^m\,\pi^{2m}\,\prod_{j=1}^m j!}$;
- $\Delta(\lambda)=\Pi_{i<j}(\lambda_i-\lambda_j)$,
- dg *is unit mass Haar measure on* $\mathrm{U}(m)$,
- $\widehat{D}(\xi)$ *is the Hermitian operator on* $\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}$ *given by*
$$\widehat{D}(\xi)\big((H_{jk}),x\big)=\left(\left(\frac{\xi_j+\xi_k}{2}\,H_{jk}\right),\ -\left(\textstyle\sum_{q=1}^m\xi_q\right)x\right)\ ,$$
- ρ *is the representation of* $\mathrm{U}(m)$ *on* $\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}$ *given by*
$$\rho(g)(H,x)=(gHg^t,x)\ .$$
- $\mathcal{H}_Z$ *is a real (but not complex) subspace of* $\mathrm{Sym}(m,\mathbb{C})$.

The proof is similar to the one in [DSZ2], but we sketch the proof here to provide a published reference. Some care must be taken since the Gaussian integrals are over real but not complex spaces of complex symmetric matrices.

Proof. We first rewrite the integral in Theorem 8.1 as a Gaussian integral over $\mathcal{H}_Z\oplus\mathbb{C}$ (viewed as a real vector space):

$$\mathcal{K}^{\rm crit}(Z)=\int_{\mathcal{H}_Z\oplus\mathbb{C}}|\det H^*H-|x|^2I|\ \ \chi_{\{\langle\Lambda_Z^{-1}H,H\rangle\leq 1\}}dHdx=\frac{\pi^m\sqrt{\det\Lambda_Z}}{b_3!}\,\mathcal{I}(Z)\,,$$

where

$$\mathcal{I}(Z)=\frac{1}{\pi^m\sqrt{\det\Lambda_Z}}\int_{\mathcal{H}_Z\times\mathbb{C}}\left|\det(HH^*-|x|^2I)\right|\exp\left(-\langle\Lambda_Z^{-1}(H,x),(H,x)\rangle\right)dHdx\ . \tag{28}$$

Here, H is a complex $m\times m$ symmetric matrix, so $H^*=\overline{H}$. The inner product in the exponent is the real part of the Hilbert-Schmidt inner product, $\langle A,B\rangle=\mathrm{ReTr}AB^*$.

As in [DSZ2], we rewrite the integral as

$$\mathcal{I}(Z)=\lim_{\varepsilon'\to 0}\lim_{\varepsilon\to 0}\mathcal{I}_{\varepsilon,\varepsilon'}(Z)\ ,$$

where $\mathcal{I}_{\varepsilon,\varepsilon'}(Z)$ is the absolutely convergent integral,

$$\mathcal{I}_{\varepsilon,\varepsilon'}(Z) = \frac{1}{(2\pi)^{m^2}\pi^m\sqrt{\det\Lambda}}\int_{\mathcal{H}_m}\int_{\mathcal{H}_m}\int_{\mathcal{H}_Z\times\mathbb{C}} |\det P|\, e^{-\varepsilon Tr\Xi^*\Xi-\varepsilon' TrP^*P}e^{i\langle\Xi,P-HH^*+|x|^2I\rangle_{HS}} \times \exp\left(-\langle\Lambda_Z^{-1}(H,x),(H,x)\rangle\right)\, dH\, dx\, dP\, d\Xi. \quad (29)$$

Here, $\mathcal{H}_m$ denotes the space of all Hermitian matrices of rank m, and $\langle,\rangle_{HS}$ is the Hilbert-Schmidt inner product $TrAB^*$. Formula 29 is valid, since as $\varepsilon\to 0$, the $d\Xi$ integral converges to the delta function $\delta_{HH^*-|x|^2I}(P)$. Then, as $\varepsilon'\to 0$, the dP integral evaluates the integrand at $P = HH^* - |x|^2I$ and we retrieve the original integral $\mathcal{I}(Z)$.

By the same manipulations as in [DSZ2], we obtain:

$$\mathcal{I}_{\varepsilon,\varepsilon'}(Z) = \frac{(-i)^{m(m-1)/2}}{(2\pi)^m(\prod_{j=1}^m j!)\pi^m\sqrt{\det\Lambda_Z}}\int_{\mathrm{U}(m)}\int_{\mathcal{H}_Z\times\mathbb{C}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\Delta(\lambda)\Delta(\xi)\ |\det(D(\lambda))| \times e^{i\langle\lambda,\xi\rangle}e^{-\varepsilon(|\xi|^2+|\lambda|^2)}e^{i\langle D(\xi),|x|^2I-gHH^*g^*\rangle_{HS}-\langle\Lambda^{-1}(H,x),(H,x)\rangle}\, d\xi\, d\lambda\, dH\, dx\, dg\ . \quad (30)$$

Further we observe that the $dHdx$ integral is a Gaussian integral. Simplifying the phase as in [DSZ2] using

$$\langle D(\xi),gHH^*g^*-|x|^2I\rangle_{HS} = Tr(D(\xi)gHg^t\bar{g}H^*g^*)-TrD(\xi)\,|x|^2 = \left\langle\widehat{D}(\xi)\rho(g)(H,x),\rho(g)(H,x)\right\rangle_{HS}$$

where $\widehat{D}(\xi)$ and $\rho(g)$ are as in the statement of the theorem, the $\mathcal{H}_Z\times\mathbb{C}$ integral becomes

$$\mathcal{I}_{\xi,g}(Z) := \int_{\mathcal{H}_Z\times\mathbb{C}}\exp\left[-i\left\langle\widehat{D}(\xi)\rho(g)(H,x),\rho(g)(H,x)\right\rangle_{HS}-\langle\Lambda_Z^{-1}(H,x),(H,x)\rangle\right]dH\,dx. \quad (31)$$

The only new points in the calculation are that this Gaussian integral is over the Hessian space $\mathcal{H}_Z$ rather than over the full space of complex symmetric matrices of this rank, and that it is a real subspace a complex vector space. Hence the Gaussian integral is a real one albeit with a complex quadratic form. We denote by $\mathcal{P}_Z$ the orthogonal projection

$$\mathcal{P}_Z : \mathrm{Sym}(m,\mathbb{C})\to\mathcal{H}_Z$$

and then we have:

$$\frac{1}{\pi^m\sqrt{\det\Lambda_Z}}\mathcal{I}_{\xi,g}(Z) = \frac{1}{\sqrt{\det\Lambda_Z}}\frac{1}{\sqrt{\det[i\mathcal{P}_Z\rho(g)^*\widehat{D}(\xi)\rho(g)+\Lambda_Z^{-1}]}} = \frac{1}{\sqrt{\det\left[i\Lambda_Z\mathcal{P}_Z\rho(g)^*\widehat{D}(\xi)\rho(g)+I_m\right]}}\ . \quad (32)$$

Substituting (32) into (30), we obtain the desired formula. We now recall that $\Lambda = C'\oplus 1$. It follows that

$$\Lambda_Z\mathcal{P}_Z\rho(g)^*\widehat{D}(\xi)\rho(g)+I_m = (C'_Z\mathcal{P}_Z\rho(g)^*D(\xi)\rho(g)+I_{h^{21}})\oplus(1-\sum_{q=1}^m\xi_q),$$

where

$$D(\xi)(H_{jk})) = \left(\frac{\xi_j + \xi_k}{2} H_{jk}\right).$$

Hence, its determinant equals

$$(1 - \sum_{q=1}^{h^{21}} \xi_q) \det(C_Z' \mathcal{P}_Z \rho(g)^* D(\xi) \rho(g) + I_{h^{21}}).$$

□

9. Black hole attractors

We close this survey with a discussion of a simpler problem analogous to counting flux vacua that arises in the quantum gravity of black holes [St, FGK], namely counting solutions of the black-hole attractor equation. For a mathematical introduction to this equation, we refer the reader to [MM]. The attractor equation is the same as the critical point equation for flux superpotentials except that $\mathcal{C} = \mathcal{M}$ and $G \in H^3(X, \mathbb{R})$. Physically, $\mathcal{N}_\psi(S)$ counts the so-called duality-inequivalent, regular, spherically symmetric BPS black holes with entropy $S \leq S_*$. The charge of a black hole is an element $Q = N^\alpha \Sigma_\alpha \in H^3(X, \mathbb{Z})$. The central charge $\mathcal{Z} = \langle Q, \Omega \rangle$ plays the role of the superpotential.

There are two main differences to the vacuum counting problems for flux superpotentials. First, the reality of the flux G in the black-hole attractor equation $\nabla W_G(z) = 0$ forces $G \in H^{3,0}_z \oplus H^{0,3}_z$ rather than $G \in H^{2,1}_z \oplus H^{0,3}_z$ as in the flux vacua equation. The space $H^{3,0} \oplus H^{0,3}$ is only 2-dimensional and that drastically simplifies the problem. Second, by a well-known computation due to Strominger, the Hessian $D\nabla G(z)$ of $|\mathcal{Z}|^2$ at a critical point is always a scalar multiple $x\Theta$ of the curvature form of the line bundle, which is the Weil-Petersson $(1,1)$ form.

We now state the analogue of Theorem 8.1 in the black hole attractor case (see also [DD]). The new feature is that the image of Hessian map from the space $\mathcal{S}_z$ of W_G with a critical point at z is the one-dimensional space of Hessians of the form

$$\begin{pmatrix} 0 & -x\Theta \\ -\overline{x\Theta} & 0 \end{pmatrix}, \tag{33}$$

and hence the pushforward under the Hessian map truly simplifies the integral in Theorem 8.1. The formula for the black-hole density becomes

$$\mathcal{K}^{\text{crit}}_{\gamma,\nabla}(z) = \int_{\mathbb{C}} |x|^{2b_3} \chi_{Q_z}(x) dx.$$

We note that the difficult absolute value in Theorem 8.1 simplifies to a perfect square in the black hole density formula and can therefore be evaluated as a Gaussian integral. Additionally, the one-dimensionality of the space of Hessians has removed the complexity of the b_3-dimensional integral in the flux vacuum setting.

We can further simplify the integral by removing Q_z, which is a scalar multiple of the Euclidean $|x|^2$. The scalar multiple involves the orthogonal projection $\Pi_{\mathcal{S}_z}(z, w)$ onto the space of $\mathcal{S}_z$ for the inner product Q_z. If we change variables $x \to \sqrt{\Pi_{\mathcal{S}_z}(z,z)}$, we get

$$\mathcal{K}^{\text{crit}}_{\gamma,\nabla}(z) = |\Pi_{\mathcal{S}_z}(z,z)| \int_{\mathbb{C}} |x|^{2b_3} e^{-\langle x,x\rangle} dx.$$

In Kähler normal coordinates, use of special geometry shows that $\Pi_z(z,z) = 1$. A simple calculation shows:

Proposition **9.1.** *The density of extremal black holes is given by:*

$$\mathcal{K}^{\text{crit}}_{\gamma,\nabla}(z) = \frac{1}{b_3} dV_{WP} \quad \Longrightarrow \quad \mathcal{N}_\psi(L) \sim L^{b_3} Vol_{WP}(\mathcal{M}).$$

The analogy between the black hole density and flux vacuum critical point density should be taken with some caution since the simplifying features are likely to have over-simplified the problem. We therefore mention another modified flux vacuum problem in which the off-diagonal entries $x\Theta$ of the Hessian matrix vanish, so that the Hessian matrix is purely holomorphic and $|\det H^*H - |x|^2\Theta| = |\det H^*H|$ again becomes a perfect square which can be evaluated by the Wick method. Namely, if one uses a flat meromorphic connection ∇ rather than the Weil-Petersson connection, the curvature vanishes away from the polar divisor. The Weil-Petersson connection arises naturally in string/M theory [CHSW], but one may view a meromorphic connection as an approximation in which the 'Planck mass' is infinitely large. In any case, it would be interesting to evaluate the density of critical points relative to meromorphic connections since they are more calculable and should have the same complexity as those for Weil-Petersson connections.

References

[Ar] N. Arkani-Hamed, S. Dimopoulos and Shamit Kachru, Predictive Landscapes and New Physics at a TeV (hep-th/0501082).

[AD] S. Ashok and M. Douglas, Counting Flux Vacua, J. High Energy Phys. 0401 (2004) 060 hep-th/0307049.

[BSZ] P. Bleher, B. Shiffman, and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds. Invent. Math. 142 (2000), no. 2, 351–395.

[BP] R. Bousso and J. Polchinski, The String Theory Landscape, Scientific American, September 2004 issue.

[Can1] P. Candelas, X. C. de la Ossa, Moduli space of Calabi-Yau manifolds. Nuclear Phys. B 355 (1991), no. 2, 455–481.

[CHSW] P. Candelas, G. Horowitz, A. Strominger, and E. Witten, Vacuum configurations for superstrings. Nuclear Phys. B 258 (1985), no. 1, 46–74.

[CQ] J. P. Conlon and F. Quevedo, On the explict construction and statistics of Calabi-Yau vacua, Jour. High Energy Physics 0410:039 (2004) (e-Print Archive: hep-th/0409215).

[DD] F. Denef and M. R. Douglas, Distributions of flux vacua, J. High Energy Phys. JHEP05(2004)072 (arxiv preprint hep-th/0404116.)

[DD2] F. Denef and M. R. Douglas, Distributions of nonsupersymmetric flux vacua, e-Print Archive: hep-th/0411183.

[DGKT] O. DeWolfe, A. Giryavets, S. Kachru and W. Taylor, Enumerating Flux Vacua with Enhanced Symmetries (hep-th/0411061).

[D] M. R. Douglas, The statistics of string/M theory vacua, J. High Energy Phys. 2003, no. 5, 046, 61 pp. (hep-th/0303194).

[DSZ1] M. R. Douglas, B. Shiffman and S. Zelditch, Critical points and supersymmetric vacua I, Communications in Mathematical Physics 252, Numbers 1-3 (2004), pp. 325 - 358 (arxiv preprint arxiv.org/math.CV/0402326).

[DSZ2] M. R. Douglas, B. Shiffman and S. Zelditch, Critical Points and supersymmetric vacua II: asymptotics and extremal metrics, math.CV/0406089.

[DSZ3] M. R. Douglas, B. Shiffman and S. Zelditch, Critical Points and supersymmetric vacua III: string/M models (in preparation).

[DO] W. Duke and O. Imamoglu, Lattice points in cones and Dirichlet series, IMRN 53 (2004)

[FGK] S. Ferrara, G. W. Gibbons, and R. Kallosh, Black holes and critical points in moduli space. Nuclear Phys. B 500 (1997), no. 1-3, 75–93.

[F] Y. V. Fyodorov, Complexity of random energy landscapes, glass transition, and absolute value of the spectral determinant of random matrices. Phys. Rev. Lett. 92 (2004), no. 24, 240601.

[GKP] S.B. Giddings, S. Kachru, and J. Polchinski, Hierarchies from fluxes in string compactifications. Phys. Rev. D (3) 66 (2002), no. 10, 106006.

[GKT] A. Giryavets, S. Kachru and P. K. Tripathy, On the taxonomy of flux vacua, JHEP 0408:002,2004 e-Print Archive: hep-th/0404243.

[GKTT] A. Giryavets, S. Kachru, P. K. Tripathy, and S. P. Trivedi, Flux Compactifications on Calabi-Yau threefolds, JHEP 0404:003,2004 e-Print Archive: hep-th/0312104.

[Gr] M. Gross, Calabi-Yau manifolds and mirror symmetry. Calabi-Yau manifolds and related geometries, in [GHJ], 69–159.

[GHJ] M. Gross, D. Huybrechts, and D. Joyce, *Calabi-Yau manifolds and related geometries.* Lectures from the Summer School held in Nordfjordeid, June 2001. Universitext. Springer-Verlag, Berlin, 2003.

[GVW] S. Gukov, C. Vafa, and E. Witten, CFT's from Calabi-Yau four-folds. Nuclear Phys. B 584 (2000), no. 1-2, 69–108.

[KKLT] S. Kachru, R. Kallosh, A. Linde, and S. P. Trivedi, de Sitter Vacua in String Theory, Phys.Rev. D68 (2003) 046005 (hep-th/0301240).

[KST] S. Kachru, M. Schulz, and S. Trivedi, Moduli Stabilization from Fluxes in a Simple IIB Orientifold, JHEP 0310 (2003) 007 (hep-th/0201028).

[MM] S.D. Miller and G. Moore, Landau-Siegel zeros and black hole entropy, e-Print Archive: hep-th/9903267.

[SZ] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles. Comm. Math. Phys. 200 (1999), no. 3, 661–683.

[St] A. Strominger, Kaluza-Klein compactifications, supersymmetry and Calabi-Yau manifolds, p. 1091-1115 of *Quantum fields and strings: a course for mathematicians. Vol. 1, 2.* Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. Edited by Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison and Edward Witten. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999.

[St3] A. Strominger, Special geometry. Comm. Math. Phys. 133 (1990), no. 1, 163–180.

[S] L. Susskind, The anthropic landscape of string theory, e-Print Archive: hep-th/0302219.

[WB] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Second Edition, Princeton Series in Physics, Princeton U. Press (1992).

[Z] S. Zelditch, Random Complex Geometry and Vacua, or: How to count universes in string/M theory (preprint, 2005).

[Z2] S. Zelditch, Angular distribution of lattice points (in preparation).

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