Geometry and large deviations for zeros of Gaussian random holomorphic polynomials on Riemann surfaces

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Based on joint work (partly in progress) with O. Zeitouni and B.Shiffman

Themes of talk

This talk is about large deviations (LD) for empirical measures of zeros of random holomorphic polynomials in one variable, and more generally about holomorphic sections of line bundles over Riemann surfaces.

- Use geometry (volume forms, hermitian metrics) to define inner products, hence Gaussian measures.

- How does the geometry influence zeros of random holomorphic polynomials (or sections)?

- Large deviations of empirical measures of zeros as the degree $N \to \infty$.

Complex Kac-Hammersley polynomials

To introduce our problem, consider

$$f(z) = \sum_{j=1}^{N} c_j z^j$$

where the coefficients c_j are independent complex Gaussian random variables of mean zero and variance one. Complex Gaussian:

$$\mathbf{E}\,(c_j) = 0 = E(c_j c_k), \quad E(c_j \bar{c}_k) = \delta_{jk}.$$

This defines a Gaussian measure γ_{KAC} on $\mathcal{P}_N^{(1)}$:

$$d\gamma_{KAC}(f) = e^{-|c|^2/2} dc.$$

Expected distribution of zeros

The empirical measure of zeros of a polynomial of degree N is the probability measure on $\mathbb{C}$ defined by

$$Z_f = \mu_\zeta = \frac{1}{N} \sum_{z:f(z)=0} \delta_z,$$

where δ_z is the Dirac delta-function at z.

Definition: The expected distribution of zeros of random polynomials of degree N with measure P is the probability measure $\mathbf{E}_P Z_f$ on $\mathbb{C}$ defined by

$$\langle \mathbf{E}_P Z_f, \varphi \rangle = \int_{\mathcal{P}_N^{(1)}} \{ \frac{1}{N} \sum_{z:f(z)=0} \varphi(z) \} dP(f),$$

for $\varphi \in C_c(\mathbb{C})$.

How are zeros of complex Kac polynomials distributed?

Complex zeros concentrate in small annuli around the unit circle S^1. In the limit as the degree $N \to \infty$, the zeros asymptotically concentrate exactly on S^1:

Theorem **1 (Kac-Hammersley-Shepp-Vanderbei)** *The expected distribution of zeros of polynomials of degree N in the Kac ensemble has the asymptotics:*

$$\mathbf{E}^N_{KAC}(Z^N_f) \to \delta_{S^1} \quad \text{as} \quad N \to \infty \,,$$

$$\text{where} \qquad (\delta_{S^1}, \varphi) := \frac{1}{2\pi} \int_{S^1} \varphi(e^{i\theta})\, d\theta.$$

Gaussian measure and inner product

It was the (implicit) choice of inner product that produced this concentration of zeros on S^1.

The inner product underlying the Kac Gaussian measure on $\mathcal{P}_N^{(1)}$ is defined by the basis $\{z^j\}$ being orthonormal. Thus, they were orthonormalized on S^1. An inner product induces an orthonormal basis $\{S_j\}$ and associated associated Gaussian measure dγ:

$$S = \sum_{j=1}^{d} c_j S_j,$$

where $\{c_j\}$ are independent complex normal random variables.

Orthonormalizing on S^1 made zeros concentrate on S^1.

Gaussian random polynomials adapted to domains and weights

We now orthonormalize polynomials on the interior Ω or boundary $\partial\Omega$ of any simply connected, bounded domain $\Omega \subset \mathbb{C}$. Introduce a weight $e^{-N\varphi}$ and a probability measure $d\nu$ on Ω and define

$$\langle f, \bar{g} \rangle_{\Omega,\varphi} := \int_{\Omega} f(z)\overline{g(z)}\, e^{-N\varphi(z)} d\nu \ .$$

Let $\gamma^N_{\Omega,\varphi}$ = the Gaussian measure induced by $\langle f, \bar{g} \rangle_{\Omega,\varphi}$ on $\mathcal{P}^{(1)}_N$.

How do zeros of random polynomials adapted to Ω concentrate?

Equilibrium distribution of zeros

Denote the expectation relative to the ensemble $(\mathcal{P}_N, \gamma_\Omega^N)$ by $\mathbf{E}_\Omega^N$.

Theorem **2** *(Shiffman-Z, 2003)*

$$\mathbf{E}_\Omega^N(Z_f^N) = \nu_\Omega + O\left(1/N\right) \ ,$$

where ν_Ω is the equilibrium measure of $\bar{\Omega}$ with respect to φ.

The equilibrium measure of a compact set K is the unique probability measure $d\nu_K$ which minimizes the energy

$$E(\mu) = -\int_K \int_K \log|z-w|\, d\mu(z)\, d\mu(w) + \int_K \varphi d\mu.$$

Thus, zeros behave like electric charges in the potential φ.

Hermitian metrics and line bundles: $SU(2)$ polynomials

There exists an inner product in which the expected distribution of zeros is 'uniform' on $\mathbb{CP}^1$ w.r.t. to the usual Fubini-Study area form ω_{FS}.

We define an inner product on $\mathcal{P}_N^{(1)}$ which depends on N:

$$\langle z^j, z^k \rangle_N = \frac{1}{\binom{N}{j}} \delta_{jk}.$$

Thus, a random $SU(2)$ polynomial has the form

$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \sqrt{\binom{N}{\alpha}}\, z^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \overline{\lambda}_\beta) = \delta_{\alpha\beta}.$$

Proposition **3** *In the* $SU(2)$ *ensemble,* $\mathbf{E}\,(Z_f) = \omega_{FS}$*, the Fubini-Study area form on* $\mathbb{CP}^1$*.*

$SU(2)$ **and holomorphic line bundles**

The $SU(2)$ inner products may be written in the form

$$\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-N\log(1+|z|^2)}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}.$$

The factor $e^{-N\log(1+|z|^2)}$ defines a Hermitian metric on the line bundle $\mathcal{O}(N)$, and its curvature form is $\omega = \frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$.

This gives a geometric interpretation of the inner product

$$\langle f, \bar{g}\rangle_{\Omega,\varphi} := \int_{\Omega} f(z)\overline{g(z)}\, e^{-N\varphi(z)} d\nu \ \ .$$

We should regard f, g as sections of the Nth power of a line bundle with Hermitian metric $e^{-N\varphi}$.

Gaussian random holomorphic sections of line bundles

We now consider more general Hermitian metrics $h = e^{-\varphi}$ on $\mathcal{O}(1) \to \mathbb{CP}^1$ and area forms on $\mathbb{CP}^1$. In fact, everything we do generalizes to any Riemann surface M of any genus.

The Hermitian metric h on $\mathcal{O}(1)$ induces Hermitian metrics $h^N = e^{-N\varphi}$ on the powers $\mathcal{O}(N)$, a volume form dV, and an inner product

$$\langle s_1, s_2 \rangle_N = \int_M s_1(z)\overline{s_2(z)} e^{-N\varphi} dV(z).$$

We let $\{S_j\}$ denote an orthonormal basis of the space $H^0(M, L^N)$ of holomorphic sections of L^N.

Then define the Gaussian measure γ_{h^N} on $s \in H^0(M, L^N)$ by

$$s = \sum_j c_j S_j, \quad \langle S_j, S_k \rangle = \delta_{jk}$$

with $\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k)$, $\mathbf{E}(c_j \overline{c_k}) = \delta_{jk}$.

Why and wherefore line bundles?

Compact complex manifolds have no non-constant holomorphic functions. The replacement for them is ‘holomorphic sections of line bundles’. I.e. twisted holomorphic functions. Line bundles (and their holomorphic sections) have a degree, just like polynomials.

Examples:

- $g = 0$: polynomials of degree N
- $g = 1$: theta functions θ_N of degree N;
- $g \geq 2$: holomorphic differentials of type $(dz)^N$.

Expected distribution of zeros

Here is a general result on the expected distribution of zeros of random holomorphic sections over all of C:

Theorem 1 (Shiffman-Z) Let $(L, h) \to C$ be any (positive) Hermitian line bundle over any Riemann surface C and let $\omega = i\partial\bar{\partial}\varphi$ be the curvature form of $h = e^{-\varphi}$. Consider the 'powers' L^N- analogous to polynomials of degree N. Then,

$$\frac{1}{(N)^m}\mathbf{E}_N(Z_f) \to \omega$$

in the sense of weak convergence; i.e., for any open $U \subset \mathbb{C}^{*m}$, we have

$$\frac{1}{N}\mathbf{E}_N\Big(\#\{z \in U : f(z)) = 0\}\Big)$$

$$\to \omega(U)\ .$$

Zeros concentrate in curved regions. Curvature causes sections to oscillate and hence zeros to occur.

Almost sure distribution of zeros

The distribution of zeros is 'self-averaging': typical sections behave in the expected way. To prove this, we define the space of sequences of sections as the Cartesian product probability space

$$\Pi_{N=1}^{\infty} H^0(M, L^N), \quad \gamma_\infty := \Pi_{N=1}^{\infty} d\gamma_N.$$

THEOREM. (S–Zelditch, 1998) Consider a *random sequence* $\{f_N\}$ of sections of L^N (or polynomials of degree N), $N = 1, 2, 3, \ldots$. Then

$$\frac{1}{N} Z_{f_N} \to \omega \qquad \textit{almost surely}\, w.r.t. \gamma_\infty.$$

A measure on the space of measures

We turn to large deviations for empirical measures of zeros.

The empirical measure of zeros defines a map $\delta : (\mathbb{CP}^1)^N \to \mathcal{M}_1(\mathbb{CP}^1)$

$$\delta(\zeta_1, \ldots, \zeta_N) = d\mu_\zeta := \frac{1}{N} \sum_{j=1}^{N} \delta_{\zeta_j},$$

to the space $\mathcal{M}_1(\mathbb{CP}^1)$ of probability measures on $\mathbb{CP}^1$.

A Gaussian probability measure γ_N on $\mathcal{P}_N$ induces the probability measure the probability measure

$$\mathbf{Prob}_N = \delta_* \gamma_N \tag{1}$$

on $\mathcal{M}_1(M)$.

Main result in genus zero

Theorem **4** *(Zeitouni-Z, in progress) Let* $C = \mathbb{CP}^1$*, let* $h = e^{-\varphi}$ *be a Hermitian metric of non-negative curvature on* $\mathcal{O}(1) \to \mathbb{CP}^1$ *and let* $d\nu$ *be a positive measure on* $\mathbb{CP}^1$*. Then the sequence of probability measures* $\{Prob_N\}$ *on* $\mathcal{M}_1(\mathbb{CP}^1)$ *induced by the Gaussian ensemble* $(H^0(\mathbb{CP}^1, \mathcal{O}(N), d\gamma_{h^N,\nu})$ *satisfies a large deviations principle with speed* N^2 *and rate functional*

$$\frac{1}{2} I_\varphi(\nu) = \int_C \int_C G_\varphi(z,w) d\nu(z) d\nu(w)$$

$$+ \max_{z \in supp \nu} \{ - \int_C G_\varphi(z,w) d\nu \} - \frac{1}{2} \int_C \varphi d\nu,$$

where G_φ *is the Green's function for the metric* $\omega_h = Ric(h)$ *given by the curvature* $(1,1)$ *form of* $h = e^{-\varphi}$*.*

The two terms reflect the competition between the repulsion between zeros and the uphill climb against the potential φ which acts against the dispersion of zeros.

Discussion

Roughly speaking this means that for any Borel subset $E \subset \mathcal{M}_1(\Omega)$,

$$\frac{1}{N^2} \log \mathbf{Prob}_N\{\nu \in \mathcal{M}_1 : \nu \in E\} \sim - \inf_{\nu \in E} I(\nu),$$

i.e. the configurations of zeros concentrate exponentially fast in the equilibrium configuration, i.e. according to the equilibrium measure which minimizes the rate functional. This strengthens the conclusion that the expected distribution of zeros is the equlibrium measure.

Most of the work goes into higher genus!

Outline of proof

The proof involves three main ingredients:

- An explicit formula for the JDP of zeros. In genus $g = 0$ it has the form

  $$dV(\zeta_1, \ldots, \zeta_N) = \frac{|\Delta(\zeta_1, \ldots, \zeta_N)|^2 d^2\zeta_1 \cdots d^2\zeta_N}{\left(\int_{\mathbb{CP}^1} \prod_{j=1}^N |(z - \zeta_j)|^2 d\nu(z)\right)^{N+1}}$$

  This is much harder in higher genus.

- extraction of a rate functional out of the JPD.

- Proof of LDP.

JPD: Step 1

A Gaussian measure is equivalent to a Fubini-Study probability measure on the projective space $\mathbb{P}V_N$ of polynomials:

$$dV_{FS} = \frac{\wedge \left(\partial\bar{\partial}||f||^2\right)^d}{(||f||^2)^{d+1}}.$$

Here, f is the independent variable and $\partial\bar{\partial}$ is the derivative in the f variable.

In coordinates $(w_1, \ldots, w_d)$ relative to an ONB,

$$dV_{FS} = \frac{\prod_{j=1}^d dw_j \wedge d\bar{w}_j}{(1 + ||w||^2)^{(d+1)}}.$$

JPD: Step 2

Generalize the Vieta formula Vieta's formula:

$$\Pi_{j=1}^{N}(z-\zeta_j) = \sum_{k=0}^{N}(-1)^k e_{N-k}(\zeta_1,\ldots,\zeta_N)z^k$$

so that the basis $\{z^k\}$ is replaced by the ONB $\{\psi_j\}$w.r.t the inner product:

$$\prod_{j=1}^{N}(z-\zeta_j) := \sum_{j=1}^{N+1}\mathcal{E}_{N+1-j}(\zeta_1,\ldots,\zeta_N)\psi_j(z).$$

Then the numerator of the JPD is

$$\Pi_{j=1}^{N} dw_j \wedge d\bar{w}_j = \; d\mathcal{E}_1 \wedge d\overline{\mathcal{E}}_1 \wedge \cdots \wedge d\mathcal{E}_N \wedge d\overline{\mathcal{E}}_N$$

$$= A_N(h)|\Delta(\zeta)|^2 \, d\zeta_1 \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\zeta_r \wedge d\bar{\zeta}_N,$$

for a non-zero constant $A_N(h)$ depending only on N and the Hermitian metric h.

JPD: Step 3 The denominator is the $(N+1)$st power of

$$||s||^2_{L^2} = \int_C \exp\left(-2N \int_C G_0(z,w) d\mu_\zeta\right) e^{-N\varphi} d\nu(z),$$

where $G_0(z,w) = -\log|z-w|$. In more general settings, G_0 is the Green's function of the metric. We then use:

$$-\frac{1}{N^2} \log \frac{|\Delta(\zeta_1,\ldots,\zeta_N)|^2}{\left(\int_D \prod_{j=1}^N |(z-\zeta_j)|^2 e^{-N\varphi} d\mu\right)^{N+1}} \sim \Sigma(\mu_\zeta) + J^\varphi(\mu_\zeta$$

where

$$\Sigma(\mu) = \int_{\mathbb{C}} \log \frac{1}{|z-w|} d\mu(z) d\mu(w)$$

is the logarithmic energy and where

$$J^\varphi(\mu) := \lim \frac{N+1}{N^2} \log \int e^{-2N \int G_0 d\mu} e^{-N\varphi} d\nu$$

$$\to \max_{z \in supp \mu} \{-2 \int_M G_0(z,w) d\mu - \varphi(z)\}.$$