BERGMAN METRICS AND GEODESICS IN THE SPACE OF KÄHLER METRICS ON TORIC VARIETIES

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Abstract. A guiding principle in Kähler geometry is that the infinite dimensional symmetric space $\mathcal{H}$ of Kähler metrics in a fixed Kähler class on a polarized projective Kähler manifold M should be well approximated by finite dimensional submanifolds $\mathcal{B}_k \subset \mathcal{H}$ of Bergman metrics of height k (Yau, Tian, Donaldson). The Bergman metric spaces are symmetric spaces of type $G_{\mathbb{C}}/G$ where $G = U(d_k+1)$ for certain d_k. This article establishes the basic estimates for Bergman approximations for geometric families of toric Kähler manifolds.

The approximation results are applied to the endpoint problem for geodesics of $\mathcal{H}$, which are solutions of a homogeneous complex Monge-Ampère equation in $A \times X$, where $A \subset \mathbb{C}$ is an annulus. Donaldson, Arezzo-Tian and Phong-Sturm raised the question whether $\mathcal{H}$-geodesics with fixed endpoints can be approximated by geodesics of $\mathcal{B}_k$. Phong-Sturm proved weak C^0-convergence of Bergman to Monge-Ampère geodesics on a general Kähler manifold. Our approximation results show that one has $C^2(A \times X)$ convergence in the case of toric Kähler metrics, extending our earlier result on $\mathbb{CP}^1$.

In subsequent papers, the techniques of this article are applied to approximations for harmonic maps into $\mathcal{H}$, to test configuration geodesic rays and to the smooth initial value problem.

Contents

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1. Introduction

This is the first in a series of articles on the Riemannian geometry of the space

$$\mathcal{H} \;=\; \{\varphi \in C^\infty(M) : \omega_\varphi \;=\; \omega_0 + dd^c\varphi > 0 \;\} \tag{1}$$

of Kähler metrics in the class $[\omega_0]$ of a polarized projective Kähler manifold (M, ω_0, L), equipped with the Riemannian metric $g_{\mathcal{H}}$ of Mabuchi-Semmes-Donaldson [M1, S, D2],

$$||\psi||^2_{g_{\mathcal{H}},\varphi} \;=\; \int_M |\psi|^2 \,\frac{\omega_\varphi^m}{m!}, \qquad \text{where } \varphi \in \mathcal{H} \text{ and } \psi \in T_\varphi\mathcal{H} \simeq C^\infty(M). \tag{2}$$

Here, $L \to M$ is an ample line bundle with $c_1(L) = [\omega_0]$. Formally, $(\mathcal{H}, g_{\mathcal{H}})$ is an infinite dimensional non-positively curved symmetric space of the type $G_{\mathbb{C}}/G$, where $G = SDiff_{\omega_0}(M)$ is the group of Hamiltonian symplectic diffeomorphisms of (M, ω_0). This statement is only formal since G does not possess a complexification and $\mathcal{H}$ is an incomplete, infinite dimensional space. An attractive approach to the infinite dimensional geometry is to approximate it by a sequence of finite dimensional submanifolds $\mathcal{B}_k \subset \mathcal{H}$ of so-called Bergman (or Fubini-Study) metrics. The space $\mathcal{B}_k$ of Bergman metrics may be identified with the finite dimensional symmetric space $GL(d_k+1, \mathbb{C})/U(d_k+1)$ where d_k is a certain dimension. Thus, $\mathcal{B}_k$ is equipped with a finite dimensional symmetric space metric $g_{\mathcal{B}_k}$, which is not the same as the submanifold Riemannian metric induced on it by $g_{\mathcal{H}}$. The purpose of the series is to show that much of the symmetric space geometry of $(\mathcal{B}_k, g_{\mathcal{B}_k})$ tends to the infinite dimensional symmetric space geometry of $(\mathcal{H}, g_{\mathcal{H}})$ as $k \to \infty$.

To put the problem and results in perspective, we recall that at the level of individual metrics $\omega \in \mathcal{H}$, there exists a well-developed approximation theory: Given ω, one can define a canonical sequence of Bergman metrics $\omega_k \in \mathcal{B}_k$ which approximates ω in the C^∞ topology (see (9)), in much the same way that smooth functions can be approximated by Bernstein polynomials (Yau [Y] and Tian [T]; see also [C, Ze1, Ze2]). The approximation theory is based on microlocal analysis in the complex domain, specifically Bergman kernel asymptotics on and off the diagonal [BSj, C, Ze1, D1, PS3]. As will be shown in [RZ3], one may use the same methods to prove that the geometry of $(\mathcal{B}_k, g_{\mathcal{B}_k})$ tends to the geometry of $(\mathcal{H}, g_{\mathcal{H}})$ at the infinitesimal level: e.g. that the Riemann metric, connection and curvature tensor of $\mathcal{B}_k$ tend to the Riemann metric, connection and curvature of $\mathcal{H}$. But our principal aim in this series is to extend the approximation from pointwise or infinitesimal objects to more global aspects of the geometry, such as such as $\mathcal{B}_k$ -geodesics or harmonic maps to $(\mathcal{B}_k, g_{\mathcal{B}_k})$. These more global approximation problems are much more difficult than the infinitesimal ones. The obstacles are analogous to those involved in complexifying $SDiff_{\omega_0}(M)$. We will explain this comparison in more detail in §1.6 at the end of the introduction.

This article is concerned with the approximation of $g_{\mathcal{H}}$-geodesic segments ω_t in $\mathcal{H}$ with fixed endpoints by $g_{\mathcal{B}_k}$-geodesic segments in $\mathcal{B}_k$. As recalled in §1.1, the geodesic equation for the Kähler potentials φ_t of ω_t is a complex homogeneous Monge-Ampère equation. Little is known about the solutions of the Dirichlet problem at present beyond the regularity result that $\varphi_t \in C^{1,\alpha}([0,T] \times M)$ for all $\alpha < 1$ if the endpoint metrics are smooth (see X. Chen [Ch] and Chen-Tian [CT] for results and background). It is therefore natural to study the approximation of Monge-Ampère $g_{\mathcal{H}}$-geodesics φ_t by the much simpler $g_{\mathcal{B}_k}$-geodesics $\varphi_k(t,z)$, which are defined by one parameter subgroups of $GL(d_k+1, \mathbb{C})$ (see (24)). The problem of approximating $\mathcal{H}$-geodesic segments between two smooth endpoints by $\mathcal{B}_k$-geodesic segments was raised by Donaldson [D1], Arezzo-Tian [AT] and Phong-Sturm [PS1] and was studied in depth by Phong-Sturm in [PS1, PS2]. They proved in [PS1] that $\varphi_k(t,z) \to \varphi_t$ in a weak C^0 sense on $[0,1] \times M$ (see (13)); a C^0 result with a remainder estimate was later proved by Berndtsson [B] for a somewhat different approximation.

In this article, we study the $g_{\mathcal{B}_k}$-approximation of $g_{\mathcal{H}}$-geodesics in the case of a polarized projective toric Kähler manifold. Our main result is that a $g_{\mathcal{H}}$ geodesic segment of toric Kähler metrics with fixed endpoints is approximated in C^2 by a sequence $\varphi_k(t,z)$ of toric $g_{\mathcal{B}_k}$- geodesic segments. More precisely, for any $T \in \mathbb{R}_+$, $\varphi_k(t,z) \to \varphi_t(z)$ in $C^2([0,T] \times M)$, generalizing the results of [SoZ1] in the case of $\mathbb{CP}^1$. It is natural to study convergence of two (space-time) derivatives since the Kähler metric $\omega_\varphi = \omega_0 + dd^c\varphi$ involves two derivatives. In the course of the proof, we introduce methods which have many other applications to global approximation problems on toric Kähler manifolds, and which should also have applications to non-toric Kähler manifolds.

Here, as in [SoZ2, RZ1, RZ2], we restrict to the toric setting because, at this stage, it is possible to obtain much stronger results than for general Kähler manifolds and because it is one of the few settings where we can see clearly what is involved in the classical limit as $k \to \infty$. The simplifying feature of toric Kähler manifolds is that they are completely integrable on both the classical and quantum level. In Riemannian terms, the submanifolds of toric metrics of $\mathcal{H}$ and $\mathcal{B}_k$ form totally geodesic flats. Hence in the toric case, the geodesic equation along the flat is linearized by the Legendre transform, with the consequence that there exists an explicit formula for the Monge-Ampère geodesic φ_t between two smooth endpoint metrics. In particular, the explicit formula shows that geodesics between smooth endpoints are smooth. We use this explicit solution throughout the article, starting from (29). Thus, in the toric case we only need to prove C^2-convergence of the Bergman approximation. An analogous result on a general Kähler manifold would require an improvement on the known regularity results on Monge-Ampère geodesics in addition to a convergence result. We refer to [CT] for the state of the art on the regularity theory.

1.1. **Background.** To state our results, we need some notation and background. Let $L \to M^m$ be an ample holomorphic line bundle over a compact complex manifold of dimension m. Let $\omega_0 \in H^{(1,1)}(M, \mathbb{Z})$ denote an integral Kähler form. Fixing a reference hermitian metric h_0 on L, we may write other hermitian metrics on L as $h_\varphi = e^{-\varphi} h_0$, and then the space of hermitian metrics h on L with curvature $(1,1)$-forms ω_h in the class of ω_0 may (by the $\partial\bar\partial$ lemma) be identified with the space $\mathcal{H}$ of relative Kähler potentials (1). We may then identify the tangent space $T_\varphi \mathcal{H}$ at $\varphi \in \mathcal{H}$ with $C^\infty(M)$. Following [M1, S, D1], we define the Riemannian metric (2) on $\mathcal{H}$. With this Riemannian metric, $\mathcal{H}$ is formally an infinite dimensional non-positively curved symmetric space.

The space $\mathcal{B}_k$ of Bergman (or Fubini-Study) metrics of height k is defined as follows: Let $H^0(M, L^k)$ denote the space of holomorphic sections of the kth power $L^k \to M$ of L and let $d_k + 1 = \dim H^0(M, L^k)$. We let $\mathcal{B}H^0(M, L^k)$ denote the manifold of all bases $\underline{s} = \{s_0, \dots, s_{d_k}\}$ of $H^0(M, L^k)$. Given a basis, we define the Kodaira embedding

$$\iota_{\underline{s}} : M \to \mathbb{CP}^{d_k}, \quad z \to [s_0(z), \dots, s_{d_k}(z)]. \tag{3}$$

We then define a Bergman metric (or equivalently, Fubini-Study) metric of height k to be a metric of the form

$$h_{\underline{s}} := (\iota_{\underline{s}}^* h_{FS})^{1/k} = \frac{h_0}{\left(\sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}\right)^{1/k}}, \tag{4}$$

where h_{FS} is the Fubini-Study Hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^{d_k}$. We then define

$$\mathcal{B}_k = \{h_{\underline{s}}, \ \ \underline{s} \in \mathcal{B}H^0(M, L^k)\}. \tag{5}$$

We use the same notation for the associated space of potentials φ such that $h_{\underline{s}} = e^{-\varphi} h_0$ and for the associated Kähler metrics ω_φ. We observe that with a choice of basis of $H^0(M, L^k)$ we may identify $\mathcal{B}_k$ with the symmetric space $GL(d_k + 1, \mathbb{C})/U(d_k + 1)$ since $GL(d_k + 1, \mathbb{C})$ acts transitively on the set of bases, while $\iota_{\underline{s}}^* h_{FS}$ is unchanged if we replace the basis $\underline{s}$ by a unitary change of basis.

Several further identifications are important. The first is that $\mathcal{B}_k$ may be identified with the space $\mathcal{I}_k$ of Hermitian inner products on $H^0(M, L^k)$, the correspondence being that a basis is identified with an inner product for which the basis is Hermitian orthonormal. As in [D1, D4], we define maps

$$Hilb_k : \mathcal{H} \to \mathcal{I}_k,$$

by the rule that a Hermitian metric $h \in \mathcal{H}$ induces the inner products on $H^0(M, L^k)$,

$$||s||^2_{Hilb_k(h)} = R \int_M |s(z)|^2_{h^k} dV_h, \tag{6}$$

where $dV_h = \frac{\omega_h^m}{m!}$, and where $R = \frac{d_k+1}{Vol(M,dV_h)}$. Also, h^k denotes the induced metric on L^k. Further, we define the identifications

$$FS_k : \mathcal{I}_k \simeq \mathcal{B}_k$$

as follows: an inner product $G = \langle\ ,\ \rangle$ on $H^0(M, L^k)$ determines a G-orthonormal basis $\underline{s} = \underline{s}_G$ of $H^0(M, L^k)$ and an associated Kodaira embedding (3) and Bergman metric (4). Thus,

$$FS_k(G) = h_{\underline{s}_G}. \tag{7}$$

The right side is independent of the choice of h_0 and the choice of orthonormal basis. As observed in [D1, PS1], $FS_k(G)$ is characterized by the fact that for any G-orthonormal basis $\{s_j\}$ of $H^0(M, L^k)$, we have

$$\sum_{j=0}^{d_k} |s_j(z)|^2_{FS_k(G)} \equiv 1, \quad (\forall z \in M). \tag{8}$$

Metrics in $\mathcal{B}_k$ are defined by an algebro-geometric construction. By analogy with the approximation of real numbers by rational numbers, we say that $h \in \mathcal{H}$ (or its curvature form ω_h) has *height* k if $h \in \mathcal{B}_k$. A basic fact is that the union

$$\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$$

of Bergman metrics is dense in the C^∞-topology in the space $\mathcal{H}$ (see [T, Ze1]). Indeed,

$$\frac{FS_k \circ Hilb_k(h)}{h} = 1 + O(k^{-2}), \tag{9}$$

where the remainder is estimated in $C^r(M)$ for any $r > 0$; left side moreover has a complete asymptotic expansion (see [D3, PS1] for precise statements).

Now that we have defined the spaces $\mathcal{H}$ and $\mathcal{B}_k$, we can compare Monge-Ampère geodesics and Bergman geodesics. Geodesics of $\mathcal{H}$ satisfy the Euler-Lagrange equations for the energy functional determined by (2); see (68). By [M1, S, D2], the geodesics of $\mathcal{H}$ in this metric are the paths $h_t = e^{-\varphi_t} h_0$ which satisfy the equation

$$\ddot{\varphi} - \frac{1}{2}|\nabla \dot{\varphi}|^2_{\omega_\varphi} = 0, \tag{10}$$

which may be interpreted as a homogeneous complex Monge-Ampère equation on $A \times M$ where A is an annulus [S, D2].

Geodesics in $\mathcal{B}_k$ with respect to the symmetric space metric are given by orbits of certain one-parameter subgroups $\sigma_k^t = e^{tA_k}$ of $GL(d_k + 1, \mathbb{C})$. In the identification of $\mathcal{B}_k$ with the symmetric space $\mathcal{I}_k \simeq GL(d_k + 1, \mathbb{C})/U(d_k + 1)$ of inner products, the 1 PS (one parameter subgroup) $e^{tA_k} \in GL(d_k + 1)$ changes an orthonormal basis $\hat{\underline{s}}^{(0)}$ for the initial inner product G_0 to an orthonormal basis $e^{tA_k} \cdot \hat{\underline{s}}^{(0)}$ for G_t where G_t is a geodesic of $\mathcal{I}_k$. Geometrically, a Bergman geodesic may be visualized as the path of metrics on M obtained by holomorphically embedding M using a basis of $H^0(M, L^k)$ and then moving the embedding under the 1 PS subgroup e^{tA_k} of motions of $\mathbb{CP}^{d_k}$. The difficulty is to interpret this simple extrinsic motion in intrinsic terms on M.

In this article, we only study the endpoint problem for the geodesic equation. We assume given $h_0, h_1 \in \mathcal{H}$ and let $h(t)$ denote the Monge-Ampère geodesic between them. We then consider the geodesic $G_k(t)$ of $\mathcal{I}_k$ between $G_k(0) = Hilb_k(h_0)$ and $G_k(1) = Hilb_k(h_1)$ or equivalently between $FS_k \circ Hilb_k(h_0)$ and $FS_k \circ Hilb_k(h_1)$. Without loss of generality, we may assume that the change of orthonormal basis (or change of inner product) matrix $\sigma_k = e^{A_k}$ between $Hilb_k(h_0), Hilb_k(h_1)$ is diagonal with entries $e^{\lambda_0}, ..., e^{\lambda_{d_k}}$ for some $\lambda_j \in \mathbb{R}$. Let $\underline{\hat{s}}^{(t)} = e^{tA_k} \cdot \underline{\hat{s}}^{(0)}$ where e^{tA_k} is diagonal with entries $e^{\lambda_j t}$. Define

$$h_k(t) := FS_k \circ G_k(t) = h_{\underline{\hat{s}}^{(t)}} =: h_0 e^{-\varphi_k(t)}. \tag{11}$$

It follows immediately from (8) that

$$\varphi_k(t; z) \ = \ \frac{1}{k} \log \left(\sum_{j=0}^{N} e^{2\lambda_j t} |\hat{s}_j^{(0)}|^2_{h_0^k} \right). \tag{12}$$

We emphasize that $\varphi_k(t;z)$ is the intrinsic $\mathcal{B}_k$ geodesic between the endpoints $FS_k \circ Hilb_k(h_0)$ and $FS_k \circ Hilb_k(h_1)$. It is of course quite distinct from the $Hilb_k$-image of the Monge-Ampère geodesic; the latter is not intrinsic to $\mathcal{B}_k$ and one cannot gain any information on the $\mathcal{H}$-geodesic by studying it.

Let us summarize the notation for hermitian metrics and geodesics of metrics:

- For any metric h on L, h^k denotes the induced metric on L^k, and for any metric H on L^k, $H^{\frac{1}{k}}$ is the induced metric on L;
- Given $h_0 \in \mathcal{H}$, $h_t = e^{-\varphi_t} h_0$ is the Monge-Ampère geodesic;
- $h_k = FS \circ Hilb_k(h) \in \mathcal{B}_k$ is the natural approximating Bergman metric to h, and $h_k(t) = e^{-\varphi_k(t)} h_0$ is the Bergman geodesic (11).

The main result of Phong-Sturm [PS1] is that the Monge-Ampère geodesic φ_t is approximated by the 1PS Bergman geodesic $\varphi_k(t,z)$ in the following weak C^0 sense:

$$\varphi_t(z) = \lim_{\ell \to \infty} \left[\sup_{k \geq \ell} \varphi_k(t,z) \right]^*, \quad \text{uniformly as } \ell \to \infty, \tag{13}$$

where u^* is the upper envelope of u, i.e., $u^*(\zeta_0) = \lim_{\epsilon \to 0} \sup_{|\zeta - \zeta_0| < \epsilon} u(\zeta)$. In particular, without taking the upper envelope, $\sup_{k \geq \ell} \varphi_k(t,z) \to \varphi(t,z)$ almost everywhere as $\ell \to \infty$. See also [B] for the subsequent proof of an analogous result for the adjoint bundle $L^k \otimes K$ (where K is the canonical bundle) with an error estimate $||\varphi_k(t) - \varphi(t)||_{C^0} = O(\frac{\log k}{k})$.

1.2. **Statement of results.** Our purpose is to show that the degree of convergence of $h_k(t) \to h_t$ or equivalently of $\varphi_k(t,z) \to \varphi_t(z)$ is much stronger for toric hermitian metrics on the invariant line bundle $L \to M$ over a smooth toric Kähler manifold. We recall that a toric variety M of dimension m carries the holomorphic action of a complex torus $(\mathbb{C}^*)^m$ with an open dense orbit. The associated real torus $\mathbf{T}^m = (S^1)^m$ acts on M in a Hamiltonian fashion with respect to any invariant Kähler metric ω, i.e., it possesses a moment map $\mu : M \to P$ with image a convex lattice polytope. Here, and henceforth, P denotes the closed polytope; its interior is denoted P^o (see §2 for background). Objects associated to M are called toric if they are invariant or equivariant with respect to the torus action (real or complex, depending on the context). We define the space of toric Hermitian metrics by

$$\mathcal{H}_{\mathbf{T}^m} = \{ \varphi \in \mathcal{H} : (e^{i\theta})^* \varphi = \varphi, \quad \text{for all } e^{i\theta} \in \mathbf{T}^m \}. \tag{14}$$

Here, we assume the reference metric h_0 is $\mathbf{T}^m$-invariant. We note that since $\mathbf{T}^m$ has a moment map, it automatically lifts to L and hence it makes sense to say that $h_0 : L \to \mathbb{C}$ is invariant under it. With a slight abuse of notation carried over from [D1], we also let φ denote the full Kähler potential on the open orbit, i.e., $\omega_\varphi = dd^c\varphi$ on the open orbit. It is clearly $\mathbf{T}^m$-invariant.

Our main result is

THEOREM **1.1.** *Let $L \to M$ be a very ample toric line bundle over a smooth compact toric variety M. Let $\mathcal{H}_T$ denote the space of toric Hermitian metrics on L. Let $h_0, h_1 \in \mathcal{H}_T$ and let h_t be the Monge-Ampère geodesic between them. Let $h_k(t)$ be the Bergman geodesic between $Hilb_k(h_0)$ and $Hilb_k(h_1)$ in $\mathcal{B}_k$. Let $h_k(t) = e^{-\varphi_k(t,z)}h_0$ and let $h_t = e^{-\varphi_t(z)}h_0$. Then*

$$\lim_{k\to\infty} \varphi_k(t,z) = \varphi_t(z)$$

in $C^2([0,1] \times M)$. In fact, there exists C independent of k such that

$$||\varphi_k - \varphi||_{C^2([0,1]\times M)} \leq C\, k^{-1/3+\epsilon}, \ \ \forall \epsilon > 0.$$

Our methods show moreover that away from the divisor at infinity $\mathcal{D}$ (cf. §2), the function $\varphi_k(t,z)$ has an asymptotic expansion in powers of k^{-1}, and converges in C^∞ to φ_t. But the asymptotics become complicated near $\mathcal{D}$, and require a 'multi-scale' analysis involving distance to boundary facets. It is therefore not clear whether φ_k has an asymptotic expansion in k^{-1} globally on M. At least, no such asymptotics follow from the known Bergman kernel asymptotics, on or off the diagonal. The analysis of these regimes for general toric varieties seems to be fundamental in 'quantum mechanical approximations' on toric varieties.

As mentioned above, the Monge-Ampère equation can be linearized in the toric case and solved explicitly (17); we give a simple new proof in §2. The geodesic arcs are easily seen to be C^∞ when the endpoints are C^∞. Hence the C^2-convergence result does not improve the known regularity results on Monge-Ampère geodesics of toric metrics, but pertain only to the degree of convergence of Bergman to Monge-Ampère geodesics in a setting where the latter are known to be smooth; it is possible that the methods can be developed to give regularity results, but this is a distant prospect (see the remarks at the end of this introduction).

1.3. **Outline of the proof.** Let us now outline the proof of Theorem 1.1. We start with the fact that the Legendre transform of the Kähler potential linearizes the Monge-Ampère equation (cf. §2.7 and [A, G, D3]). The Legendre transform $\mathcal{L}\varphi$ of the open-orbit Kähler potential φ, a convex function on $\mathbb{R}^m$ in logarithmic coordinates, is the so-called dual symplectic potential

$$u_\varphi(x) = \mathcal{L}\varphi(x), \tag{15}$$

a convex function on the convex polytope P. Under this Legendre transform, the complex Monge-Ampère equation on $\mathcal{H}_{\mathbf{T}^m}$ linearizes to the equation $\ddot{u} = 0$ and is thus solved by

$$u_t = u_{\varphi_0} + t(u_{\varphi_1} - u_{\varphi_0}). \tag{16}$$

Hence the solution φ_t of the geodesic equation on $\mathcal{H}$ is solved in the toric setting by

$$\varphi_t = \mathcal{L}^{-1}u_t. \tag{17}$$

Our goal is to show that $\varphi_k(t;z) \to \mathcal{L}^{-1}u_t$ as in (16) in a strong sense.

The second simplifying feature of the toric setting occurs on the quantum level. The Bergman geodesic is obtained by applying the FS_k map to the one-parameter subgroup

e^{tA_k}. In general, it is difficult to understand what kind of asymptotic behavior is possessed by the operators e^{tA_k}. But on a toric variety, there exists a natural basis of the space of holomorphic sections $H^0(M, L^k)$ furnished by monomial sections z^α which are orthogonal with respect to all torus-invariant inner products, and with respect to which all change of basis operators e^{tA_k} are diagonal; we refer to §2 or to [STZ1] for background. Hence, we only need to analyze the eigenvalues of e^{A_k}. The exponents α of the monomials are lattice points $\alpha \in kP$ in the kth dilate of the polytope P corresponding to M. The eigenvalues in the toric case are given by

$$\lambda_\alpha := \frac{1}{2} \log \left(\frac{\mathcal{Q}_{h_0^k}(\alpha)}{\mathcal{Q}_{h_1^k}(\alpha)} \right), \tag{18}$$

where $\mathcal{Q}_{h_0^k}(\alpha)$ is a 'norming constant' for a toric inner product. By a norming constant for a toric Hermitian inner product G on $H^0(M, L^k)$ we mean the associated L^2 norm-squares of the monomials

$$\mathcal{Q}_G(\alpha) = ||s_\alpha||_G^2. \tag{19}$$

In particular, if $h \in \mathcal{H}_{\mathbf{T}^m}$, the norming constants for $Hilb_k(h)$ are given by

$$\mathcal{Q}_{h^k}(\alpha) = ||s_\alpha||_{h^k}^2 := \int_{M_P} |s_\alpha(z)|_{h^k}^2 dV_h. \tag{20}$$

Thus, an orthonormal basis of $H^0(M, L^k)$ with respect to $Hilb_k(h)$ for $h \in \mathcal{H}_T$ is given by $\{\frac{s_\alpha}{\sqrt{\mathcal{Q}_{h^k}(\alpha)}}, \quad \alpha \in kP \cap \mathbb{Z}^m\}$. An equivalent, and in a sense dual (cf. §3), formulation is in terms of the functions

$$\mathcal{P}_{h^k}(\alpha, z) := \frac{|s_\alpha(z)|_{h^k}^2}{\mathcal{Q}_{h^k}(\alpha)}, \tag{21}$$

and their special values

$$\mathcal{P}_{h^k}(\alpha) := \mathcal{P}_{h^k}(\alpha, \mu_h^{-1}(\frac{\alpha}{k})) = \frac{|s_\alpha(\mu_h^{-1}(\frac{\alpha}{k}))|_{h^k}^2}{\mathcal{Q}_{h^k}(\alpha)}. \tag{22}$$

Given two toric hermitian metrics $h_0, h_1 \in \mathcal{H}_{\mathbf{T}^m}$, the change of basis matrix $e^{A_k} = \sigma_{h_0, h_1, k}$ from the monomial orthonormal basis for $Hilb_k(h_0)$ to that for $Hilb_k(h_1)$ is diagonal, and the eigenvalues are given by

$$Sp(e^{A_k} e^{A_k^*}) := \{e^{2\lambda_\alpha(k)} = \frac{\mathcal{Q}_{h_0^k}(\alpha)}{\mathcal{Q}_{h_1^k}(\alpha)}, \quad \alpha \in kP\}. \tag{23}$$

Hence, for a $\mathcal{B}_k$-geodesic, (12) becomes

$$\varphi_k(t, z) = \frac{1}{k} \log Z_k(t, z) \tag{24}$$

where

$$Z_k(t, z) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \left(\frac{\mathcal{Q}_{h_0^k}(\alpha)}{\mathcal{Q}_{h_1^k}(\alpha)} \right)^t \frac{|s_\alpha(z)|_{h_0^k}^2}{\mathcal{Q}_{h_0^k}(\alpha)}. \tag{25}$$

It is interesting to observe that the relative Kähler potential (24) is the logarithm of an exponential sum, hence has the form of a free energy of a statistical mechanical problem with states parameterized by $\alpha \in kP$ and with Boltzmann weights $\left(\frac{\mathcal{Q}_{h_0^k}(\alpha)}{\mathcal{Q}_{h_1^k}(\alpha)}\right)^t$.

Thus, our goal is to prove that

$$\frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} \left(\frac{\mathcal{Q}_{h_0^k}(\alpha)}{\mathcal{Q}_{h_1^k}(\alpha)}\right)^t \frac{|s_\alpha(z)|^2_{h_0^k}}{\mathcal{Q}_{h_0^k}(\alpha)} \to \varphi_t(z) \text{ in } C^2(A \times M). \tag{26}$$

1.4. **Heuristic proof.** Let us next sketch a heuristic proof which makes the pointwise convergence obvious. The first step is to obtain good asymptotics of the norming constants (20). As in [SoZ1], they may be expressed in terms of the symplectic potential by

$$Q_{h^k}(\alpha) = \int_P e^{-k\left(u_\varphi(x) + \langle \frac{\alpha}{k} - x, \nabla u_\varphi(x)\rangle\right)} dx \tag{27}$$

As $k \to \infty$ the integral is dominated by the unique point $x = \frac{\alpha}{k}$ where the 'phase function' is maximized. The Hessian is always non-degenerate and by complex stationary phase we obtain the asymptotics

$$Q_{h^k}(\alpha_k) \sim k^{-m/2} e^{2k u_\varphi(\alpha)}.$$

The complex stationary phase (or steepest descent) method does not apply near the boundary ∂P, causing serious complications, but in this heuristic sketch we ignore this aspect.

If we then replace each term in Z_k by its asymptotics, we obtain

$$\varphi_k(t, e^{\rho/2}) \sim \frac{1}{k} \log \sum_{\alpha \in P \cap \frac{1}{k}\mathbb{Z}^m} e^{2k(u_0(\alpha) + t(u_1(\alpha) - u_0(\alpha)) + \langle \rho, \alpha \rangle)}. \tag{28}$$

The exponent $(u_0(\alpha) + t(u_1(\alpha) - u_0(\alpha)) + \langle \rho, \alpha \rangle)$ is convex and therefore has a unique minimum point. This suggests applying a discrete analogue of complex stationary phase to the sum (28), a Dedekind-Riemann sum which is asymptotic to the integral

$$\int_P e^{2k(u_0(\alpha) + t(u_1(\alpha) - u_0(\alpha)) + \langle \rho, \alpha \rangle)} d\alpha.$$

Taking $\frac{1}{k}\log$ of the integral and applying complex stationary phase gives the asymptote

$$\max_{\alpha \in P} \{u_0(\alpha) + t(u_1(\alpha) - u_0(\alpha)) + \langle \rho, \alpha \rangle\}.$$

But this is the Legendre transform of the ray of symplectic potentials

$$u_{\varphi_0}(\alpha) + t(u_{\varphi_1}(\alpha) - u_{\varphi_0}(\alpha)),$$

and thus is the Monge-Ampère geodesic.

This is the core idea of the proof. We now give the rigorous version.

1.5. **Outline of the rigorous proof.** The main difficulty in the proof of Theorem 1.1 is that the norms have very different asymptotic regimes according to the position of the normalized lattice point $\frac{\alpha}{k}$ relative to the boundary ∂P of the polytope. Even in the simplest case of $\mathbb{CP}^m$, the different positions correspond to the regimes of the central limit theorem, large deviations theorems and Poisson law of rare events for multi-nomial coefficients. In determining the asymptotics of (24), we face the difficulty that these Boltzmann weights might be exponentially growing or decaying in k as $k \to \infty$.

To simplify the comparison between the Bergman and Monge-Ampère geodesics, we take advantage of the explicit solution (17) of geodesic equation to re-write $Z_k(t,z)$ in the form

$$e^{-k\varphi_t(z)} Z_k(t,z) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \mathcal{R}_k(t,\alpha) \frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \mathcal{R}_k(t,\alpha) \mathcal{P}_{h_t^k}(\alpha, z), \tag{29}$$

where as usual $h_t = e^{-\varphi_t} h_0$ (with φ_t as in (17)), and where

$$\mathcal{R}_k(t,\alpha) := \frac{\mathcal{Q}_{h_t^k}(\alpha)}{(\mathcal{Q}_{h_0^k}(\alpha))^{1-t} (\mathcal{Q}_{h_1^k}(\alpha))^t} \tag{30}$$

One of the key ideas is that $\mathcal{R}_k(t,\alpha)$ is to at least one order a semi-classical symbol in k, i.e., has at least to some extent an asymptotic expansion in powers of k. Once this is established, it is possible to prove that

$$\frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} \mathcal{R}_k(t,\alpha) \mathcal{P}_{h_t^k}(\alpha, z) \to 0 \tag{31}$$

in the C^2-topology on $[0,1] \times M$.

The proof of Theorem 1.1 consists of four main ingredients:

- The Localization Lemma 1.2, which states that the sum over α localizes to a ball of radius $O(k^{-\frac{1}{2}+\delta})$ around the point $\mu_t(z)$. Here and hereafter, δ can be taken to be any sufficiently small positive constant.
- Bergman/Szegö asymptotics (see §4.2), which allow one to make comparisons between the sum in Z_k and sums with known asymptotics.
- The Regularity Lemma 1.3, which states that the summands $\mathcal{R}_k(t,\alpha)$ one is averaging have sufficiently smooth asymptotics as $k \to \infty$, allowing one to Taylor expand to order at least one around the point $\mu_t(z)$.
- Joint asymptotics of the Fourier coefficients (21) and particularly their special values $\mathcal{P}_{h^k}(\alpha)$ in the parameters k and distance to ∂P (see Proposition 6.1). We use a complex stationary phase method in the 'interior region' far from ∂P and local Bargmann-Fock models near ∂P.

The Localization Lemma is needed not just for $\mathcal{R}_k(t,\alpha)$ but also for summands which arise from differentiation with respect to (t,z):

Lemma **1.2.** *(Localization of Sums) Let $B_k(t,\alpha) : \mathbb{Z}^m \cap kP \to \mathbb{C}$ be a family of lattice point functions satisfying $|B_k(t,\alpha)| \leq C_0 k^M$ for some $C_0, M \geq 0$. Then, there exists $C > 0$ so* *that for any* $\delta > 0$,

$$\sum_{\alpha \in kP \cap \mathbb{Z}^m} B_k(t,\alpha) \frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \sum_{\alpha : |\frac{\alpha}{k} - \mu_t(z)| \le k^{-\frac{1}{2}+\delta}} B_k(t,\alpha) \frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} \ + \ O_\delta(k^{-C}).$$

The proof is an integration by parts argument. One could localize to the smaller scale $|\frac{\alpha}{k} - \mu_t(z)| \le C \frac{\log k}{\sqrt{k}}$ but then the argument only brings errors of the order $(\log k)^{-M}$ for all M and that complicates later applications.

The regularity Lemma concerns the behavior of the 'Fourier multiplier' $R_k(t,\alpha)$ (30). The sum (25) formally resembles the Berezin covariant symbol of a Toeplitz Fourier multiplier, i.e., the restriction to the diagonal of the Schwartz kernel of the operator; we refer to ([STZ2, Ze2]) for discussion of such Toeplitz Fourier multipliers operators on toric varieties and their Berezin symbols. However, the resemblance is a priori just formal – it is not obvious that $R_k(t,\alpha)$ has asymptotics in k. As mentioned above, the nature of the asymptotics is most difficult near ∂P; it is not obvious that smooth convergence holds along $\mathcal{D}$, the divisor at infinity.

The purpose of introducing $R_k(t,\alpha)$ is explained by the following result. First, we make the

Definition: We define the metric volume ratio to be the function on $[0,1] \times P$ defined by

$$\mathcal{R}_\infty(t,x) := \left(\frac{\det \nabla^2 u_t(x)}{(\det \nabla^2 u_0(x))^{1-t} (\det \nabla^2 u_1(x))^t} \right)^{1/2}.$$

Lemma **1.3.** *(Regularity) The volume ratio* $\mathcal{R}_\infty(t,x) \in C^\infty([0,1] \times P)$*. Further, for* $0 \le j \le 2$*,*

$$(\frac{\partial}{\partial t})^j \mathcal{R}_k(t,\alpha) = (\frac{\partial}{\partial t})^j \mathcal{R}_\infty(t, \frac{\alpha}{k}) + O(k^{-\frac{1}{3}}),$$

where the O *symbol is uniform in* (t,α).

This Lemma is the subtlest part of the analysis. If the $\mathcal{R}_k$ function were replaced by a fixed function $f(x)$ evaluated at $\frac{\alpha}{k}$ then the convergence problem reduces to generalizations of convergence of Bernstein polynomial approximations to smooth functions [Ze2], and only requires now standard Bergman kernel asymptotics. However, the actual $R_k(t,\alpha)$ do not apriori have this form, and much more is required for their analysis than asymptotics (on and off diagonal) of Bergman kernels. The analysis uses a mixture of complex stationary phase arguments in directions where $\frac{\alpha}{k}$ is 'not too close' to ∂P, while directions 'close to' ∂P we use an approximation by the 'linear' Bargmann-Fock model (see §2.6 and §6.4).

The somewhat unexpected $k^{-1/3}$ remainder estimate has its origin in this mixture of complex stationary phase and Bargmann-Fock asymptotics. Both methods are valid for k satisfying $\frac{C \log k}{k} \le \delta_k \le C' \frac{1}{\sqrt{k} \log k}$. In this region, the stationary phase remainder is of order $(k\delta_k)^{-1}$ while the Bargmann-Fock remainder is of order $k\delta_k^2$; the two remainders agree when $\delta_k = k^{-\frac{2}{3}}$, and then the remainder is $O(k^{-1/3})$. For smaller δ_k the Bargmann-Fock approximation is more accurate and for larger δ_k the stationary phase approximation is more accurate. This matter is discussed in detail in §6.4.

The rest of the proof of the C^2-convergence may be roughly outlined as follows: We calculate two logarithmic derivatives of $e^{-k\varphi_t(z)}Z_k(t,z)$ of (29) with respect to (t,ρ). Using the Localization Lemma 1.2 we can drop the terms in the resulting sums corresponding to α for which $|\frac{\alpha}{k}-\mu_t(z)|>k^{-\frac{1}{2}+\delta}$. In the remaining terms we use the Regularity Lemma 1.3 to approximate the summands by their Taylor expansions to order one around $\mu_t(z)$. This reduces the expressions to derivatives of the diagonal Szegö kernel

$$\Pi_{h_t^k}(z,z)=\sum_{\alpha\in kP\cap\mathbb{Z}^m}\frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} \tag{32}$$

for the metric h_t^k on $H^0(M,L^k)$ induced by Monge-Ampère geodesic h_t. Here, we use the smoothness of h_t. The known asymptotic expansion of this kernel (§4.2) implies the C^2-convergence of $e^{k\varphi_t(z)}Z_k(t,z)$.

As indicated in this sketch, the key problem is to analyze the joint asymptotics of norming constants $\mathcal{Q}_h^k(\alpha)$ and the dual constants $\mathcal{P}_{h^k}(\alpha)$ (22) in (k,α). Norming constants are a complete set of invariants of toric Kähler metrics. Initial results (but not joint asymptotics in the boundary regime) were obtained in [STZ1]; norms are also an important component of Donaldson's numerical analysis of canonical metrics [D4] on toric varieties. In [SoZ1] the joint asymptotics of $\mathcal{Q}_h^k(\alpha)$ were studied up to the boundary of the polytope $[0,1]$ associated to $\mathbb{CP}^1$. In this article, we emphasize the dual constants (22).

1.6. **Bergman approximation and complexification.** Having described our methods and results, we return to the discussion of their relation to Kähler quantization and to the obstacles in complexifying $Diff_{\omega_0}(M)$. Further discussion is given in [RZ2].

We may distinguish two intuitive ideas as to the nature of Monge-Ampère geodesics. The first heuristic idea, due to Semmes [S] and Donaldson [D1], is to view HCMA geodesics as one parameter subgroups of $G_{\mathbb{C}}$ where $G=SDiff_{\omega_0}(M)$. One parameter subgroups of $SDiff_{\omega_0}(M)$ are defined by Hamiltonian flows of initial Hamiltonians $\dot{\varphi}_0$ with respect to ω_0. A complexified one parameter subgroup is the analytic continuation in time of such a Hamiltonian flow [S, D1]. This idea is heuristic inasmuch as Hamiltonian flows need not possess analytic continuations in time; moreover, no genuine complexification of $SDiff_{\omega_0}(M)$ exists.

The second intuitive idea (backed up by the results of [PS1] and this article) is to view HCMA geodesics as classical limits of $\mathcal{B}_k$ geodesics. The latter have a very simple extrinsic interpretation as one parameter motions $e^{tA_k}\iota_{\underline{s}}(M)$ of a holomorphic embedding $\iota_{\underline{s}}:M\to\mathbb{CP}^{d_k}$. But the passage to the classical limit is quite non-standard from the point of view of Kähler quantization. The problem is that the approximating one parameter subgroups e^{tA_k} of operators on $H^0(M,L^k)$, which change an orthonormal basis for an initial inner product to a path of orthonormal bases for the geodesic of inner products, are not apriori complex Fourier integral operators or any known kind of quantization of classical dynamics.

The heuristic view taken in this article and series is that e^{tA_k} should be approximately the analytic continuation of the Kähler quantization of a classical Hamiltonian flow. To explain this, let us recall the basic ideas of Kähler quantization.

Traditionally, Kähler quantization refers to the quantization of a polarized Kähler manifold (M,ω,L) by Hilbert spaces $H^0(M,L^k)$ of holomorphic sections of high powers of a holomorphic line bundle $L\to M$ with Chern class $c_1(L)=[\omega]$. The Kähler form determines a Hermitian metric h such that $Ric(h) = \omega$. The Hermitian metric induces inner products $Hilb_k(h)$ on $H^0(M, L^k)$. In this quantization theory, functions H on M are quantized as Hermitian (Toeplitz) operators $\hat{H} := \Pi_{h^k} H \Pi_{h^k}$ on $H^0(M, L^k)$, and canonical transformations of (M, ω) are quantized as unitary operators on $H^0(M, L^k)$. Quantum dynamics is given by unitary groups $e^{itk\hat{H}}$ (see [BBS, BSj, Ze1] for references).

In the case of Bergman geodesics with fixed endpoints, H should be $\dot{\varphi}_0$, the initial tangent vector to the HCMA geodesic with the fixed endpoints. The quantization of the Hamiltonian flow of $\dot{\varphi}$ should then be $e^{itk\hat{H}}$ and its analytic continuation should be $e^{tk\hat{H}}$. The change of basis operator e^{tA_k} should then be approximately the same as $e^{tk\hat{H}}$. But proving this and taking the classical limit is necessarily non-standard when the classical analytic continuation of the Hamiltonian flow of $\dot{\varphi}$ does not exist. Moreover, we only know that $\dot{\varphi} \in C^{1,0}$.

This picture of the Bergman approximation to HCMA geodesics is validated in this article in the case of the Dirichlet problem on projective toric Kähler manifolds. It is verified for the initial value problem on toric Kähler manifolds in [RZ2]. In work in progress, we are investigating the same principle for general Kähler metrics on Riemann surfaces [RZ4].

1.7. **Final remarks and further results and problems.** An obvious question within the toric setting is whether $\varphi_k(t) \to \varphi_t$ in a stronger topology than C^2 on a toric variety. It seems possible that the methods of this paper could be extended to C^k-convergence. The methods of this paper easily imply C^k convergence for all k away from ∂P or equivalently the divisor at infinity, but the degree of convergence along this set has yet to be investigated. As mentioned above, we do not see why φ_k should have an asymptotic expansion in k, but this aspect may deserve further exploration. We also mention that our methods can be extended to prove C^2-convergence of Berndtsson's approximations in [B].

In subsequent articles on the toric case, we build on the methods introduced here to prove convergence theorems for other geodesics and for general harmonic maps [RZ1] (including the Wess-Zumino-Witten equation). In [SoZ2], we develop the methods of this article to prove that the geodesic rays constructed in [PS2] from test configurations are $C^{1,1}$ and no better on a toric variety. Test configuration geodesic rays are solutions of a kind of initial value problem; we refer to the articles [PS2, SoZ2] for the definitions and results. For test configuration geodesics, the analogue of $\mathcal{R}_k$ is not even smooth in t. The smooth initial value problem is studied in [RZ2]. In a different direction, one of the authors and Y. Rubinstein prove a C^2 convergence result for completely general harmonic maps of Riemannian manifolds with boundary into toric varieties (see [R, RZ1]). This includes the Wess-Zumino-Witten model where the manifold is a Riemann surface with boundary.

We believe that the techniques of this paper extend to other settings with a high degree of symmetry, such as Abelian varieties and other settings discussed in [D5]. The general Kähler case involves significant further obstacles. A basic problem in generalizing the results is to construct a useful localized basis of sections on a general (M, ω). In the toric case, we use the basis of $\mathbf{T}^m$-invariant states $\hat{s}_\alpha = z^\alpha$, which 'localize' on the so-called 'Bohr-Sommerfeld tori', i.e. the inverse images $\mu^{-1}(\frac{\alpha}{k})$ of lattice points under the moment map μ. Such Bohr-Sommerfeld states also exist on any Riemann surface; in [RZ4], we relate them to the convergence problem for HCMA geodesics on Riemann surfaces.

We briefly speculate on the higher dimensional general Kähler case. There are a number of plausible substitutes for the Bohr-Sommerfeld basis on a general Kähler manifold. A rather traditional one is to study the asymptotics of e^{A_k} on a basis of *coherent states* $\Phi^w_{h^k}$. Here, $\Phi^w_{h^k}(z) = \frac{\Pi_{h^k}(z,w)}{\sqrt{\Pi_{h^k}(w,w)}}$ are L^2 normalized Szegö kernels pinned down in the second argument. Intuitively, $\Phi^w_{h^k}$ is like a Gaussian bump centered at w with shape determined by the metric h. It is thus more localized than the monomials z^α, which are only Gaussian transverse to the tori. Under the change of basis operators e^{tA_k}, both the center and shape should change. Like the monomials z^α, coherent states have some degree of orthogonality. There are in addition other well-localized bases depending on the Kähler metric which may be used in the analysis.

Our main result (Theorem 1.1) may be viewed heuristically as showing that as $k \to \infty$ the change of basis operators e^{tA_k} tend to a path f_t of diffeomorphisms changing the initial Kähler metric ω_0 into the metric ω_t along the Monge-Ampère geodesic. We conjecture that $e^{tA_k}\Phi^w_{h^k} \sim \Phi^{f_t(w)}_{h_t^k}$, where h_t is the Monge-Ampère geodesic and f_t is the Moser path of diffeomorphisms such that $f_t^*\omega_0 = \omega_t$. We leave the exact degree of asymptotic similarity vague at this time since even the regularity of the Moser path is currently an open problem.

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2. Background on toric varieties

In this section, we review the necessary background on toric Kähler manifolds. In addition to standard material on Kähler and symplectic potentials, moment maps and polytopes, we also present some rather non-standard material on almost analytic extensions of Kähler potentials and moment maps that are needed later on. We also give a simple proof that the Legendre transform from Kähler potentials to symplectic potentials linearizes the Monge-Ampére equation.

Let M be a complex manifold. We use the following standard notation: $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. We often find it convenient to use the real operators $d = \partial + \bar\partial$, $d^c := \frac{i}{4\pi}(\bar\partial - \partial)$ and $dd^c = \frac{i}{2\pi}\partial\bar\partial$.

Let $L \to M$ be a holomorphic line bundle. The Chern form of a Hermitian metric h on L is defined by

$$c_1(h) = \omega_h := -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \|e_L\|_h^2 \,, \tag{33}$$

where e_L denotes a local holomorphic frame (= nonvanishing section) of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h-norm of e_L. We say that (L,h) is positive if the (real) 2-form ω_h is a positive $(1,1)$ form, i.e., defines a Kähler metric. We write $\|e_L(z)\|_h^2 = e^{-\varphi}$ or locally $h = e^{-\varphi}$, and then refer to φ as the Kähler potential of ω_h in U. In this notation,

$$\omega_h = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi = dd^c\varphi. \tag{34}$$

If we fix a Hermitian metric h_0 and let $h = e^{-\varphi} h_0$, and put $\omega_0 = \omega_{h_0}$, then

$$\omega_h = \omega_0 + dd^c \varphi. \tag{35}$$

The metric h induces Hermitian metrics h^k on $L^k = L \otimes \cdots \otimes L$ given by $\|s^{\otimes k}\|_{h_N} = \|s\|_h^k$.

We now specialize to toric Kähler manifolds; for background, we refer to [A, D3, G, STZ1]. A toric Kähler manifold is a Kähler manifold (M, J, ω) on which the complex torus $(\mathbb{C}^*)^m$ acts holomorphically with an open orbit M^o. Choosing a basepoint m_0 on the open orbit identifies $M^o \equiv (\mathbb{C}^*)^m$ and give the point $z = e^{\rho/2 + i\varphi} m_0$ the holomorphic coordinates

$$z = e^{\rho/2 + i\varphi} \in (\mathbb{C}^*)^m, \quad \rho, \varphi \in \mathbb{R}^m. \tag{36}$$

The real torus $\mathbf{T}^m \subset (\mathbb{C}^*)^m$ acts in a Hamiltonian fashion with respect to ω. Its moment map $\mu = \mu_\omega : M \to P \subset \mathbf{t}^* \simeq \mathbb{R}^m$ (where $\mathbf{t}$ is the Lie algebra of $\mathbf{T}^m$) with respect to ω defines a singular torus fibration over a convex lattice polytope P; as in the introduction, P is understood to be the closed polytope. We recall that the moment map of a Hamiltonian torus action with respect to a symplectic form ω is the map $\mu_\omega : M \to \mathbf{t}^*$ defined by $d\langle \mu_\omega(z), \xi \rangle = \iota_{\xi^\#} \omega$ where $\xi^\#$ is the vector field on M induced by the vector $\xi \in \mathbf{t}$. Over the open orbit one thus has a symplectic identification

$$\mu : M^o \simeq P^o \times \mathbf{T}^m.$$

We let x denote the Euclidean coordinates on P. The components $(I_1, \dots, I_m)$ of the moment map are called action variables for the torus action. The symplectically dual variables on $\mathbf{T}^m$ are called the angle variables. Given a basis of $\mathbf{t}$ or equivalently of the action variables, we denote by $\{\frac{\partial}{\partial \theta_j}\}$ the corresponding generators (Hamiltonian vector fields) of the $\mathbf{T}^m$ action. Under the complex structure J, we also obtain generators $\frac{\partial}{\partial \rho_j}$ of the $\mathbb{R}_+^m$ action.

The action variables are globally defined smooth functions but fail to be coordinates at points where the generators of the $\mathbf{T}^m$ action vanish. We denote the set of such points by $\mathcal{D}$ and refer to it as the divisor at infinity. If $p \in \mathcal{D}$ and $\mathbf{T}_p^m$ denotes the isotropy group of p, then the generating vector fields of $\mathbf{T}_p^m$ become linearly dependent at P. Since we are proving C^2 estimates, we need to replace them near points of $\mathcal{D}$ by vector fields with norms bounded below. We discuss good choices of coordinates near points of $\mathcal{D}$ below.

We assume M is smooth and that P is a Delzant polytope. It is defined by a set of linear inequalities

$$\ell_r(x) := \langle x, v_r \rangle - \lambda_r \geq 0, \ r = 1, ..., d,$$

where v_r is a primitive element of the lattice and inward-pointing normal to the r-th $(m-1)$-dimensional facet $F_r = \{\ell_r = 0\}$ of P. We recall that a facet is a highest dimensional face of a polytope. The inverse image $\mu^{-1}(\partial P)$ of the boundary of P is the divisor at infinity $\mathcal{D} \subset M$. For $x \in \partial P$ we denote by

$$\mathcal{F}(x) = \{r : \ell_r(x) = 0\}$$

the set of facets containing x. To measure when $x \in P$ is near the boundary we further define

$$\mathcal{F}_\epsilon(x) = \{r : |\ell_r(x)| < \epsilon\}. \tag{37}$$

The simplest toric varieties are linear Kähler manifolds (V, ω) carrying a linear holomorphic torus action. They provide local models near a corner of P or equivalently near a fixed point of the $\mathbf{T}^m$ action. As discussed in [GS, LT], a linear symplectic torus action is determined by a choice of m elements β_j of the weight lattice of the Lie algebra of the torus. The vector space then decomposes $(V,\omega) = \bigoplus (V_i, \omega_i)$ of orthogonal symplectic subspaces so that the moment map has the form

$$\mu_{BF}(v_1, \dots, v_m) = \sum |v_j|^2 \beta_j. \tag{38}$$

The image of the moment map is the orthant $\mathbb{R}_+^m$. This provides a useful local model at corners. We refer to these as Bargmann-Fock models; they play a fundamental role in this article (cf. §2.6).

2.1. **Slice-orbit coordinates.** We will also need local models at points near codimension r faces, and therefore supplement the coordinates (36) on the open orbit with holomorphic coordinates valid in neighborhoods of points of $\mathcal{D}$. An atlas of coordinate charts for M generalizing the usual affine charts of $\mathbb{CP}^m$ is given in [STZ1], §3.2 and we briefly recall the definitions. For each vertex $v_0 \in P$, we define the chart U_{v_0} by

$$U_{v_0} := \{z \in M_P \,;\, \chi_{v_0}(z) \neq 0\}, \tag{39}$$

where

$$\chi_\alpha(z) = z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

Throughout the article we use standard multi-index notation, and put $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Since P is Delzant, we can choose lattice points $\alpha^1, \dots, \alpha^m$ in P such that each α^j is in an edge incident to the vertex v_0, and the vectors $v^j := \alpha^j - v_0$ form a basis of $\mathbb{Z}^m$. We define

$$\eta : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m, \quad \eta(z) = \eta_j(z) := (z^{v^1}, \dots, z^{v^m}). \tag{40}$$

The map η is a $\mathbf{T}^m$-equivariant biholomorphism with inverse

$$z : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m, \quad z(\eta) = (\eta^{\Gamma e^1}, \dots, \eta^{\Gamma e^m}), \tag{41}$$

where e^j is the standard basis for $\mathbb{C}^m$, and Γ is an $m \times m$-matrix with $\det \Gamma = \pm 1$ and integer coefficients defined by

$$\Gamma v^j = e^j, \quad v^j = \alpha^j - v_0. \tag{42}$$

The corner of P at v_0 is transformed to the standard corner of the orthant $\mathbb{R}_+^m$ by the affine linear transformation

$$\tilde{\Gamma} : \mathbb{R}^m \ni u \to \Gamma u - \Gamma v_0 \in \mathbb{R}^m, \tag{43}$$

which preserves $\mathbb{Z}^m$, carries P to a polytope $Q_{v_0} \subset \{x \in \mathbb{R}^m \,;\, x_j \geq 0\}$ and carries the facets F_j incident at v_0 to the coordinate hyperplanes $= \{x \in Q_{v_0} \,;\, x_j = 0\}$. The map η extends a homeomorphism:

$$\eta : U_{v_0} \to \mathbb{C}^m, \quad \eta(z_0) = 0, \quad z_0 = \text{ the fixed point corresponding to } v_0. \tag{44}$$

By this homeomorphism, the set $\mu_P^{-1}(\bar{F}_j)$ corresponds to the set $\{\eta \in \mathbb{C}^m \,;\, \eta_j = 0\}$. If $\bar{F}$ be a closed face with $\dim F = m - r$ which contains v_0, then there are facets $F_{i_1}, \dots, F_{i_r}$ incident at v_0 such that $\bar{F} = \bar{F}_{i_1} \cap \cdots \cap \bar{F}_{i_r}$. The subvariety $\mu_P^{-1}(\bar{F})$ corresponding $\bar{F}$ is expressed by

$$\mu_P^{-1}(\bar{F}) \cap U_{v_0} = \{\eta \in \mathbb{C}^m \,;\, \eta_{i_j} = 0, \quad j = 1, \dots, r\}. \tag{45}$$

When working near a point of $\mu_P^{-1}(\bar{F})$, we simplify notation by writing

$$\eta = (\eta', \eta'') \in \mathbb{C}^m = \mathbb{C}^r \times \mathbb{C}^{m-r} \tag{46}$$

where $\eta' = (\eta_{i_j})$ as in (45) and where η'' are the remaining η_j's, so that $(0, \eta'')$ is a local coordinate of the submanifold $\mu_P^{-1}(\bar{F})$. When the point $(0, \eta'')$ lies in the open orbit of $\mu_P^{-1}(\bar{F})$, we often write $\eta'' = e^{i\theta'' + \rho''/2}$. In practice, we simplify notation by tacitly treating the corner at v_0 as if it were the standard corner of $\mathbb{R}^m_+$, omit mention of Γ and always use (z', z'') instead of η. It is straightforward to rewrite all the expressions we use in terms of the more careful coordinate charts just mentioned.

These coordinates may be described more geometrically as *slice-orbit* coordinates. Let $P_0 \in \mu_P^{-1}(\bar{F})$ and let $(\mathbb{C}^*)^m_{P_0}$ denote its stabilizer (isotropy) subgroup. Then there always exists a local slice at P_0, i.e., a local analytic subspace $S \subset M$ such that $P_0 \in S$, S is invariant under $(\mathbb{C}^*)^m_{P_0}$, and such that the natural $(\mathbb{C}^*)^m$ equivariant map of the normal bundle of the orbit $(\mathbb{C}^*)^m \cdot P_0$,

$$[\zeta, P] \in (\mathbb{C}^*)^m \times_{(\mathbb{C}^*)^m_z} S \to \zeta \cdot P \in M \tag{47}$$

is biholomorphism onto $(\mathbb{C}^*)^m \cdot S$. The terminology is taken from [Sj] (see Theorem 1.23). The slice S can be taken to be the image of a ball in the hermitian normal space $T_{P_0}((\mathbb{C}^*)^m P_0)^\perp$ to the orbit under any local holomorphic embedding $w : T_{P_0}((\mathbb{C}^*)^m P_0)^\perp \to M$ with $w(P_0) = P_0, dw_{P_0} = Id$. The affine coordinates η'' above define the slice $S = \eta^{-1}\{(z', z''(P_0)) : z' \in (\mathbb{C}^*)^r\}$. The local 'orbit-slice' coordinates are then defined by

$$P = (z', e^{i\theta'' + \rho''/2}) \iff \eta(P) = e^{i\theta'' + \rho''/2}(z', 0) \tag{48}$$

where $(z', 0) \in S$ is the point on the slice with affine holomorphic coordinates $z' = (\eta')$.

As will be seen below, toric functions are smooth functions of the variables e^{ρ_j} away from $\mathcal{D}$, and of the variables $|z_j|^2$ at points near $\mathcal{D}$. We introduce the following 'polar coordinates' centered at a point $P \in \mathcal{D}$:

$$r_j := |z_j| = e^{\rho_j/2}. \tag{49}$$

They are polar coordinates along the slice. The gradient vector field of r_j is denoted $\frac{\partial}{\partial r_j}$. As with polar vector fields, it is not well-defined at $r_j = 0$. But to prove C^ℓ estimates of functions which are smooth functions of r_j^2 it is sufficient to prove C^ℓ estimates with respect to the vector fields $\frac{\partial}{\partial r_j}$ or $\frac{\partial}{\partial (r_j^2)}$.

2.2. **Kähler potential in the open orbit and symplectic potential.** Now consider the Kähler metrics ω in $\mathcal{H}$ (cf. (1)). We recall that on any simply connected open set, a Kähler metric may be locally expressed as $\omega = 2i\partial\bar{\partial}\varphi$ where φ is a locally defined function which is unique up to the addition $\varphi \to \varphi + f(z) + \overline{f(z)}$ of the real part of a holomorphic or antiholomorphic function f. Here, $a \in \mathbb{R}$ is a real constant which depends on the choice of coordinates. Thus, a Kähler metric $\omega \in \mathcal{H}$ has a Kähler potential φ over the open orbit $M^o \subset M$. In fact, there is a canonical choice of the open-orbit Kähler potential once one fixes the image P of the moment map:

$$\varphi(z) = \log \sum_{\alpha \in P} |z^\alpha|^2 = \log \sum_{\alpha \in P} e^{\langle \alpha, \rho \rangle}. \tag{50}$$

Invariance under the real torus action implies that φ only depends on the ρ-variables, so that we may write it in the form

$$\varphi(z) = \varphi(\rho) = F(e^\rho). \tag{51}$$

The notation $\varphi(z) = \varphi(\rho)$ is an abuse of notation, but is rather standard since [D3]. For instance, the Fubini-Study Kähler potential is $\varphi(z) = \log(1 + |z|^2) = \log(1 + e^\rho) = F(e^\rho)$. Note that the Kähler potential $\log(1 + |z|^2)$ extends to $\mathbb{C}^m$ from the open orbit $(\mathbb{C}^*)^m$, although the coordinates (ρ, θ) are only valid on the open orbit. This is a typical situation.

On the open orbit, we then have

$$\omega_\varphi = \frac{i}{2} \sum_{j,k} \frac{\partial^2 \varphi(\rho)}{\partial \rho_k \partial \rho_j} \frac{dz_j}{z_j} \wedge \frac{d\bar{z}_k}{\bar{z}_k} \tag{52}$$

Positivity of ω_φ implies that $\varphi(\rho) = F(e^\rho)$ is a strictly convex function of $\rho \in \mathbb{R}^n$. The moment map with respect to ω_φ is given on the open orbit by

$$\mu_{\omega_\varphi}(z_1, \dots, z_m) = \nabla_\rho \varphi(\rho) = \nabla_\rho F(e^{\rho_1}, \dots, e^{\rho_m}), \quad (z = e^{\rho/2 + i\theta}). \tag{53}$$

Here, and henceforth, we subscript moments maps either by the Hermitian metric h or by a local Kähler potential φ. The formula (53) follows from the fact that the generators $\frac{\partial}{\partial \theta_j}$ of the $\mathbf{T}^m$ actions are Hamiltonian vector fields with respect to ω_φ with Hamiltonians $\frac{\partial \varphi(\rho)}{\partial \rho_j}$, since

$$\iota_{\frac{\partial}{\partial \theta_j}} \omega_\varphi = d \frac{\partial \varphi}{\partial \rho_j}. \tag{54}$$

The moment map is a homeomorphism from $\rho \in \mathbb{R}^m$ to the interior P^o of P and extends as a smooth map from $M \to \bar{P}$ with critical points on the divisor at infinity $\mathcal{D}$. Hence, the Hamiltonians (54) extend to $\mathcal{D}$.

Note that the local Kähler potential on the open orbit is not the same as the global smooth *relative Kähler potential* in (1) with respect to a background Kähler metric ω_0. That is, given a reference metric ω_0 with Kähler potential φ_0, it follows by the $\partial\bar{\partial}$ lemma that $\omega = \omega_0 + dd^c \varphi$ with $\varphi \in C^\infty(M)$. As discussed in [D3] (see Proposition 3.1.7), the Kähler potential φ on the open orbit defines a singular potential on M which satisfies $dd^c \varphi = \omega + H$ where H is a fixed current supported on $\mathcal{D}$. We generally denote Kähler potentials by φ and in each context explain which type we mean.

By (52), a $\mathbf{T}^m$-invariant Kähler potential defines a real convex function on $\rho \in \mathbb{R}^m$. Its Legendre dual is the *symplectic potential* u_φ: for $x \in P$ there is a unique ρ such that $\mu_\varphi(e^{\rho/2}) = \nabla_\rho \varphi = x$. Then the Legendre transform is defined to be the convex function

$$u_\varphi(x) = \langle x, \rho_x \rangle - \varphi(\rho_x), \quad e^{\rho_x/2} = \mu_\varphi^{-1}(x) \iff \rho_x = 2 \log \mu_\varphi^{-1}(x) \tag{55}$$

on P. The gradient $\nabla_x u_\varphi$ is an inverse to μ_{ω_φ} on $M_\mathbb{R}$ on the open orbit, or equivalently on P, in the sense that $\nabla u_\varphi(\mu_{\omega_\varphi}(z)) = z$ as long as $\mu_{\omega_\varphi}(z) \notin \partial P$.

The symplectic potential has canonical logarithmic singularities on ∂P. According to [A] (Proposition 2.8) or [D3] (Proposition 3.1.7), there is a one-to-one correspondence between $\mathbf{T}^m_\mathbb{R}$-invariant Kähler potentials ψ on M_P and symplectic potentials u in the class S of continuous convex functions on $\bar{P}$ such that $u - u_0$ is smooth on $\bar{P}$ where

$$u_0(x) = \sum_k \ell_k(x) \log \ell_k(x). \tag{56}$$

Thus, $u_\varphi(x) = u_0(x) + f_\varphi(x)$ where $f_\varphi \in C^\infty(\bar{P})$. We note that u_0 and u_φ are convex, that $u_0 = 0$ on ∂P and hence $u_\varphi = f_\varphi$ on ∂P. By convexity, $\max_P u_0 = 0$.

We denote by $G_\varphi = \nabla_x^2 u_\varphi$ the Hessian of the symplectic potential. It has simple poles on ∂P. It follows that $\nabla_\rho^2 \varphi$ has a kernel along $\mathcal{D}$. The kernel of $G_\varphi^{-1}(x)$ on $T_x \partial P$ is the linear span of the normals μ_r for $r \in \mathcal{F}(x)$. We also denote by $H_\varphi(\rho) = \nabla_\rho^2 \varphi(e^\rho)$ the Hessian of the Kähler potential on the open orbit in ρ coordinates. By Legendre duality,

$$H_\varphi(\rho) = G_\varphi^{-1}(x), \ \ \mu(e^\rho) = x. \tag{57}$$

This relation may be extended to $\mathcal{D} \to \partial P$. The kernel of the left side is the Lie algebra of the isotropy group G_p of any point $p \in \mu^{-1}(x)$. The volume density has the form

$$\det(G_\varphi^{-1}) = \delta_\varphi(x) \cdot \prod_{r=1}^d \ell_r(x), \tag{58}$$

for some positive smooth function δ_φ [A]. We note that $\log \prod_{r=1}^d \ell_r(x)$ is known in convex optimization as the logarithmic barrier function of P.

2.3. **Kähler potential near $\mathcal{D}$.** We also need smooth local Kähler potentials in neighborhoods of points $z_0 \in \mathcal{D}$. We note that the open orbit Kähler potential (50) is well-defined near $z = 0$. Local expressions for the Kähler potential at other points of $\mathcal{D}$ essentially amount to making an affine transformation of P to transform a given corner of P to 0, and in these coordinates the local Kähler potential near any point of $\mathcal{D}$ can be expressed in the form (50). For instance, on $\mathbb{CP}^1$, a Kähler potential valid at $z = \infty$ is given in the coordinates $w = \frac{1}{z}$ by $\log(1 + |w|^2)$. It differs on the open orbit from the canonical Kähler potential $\log(1 + |z|^2)$ by the term $\log |z|^2$ whose $i\partial\bar\partial$ is a delta function at $z = 0$, supported on $\mathcal{D}$ away from the point $w = 0$ that one is studying. In [So] the reader can find further explicit examples of toric Kähler potentials in affine coordinate charts. Hence, in what follows, we will always use (50) as the local expression of the Kähler potential, without explicitly writing in the affine change of variables.

We will however need to be explicit about the use of slice-orbit coordinates z'_j, ρ''_j (48) in the local expressions of the Kähler potential. The coordinates near z_0 depend on $\mathcal{F}_\epsilon(z_0)$ from (37). For each $z_0 \in \mathcal{D}$ corresponding to a codimension r face of P, after an affine transformation changing the face to $x' = 0$, we may write the Kähler potential as the canonical one in slice-orbit coordinates, $F(|z'|^2, e^{\rho''})$ §2.1 (48). Since $0 \in P$, F is smooth up to the boundary face $z' = 0$. The fact that F is smooth up to the boundary also follows from the general fact that a smooth $\mathbf{T}^m$-invariant function $g \in C^\infty_{\mathbf{T}^m}(M)$ may be expressed in the form $g(z) = \hat F_g(\mu_\varphi(z))$ where as $\hat F_g \in C^\infty(\mathbb{R}^m)$. This is known as the divisibility property of $\mathbf{T}^m$-invariant smooth functions (cf. [LT]). It implies that F is a smooth function of the polar coordinates r_j^2 near points of $\mathcal{D}$ in the sense of (49).

2.4. **Almost analytic extensions.** In analyzing the Bergman/Szegö kernel and the functions (21), we make use of the *almost analytic extension* $\varphi(z, w)$ to $M \times M$ of a Kähler potential for a Kähler ω; for background on almost analytic extensions, see [BSj, MSj]. It is defined near the totally real anti-diagonal $(z, \bar z) \in M \times M$ by

$$\varphi_{\mathbb{C}}(x + h, x + k) \sim \sum_{\alpha,\beta} \frac{\partial^{\alpha+\beta}\varphi}{\partial z^\alpha \partial \bar z^\beta}(x) \frac{h^\alpha}{\alpha!} \frac{k^\beta}{\beta!}. \tag{59}$$

When φ is real analytic on M, the almost analytic extension $\varphi(z,w)$ is holomorphic in z and anti-holomorphic in w and is the unique such function for which $\varphi(z) = \varphi(z,z)$. In the general C^∞ case, the almost analytic extension is a smooth function with the right side of (59) as its C^∞ Taylor expansion along the anti-diagonal, for which $\bar{\partial}\varphi(z,w) = 0$ to infinite order on the anti-diagonal. It is only defined in a small neighborhood $(M \times M)_\delta = \{(z,w) : d(z,w) < \delta\}$ of the anti-diagonal in $M \times M$, where $d(z,w)$ refers to the distance between z and w with respect to the Kähler metric ω. It is well defined up to a smooth function vanishing to infinite order on the diagonal; the latter is negligible for our purposes (cf. Proposition 1.1 of [BSj].)

The analytic continuation $\varphi(z,w)$ of the Kähler potential was used by Calabi [Ca] in the analytic case to define a Kähler distance function, known as the 'Calabi diastasis function'

$$D(z,w) := \varphi(z,w) + \varphi(w,z) - (\varphi(z) + \varphi(w)). \tag{60}$$

Calabi showed that

$$D(z,w) = d(z,w)^2 + O(d(z,w)^4), \;\; dd^c_w D(z,w)|_{z=w} = \omega. \tag{61}$$

One has the same notion in the almost analytic sense.

The gradient of the almost analytic extension of the Kähler potential in the toric case defines the almost analytic extension $\mu_{\mathbb{C}}(z,w)$ of the moment map. We are mainly interested in the case where $w = e^{i\theta}z$ lies on the $\mathbf{T}^m$-orbit of z, and by (53) we have,

$$i\mu_{\mathbb{C}}(z, e^{i\theta}z) = \nabla_\theta \varphi_{\mathbb{C}}(z, e^{i\theta}z) = \nabla_\theta F_{\mathbb{C}}(e^{i\theta}|z|^2), \tag{62}$$

where F is defined in (51). We sometimes drop the subscript in $F_{\mathbb{C}}$ and $\mu_{\mathbb{C}}$ since there is only one interpretation of their extension; but we emphasize that $\varphi(z, e^{i\theta}z) = F_{\mathbb{C}}(e^{i\theta}|z|^2)$ is very different from $\varphi(e^{i\theta}z) = F(|e^{i\theta}z|^2) = F(|z|^2)$. For example, the moment map of the Bargman-Fock model $(\mathbb{C}^m, |z|^2)$ is $\mu(z) = (|z_1|^2, \ldots, |z_m|^2)$, whose analytic extension is $(z_1\bar{w}_1, \ldots, z_m\bar{w}_m)$. Similarly that of the Fubini-Study metric on $\mathbb{CP}^m$ is (in multi-index notation) $\mu_{FS,\mathbb{C}}(z,w) = \frac{z\cdot\bar{w}}{1+z\cdot\bar{w}}$. In §2.6 we further illustrate the notation in the basic examples of Bargmann-Fock and Fubini-Study models. We also observe that (62) continues to hold for the Kähler potential $F(|z'|^2, e^{\rho''})$ in slice-orbit coordinates. That is we have,

$$i\;\mu(z', e^{\rho''/2}) = \nabla_{\theta',\theta''} F_{\mathbb{C}}(e^{i\theta'}|z'|^2, e^{i\theta''+\rho''})|_{(\theta',\theta'')=(0,0)}. \tag{63}$$

The complexified moment map is a map

$$\mu_{\mathbb{C}} \to (M \times M)_\delta \to \mathbb{C}^m. \tag{64}$$

The invariance of μ under the torus action implies that $\mu_{\mathbb{C}}(e^{i\theta}z, e^{i\theta}w) = \mu_{\mathbb{C}}(z,w)$. The following Proposition will clarify the discussion of critical point sets later on (see e.g. Lemma 5.2).

Proposition 2.1. *For δ sufficiently small so that $\mu_{\mathbb{C}}(z,w)$ is well-defined,we have*

1. $\Im \mu_{\mathbb{C}}(z, e^{i\theta}z) = \frac{1}{2}\nabla_\theta D(z, e^{i\theta}z)$.
2. $\mu_{\mathbb{C}}(z, e^{i\theta}z) = \mu_{\mathbb{C}}(z,z)$ *with* $(z, e^{i\theta}z) \in (M \times M)_\delta$ *if and only if* $e^{i\theta}z = z$.

Proof. The proof of the identity (1) is immediate from the definitions; we only note that the diastasis function is a kind of real part, and that the imaginary part originates in the factor of i in (62). One can check the factors of i in the Bargmann-Fock model, where $\mu_{\mathbb{C}}(z, e^{i\theta}z) = e^{i\theta}|z|^2$ while $D(z, e^{i\theta}z) = 2(\cos\theta - 1)|z|^2 + 2i(\sin\theta)|z|^2$ (in vector notation) .

By (61), $D(z,w)$ has a strict global minimum at $w=z$ which is non-degenerate. It is therefore isolated for each z. Since its Hessian at $w=z$ is the identify with respect to ω, the isolating neighborhood has a uniform size as z varies. Thus, there exists a $\delta>0$ so that $\mu_{\mathbb{C}}(z,w)=\mu_{\mathbb{C}}(z,z)$ in $(M\times M)_\delta$ if and only if $z=w$. This is true both in the real analytic case and the almost-analytic case.

□

2.5. **Hilbert spaces of holomorphic sections.** On the 'quantum level', a toric Kähler variety (M,ω) induces the sequence of spaces $H^0(M,L^k)$ of holomorphic sections of powers of the holomorphic toric line bundle L with $c_1(L)=\frac{1}{2\pi}[\omega]$. The $(\mathbb{C}^*)^m$ action lifts to $H^0(M,L^k)$ as a holomorphic representation which is unitary on $\mathbf{T}^m$. Corresponding to the lattice points $\alpha\in kP$, there is a natural basis $\{s_\alpha\}$ (denoted χ^P_α in [STZ1]) of $H^0(M,L^k)$ given by joint eigenfunctions of the $(\mathbb{C}^*)^m$ action. It is well-known that the joint eigenvalues are precisely the lattice points $\mathbb{Z}^m\cap kP$ in the kth dilate of P. On the open orbit $s_\alpha(z)=\chi_\alpha(z)e^k$ where e is a frame and where as above $\chi_\alpha(z)=z^\alpha=z_1^{\alpha_1}\cdots z_m^{\alpha_m}$. Hence, the s_α are referred to as monomials. For further background, we refer to [STZ1]. A hermitian metric h on L induces Hilbert space inner products (6) on $H^0(M,L^k)$.

As is evident from (21), we will need formulae for the monomials which are valid near $\mathcal{D}$. By (40) and (42), we have

$$\chi_{\alpha^j}(z)=\eta_j(z)\chi_{v^0}(z),\quad z\in(\mathbb{C}^*)^m, \tag{65}$$

and by (43) we then have

$$|\chi_\alpha(z)|^2=|\eta^{\tilde{\Gamma}(\alpha)}|^2. \tag{66}$$

As mentioned above, for simplicity of notation we suppress the transformation $\tilde{\Gamma}$ and coordinates η, and we will use the 'orbit-slice' coordinates of (48). Thus, we denote the monomials cooresponding to lattice points α near a face F by $(z')^{\alpha'}e^{\langle(i\theta''+\rho''/2),\alpha''\rangle}$, where $\tilde{\Gamma}(\alpha)=(\alpha',\alpha'')$ with α'' in the coordinate hyperplane corresponding under $\tilde{\Gamma}$ to F and with α' in the normal space.

2.6. **Examples: Bargmann-Fock and Fubini-Study models .** As mentioned above the Bargmann-Fock model is the linear model. It plays a fundamental role in this article because it provides an approximation for objects on any toric variety on balls of radius $\frac{\log k}{\sqrt{k}}$ and also near $\mathcal{D}$. Although it and the Fubini-Study model are elementary examples, we go over them because the notation is used frequently later on.

The Bargmann-Fock models on $\mathbb{C}^m$ correspond to choices of a positive definite Hermitian matrix H on $\mathbb{C}^m$. A toric Bargmann-Fock model is one in which H commutes with the standard $\mathbf{T}^m$ action, i.e., is a diagonal matrix. We denote its diagonal elements by $H_{j\bar{j}}$. The Kähler metric on $\mathbb{C}^m$ is thus $i\partial\bar\partial\varphi_{BF,H}(z)$ where the global Kähler potential is

$$\varphi_{BF,H}(z)=\sum_{j=1}^m H_{j\bar{j}}|z_j|^2=F(|z_1|^2,\ldots,|z_m|^2),\ \text{ with }\ F(y_1,\ldots,y_m)=\sum_j H_{j\bar{j}}y_j.$$

For simplicity we often only consider the case $H=I$. Putting $|z_j|^2=e^{\rho_j}$ and using (53), it follows that $\mu_{BF,H}(z_1,\ldots,z_m)=(H_{1\bar{1}}|z_1|^2,\ldots,H_{m\bar{m}}|z_m|^2):\mathbb{C}^m\to\mathbb{R}^m_+$ as in (38). The symplectic potential Legendre dual to $\varphi_{BF,H}$ is given by

$$u_{BF,H}(x) = -\varphi_{BF,H}(\mu_{BF}^{-1}(x)) + 2\langle \log \mu_{BF,H}^{-1}(x), x\rangle = -\sum_j x_j + \sum_{j=1}^m x_j \log(\frac{x_j}{H_{j\bar{j}}}). \tag{67}$$

In this case, $G_{BF,H}$ is the diagonal matrix with entries $\frac{1}{x_j H_{j\bar{j}}}$, so $\det G_{BF,H} = \frac{1}{\det H}\Pi_j \frac{1}{x_j}$.

The off-diagonal analytic extension of the Kähler potential in the sense of (59) is then

$$\varphi_{BF,H}(z,\bar{w}) = \sum_{j=1}^m H_{j\bar{j}} z_j \bar{w}_j = F(z_1\bar{w}_1, \dots, z_m \bar{w}_m)$$

and in particular,

$$\varphi_{BF,H}(z, e^{i\theta} z) = \sum_{j=1}^m H_{j\bar{j}} e^{i\theta_j} |z_j|^2 = F(e^{i\theta_1}|z_1|^2, \dots, e^{i\theta}|z_m|^2).$$

Henceforth we often write the the right side in the multi-index notation $F_{\mathbb{C}}(e^{i\theta}|z|^2)$. We observe, as claimed in (62), that $\nabla_\theta F_{BF,\mathbb{C}}(e^{i\theta}|z|^2)|_{\theta=0} = i\mu_{BF}(z)$.

Quantization of the Bargmann-Fock model with $H = I$ produces the Bargmann-Fock (Hilbert) space

$$\mathcal{H}^2(\mathbb{C}^m, (2\pi)^{-m} k^m e^{-k|z|^2} dz \wedge d\bar{z})$$

of entire functions which are L^2 relative to the weight $e^{-k|z|^2/2}$. It is infinite dimensional and a basis is given by the monomials z^α where $\alpha \in \mathbb{R}_+^m \cap \mathbb{Z}^m$. In §3.0.1 we compute their L^2 norms. For $H \neq I$ one uses the volume form $e^{-k\langle Hz,z\rangle}(i\partial\bar{\partial}\langle Hz, z\rangle)^m/m! = e^{-k\langle Hz,z\rangle}(\det H) dz \wedge d\bar{z}$.

Toric Fubini-Study metrics provide compact models which are similar to Bargmann-Fock models. In a local analysis we always use the latter. A Fubini-Study metric on $\mathbb{CP}^m$ is determined by a positive Hermitian form H on $\mathbb{C}^{m+1}$ and a toric Fubini-Study metric is a diagonal one $\sum_{j=0}^m H_{j\bar{j}}|Z_j|^2$. In the affine chart $Z_0 \neq 0$ (e.g.) a local Fubini-Study Kähler potential is $\varphi_{FS,H}(z_1, \dots, z_m) = \log(1 + \sum_j h_{j\bar{j}}|z_j|^2)$ where $h_{j\bar{j}} = \frac{H_{j\bar{j}}}{H_{0\bar{0}}}$. This is a valid Kähler potential near $z = 0$ but of course has logarithmic singularities on the hyperplane at infinity. The almost analytic extension of the Fubini-Study Kähler potential is given in the affine chart by $\log(1 + \sum_j h_{j\bar{j}} z_j \bar{w}_j)$. Thus (62) asserts that

$$i\, \frac{\sum_j h_{j\bar{j}}|z_j|^2}{1 + \sum_j h_{j\bar{j}}|z_j|^2} = \nabla_\theta \log(1 + \sum_j h_{j\bar{j}} e^{i\theta_j}|z_j|^2)|_{\theta=0}.$$

Quantization produces the Hilbert spaces $H^0(\mathbb{CP}^m, \mathcal{O}(k))$, where $\mathcal{O}(k) \to \mathbb{CP}^m$ is the kth power of the hyperplane section bundle. Sections lift to homogeneous holomorphic polynomials on $\mathbb{C}^{m+1}$, and correspond to lattice points in $k\Sigma$ where Σ is the unit simplex in $\mathbb{R}^m$.

2.7. **Linearization of the Monge-Ampère equation.** It is known that the Legendre transform linearizes the Monge-Ampère geodesic equation. Since it is important for this article, we present a simple proof that does not seem to exist in the literature.

PROPOSITION **2.2.** *Let M_P^c be a toric variety. Then under the Legendre transform $\varphi \to u_\varphi$, the complex Monge-Ampére equation on $\mathcal{H}_{\mathbf{T}^m}$ linearizes to the equation $u'' = 0$. Hence the Legendre transform of a geodesic φ_t has the form $u_t = u_0 + t(u_1 - u_0)$.*

Proof. It suffices to show that the energy functional

$$E = \int_0^1 \int_M \dot{\varphi}_t^2 d\mu_{\varphi_t} dt \tag{68}$$

is Euclidean on paths of symplectic potentials. For each t let us pushforward the integral $\int_M \dot{\varphi}_t^2 d\mu_\varphi$ under the moment map μ_{φ_t}. The integrand is by assumption invariant under the real torus action, so the pushforward is a diffeomorphism on the real points. The volume measure $d\mu_{\varphi_t}$ pushes forward to dx. The function $\partial_t \varphi_t(\rho)$ pushes forward to the function $\psi_t(x) = \dot{\varphi}_t(\rho_{x,t})$ where $\mu_{\varphi_t}(\rho_{x,t}) = x$. By (55), the symplectic potential at time t is

$$u_t(x) = \langle x, \rho_{x,t} \rangle - \varphi_t(\rho_{x,t}).$$

We note that

$$\dot{u}_t = \langle x, \partial_t \rho_{x,t} \rangle - \dot{\varphi}_t(\rho_{x,t}) - \langle \nabla_\rho \varphi_t(\rho_{x,t}), \partial_t \rho_{x,t} \rangle. \tag{69}$$

The outer terms cancel, and thus, our integral is just

$$\int_0^1 \int_P |\dot{u}_t|^2 dx.$$

Clearly the Euler-Lagrange equations are linear.

□

3. The functions $\mathcal{P}_{h^k}$ and $\mathcal{Q}_{h^k}$

We now introduce the key players in the analysis, the norming constants $\mathcal{Q}_{h^k}(\alpha)$ (20) and the dual constants $\mathcal{P}_{h^k}(\alpha)$ of (22). The duality is given in the following:

Proposition **3.1.** *We have:*

$$Q_{h_k}(\alpha) = \frac{e^{k u_\varphi(\frac{\alpha}{k})}}{\mathcal{P}_{h^k}(\alpha)},$$

Proof. By (55), it follows that

$$||s_\alpha(\mu_h^{-1}(\frac{\alpha}{k}))||_{h^k}^2 = |\chi_\alpha(\mu_h^{-1}(\frac{\alpha}{k}))|^2 e^{-k\varphi_h(\mu_h^{-1}(\frac{\alpha}{k}))} = e^{k u_{\varphi_h}(\frac{\alpha}{k})}. \tag{70}$$

□

Corollary **3.2.**

$$\mathcal{R}_k(t,\alpha) = \frac{\Big(\mathcal{P}_{h_0^k}(\alpha)\Big)^{1-t} \Big(\mathcal{P}_{h_1^k}(\alpha)\Big)^t}{\mathcal{P}_{h_t^k}(\alpha)}$$

Proof. We need to show that

$$\frac{\mathcal{Q}_{h_t^k}(\alpha)}{(\mathcal{Q}_{h_0^k}(\alpha))^{1-t}(\mathcal{Q}_{h_1^k}(\alpha))^t} = \frac{\Big(\mathcal{P}_{h_0^k}(\alpha)\Big)^{1-t}\Big(\mathcal{P}_{h_1^k}(\alpha)\Big)^t}{\mathcal{P}_{h_t^k}(\alpha)}. \tag{71}$$

By Proposition 3.1, the left side of (71) equals

$$\frac{|\chi_\alpha(\mu_t^{-1}(\frac{\alpha}{k}))|^2 e^{-k\varphi_t(\mu_t^{-1}(\frac{\alpha}{k}))}}{\mathcal{P}_{h_t^k}(\alpha)} \times \left(\frac{\mathcal{P}_{h_0^k}(\alpha)}{|\chi_\alpha(\mu_0^{-1}(\frac{\alpha}{k}))|^2 e^{-k\varphi_0(\mu_0^{-1}(\frac{\alpha}{k}))}}\right)^{1-t} \times \left(\frac{\mathcal{P}_{h_1^k}(\alpha)}{|\chi_\alpha(\mu_1^{-1}(\frac{\alpha}{k}))|^2 e^{-k\varphi_1(\mu_1^{-1}(\frac{\alpha}{k}))}}\right)^t$$

By (70), the left side of (71) equals

$$= e^{k\left(u_t(\frac{\alpha}{k})+(1-t)u_0(\frac{\alpha}{k})+tu_1(\frac{\alpha}{k})\right)} \times \frac{\left(\mathcal{P}_{h_0^k}(\alpha)\right)^{1-t}\left(\mathcal{P}_{h_1^k}(\alpha)\right)^{t}}{\mathcal{P}_{h_t^k}(\alpha)}.$$

But $u_t(x) + (1-t)u_0(x) + tu_1(x) = 0$ on a toric variety, and this gives the stated equality.
□

Further, we relate the full $\mathcal{P}_{h^k}(\alpha, z)$ to the Szegö kernel. The Szegö (or Bergman) kernels of a positive Hermitian line bundle $(L, h) \to (M, \omega)$ over a Kähler manifold are the kernels of the orthogonal projections $\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k)$ onto the spaces of holomorphic sections with respect to the inner product $Hilb_k(h)$ (6). Thus, we have

$$\Pi_{h^k} s(z) = \int_M \Pi_{h^k}(z, w) \cdot s(w) \frac{\omega_h^m}{m!}, \tag{72}$$

where the $\cdot$ denotes the h-hermitian inner product at w. Let e_L be a local holomorphic frame for $L \to M$ over an open set $U \subset M$ of full measure, and let $\{s_j^k = f_j e_L^{\otimes k} : j = 1, \dots, d_k\}$ be an orthonormal basis for $H^0(M, L^k)$ with $d_k = \dim H^0(M, L^k)$. Then the Szegö kernel can be written in the form

$$\Pi_{h^k}(z, w) := F_{h^k}(z, w)\, e_L^{\otimes k}(z) \otimes \overline{e_L^{\otimes k}(w)}\,, \tag{73}$$

where

$$F_{h^k}(z, w) = \sum_{j=1}^{d_k} f_j(z) \overline{f_j(w)}\ . \tag{74}$$

Since the Szegö kernel is a section of the bundle $(L^k) \otimes (L^k)^* \to M \times M$, it often simplifies the analysis to lift it to a scalar kernel $\hat{\Pi}_{h^k}(x, y)$ on the associated unit circle bundle $X \to M$ of (L, h). Here, $X = \partial D_h^*$ is the boundary of the unit disc bundle with respect to h^{-1} in the dual line bundle L^*. We use local product coordinates $x = (z, t) \in M \times S^1$ on X where $x = e^{it}\|e_L(z)\|_h e_L^*(z) \in X$. To avoid confusing the S^1 action on X with the $\mathbf{T}^m$ action on M we use e^{it} for the former and $e^{i\theta}$ (multi-index notation) for the latter. We note that the $\mathbf{T}^m$ action lifts to X and combines with the S^1 action to produce a $(S^1)^{m+1}$ action. We refer to [Ze1, SZ, Ze2] for background and for more on lifting the Szegö kernel of a toric variety.

The equivariant lift of a section $s = f e_L^{\otimes k} \in H^0(M, L^k)$ is given explicitly by

$$\hat{s}(z, t) = e^{ikt}\|e_L^{\otimes k}\|_{h^k} f(z) = e^{k\left[-\frac{1}{2}\varphi(z)+it\right]} f(z)\ . \tag{75}$$

The Szegö kernel thus lifts to $X \times X$ as the scalar kernel

$$\hat{\Pi}_k(z, t; w, t') = e^{k\left[-\frac{1}{2}\varphi(z)-\frac{1}{2}\varphi(w)+i(t-t')\right]} F_k(z, w)\ . \tag{76}$$

Since it is S^1- equivariant we often put $t = t' = 0$.

Proposition **3.3.** *We have*

$$\mathcal{P}_{h^k}(\alpha, z) = (2\pi)^{-m} \int_{\mathbf{T}^m} \hat{\Pi}_{h^k}(e^{i\theta} z, 0; z, 0) e^{-i\langle\alpha,\theta\rangle} d\theta.$$

Proof. We recall that $\chi_\alpha(z) = z^\alpha$ is the local representative of s_α in the open orbit with respect to an invariant frame. Since $\{\frac{\chi_\alpha}{\sqrt{\mathcal{Q}_{h^k}(\alpha)}}\}$ is the local expression of an orthonormal basis, we have

$$F_{h^k}(z,w) = \sum_{\alpha \in kP\cap\mathbb{Z}^m} \frac{\chi_\alpha(z)\overline{\chi_\alpha(w)}}{\mathcal{Q}_{h^k}(\alpha)}$$

hence

$$\hat{\Pi}_{h^k}(z,0;w,0) = \sum_{\alpha \in kP\cap\mathbb{Z}^m} \frac{\chi_\alpha(z)\overline{\chi_\alpha(w)}e^{-k(\varphi(z)+\varphi(w))/2}}{\mathcal{Q}_{h^k}(\alpha)}.$$

It follows that

$$\Pi_{h^k}(e^{i\theta}z,0;z,0) = \sum_{\alpha \in kP\cap\mathbb{Z}^m} \frac{|\chi_\alpha(z)|^2 e^{-k\varphi(z)}e^{i\langle\alpha,\theta\rangle}}{\mathcal{Q}_{h^k}(\alpha)}.$$

Integrating against $e^{-i\langle\alpha,\theta\rangle}$ sifts out the α term.

□

Corollary **3.4.** *We have*

$$\mathcal{P}_{h^k}(\alpha) = (2\pi)^{-m}\int_{\mathbf{T}^m} \hat{\Pi}_{h^k}(e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k}),0;\mu_h^{-1}(\frac{\alpha}{k}),0)e^{-i\langle\alpha,\theta\rangle}d\theta. \tag{77}$$

3.0.1. *Bargmann-Fock model.* As discussed in §2.6, the Hilbert space in this model has the orthogonal basis z^α with $\alpha \in \mathbb{R}^m_+ \cap \mathbb{Z}^m$. The Bargmann-Fock norming constants when $H = I$ are given by

$$Q_{h^k_{BF}}(\alpha) = k^{-|\alpha|-m}\alpha!, \quad (\alpha! := \alpha_1!\cdots\alpha_m!)$$

It follows that an orthonormal basis of holomorphic monomials is given by $\{k^{\frac{|\alpha|+m}{2}}\frac{z^\alpha}{\sqrt{\alpha!}}\}$.

We therefore have

$$\frac{|s_\alpha(z)|^2_{h^k_{BF}}}{\mathcal{Q}_{h^k_{BF}}(\alpha)} = k^{|\alpha|+m}\frac{|z^\alpha|^2}{\alpha!}e^{-k|z|^2}, \tag{78}$$

and in particular,

$$\mathcal{P}_{h^k_{BF}}(\alpha) = k^m e^{-|\alpha|}\frac{\alpha^\alpha}{\alpha!}, \tag{79}$$

where $\alpha^\alpha = 1$ when $\alpha = 0$. Here, we use that $u_{BF}(\frac{\alpha}{k}) = \frac{\alpha}{k}\log\frac{\alpha}{k} - \frac{\alpha}{k}$, so that $e^{ku_{BF}(\frac{\alpha}{k})} = e^{-|\alpha|}\frac{k^{-|\alpha|}}{\alpha^\alpha}$ and that $\mathcal{Q}_{h^k_{BF}}(\alpha) = k^{-m-|\alpha|}\alpha!$. We observe that $\mathcal{P}_{h^k_{BF}}(\alpha)$ depends on k only through the factor k^m.

Precisely the same formula holds if we replace I by a positive diagonal H with elements $H_{j\bar{j}}$. By a change of variables, $\mathcal{Q}_{h^k_{BF,H}}(\alpha) = \Pi_{j=1}^m H_{j\bar{j}}^{-\alpha_j}\mathcal{Q}_{h^k_{BF}}(\alpha)$, and also by (67) $u_{BF,H}(x) = u_{BF}(x) + \sum_j x_j \log H_{j\bar{j}}$. Hence, by Proposition 3.1,

$$\mathcal{P}_{h^k_{BF,H}}(\alpha) = \mathcal{P}_{h^k_{BF}}(\alpha)\Pi_{j=1}^m H_{j\bar{j}}^{-\alpha_j}e^{\sum_j \alpha_j \log H_{j\bar{j}}} = \mathcal{P}_{h^k_{BF}}(\alpha).$$

3.0.2. $\mathbb{CP}^m$. In the Fubini-Study model, a basis of $H^0(\mathbb{CP}^m, \mathcal{O}(k))$ is given by monomials with $\alpha \in k\Sigma$ (see §2.6), and the norming constants are given by

$$\mathcal{Q}_{h_{FS}^k}(\alpha) = \binom{k}{\alpha} := \binom{k}{\alpha_1, \dots, \alpha_m}^{-1}. \tag{80}$$

Recall that multinomial coefficients are defined for $\alpha_1 + \cdots + \alpha_m \leq k$ by

$$\binom{k}{\alpha_1, \dots, \alpha_m} = \frac{k!}{\alpha_1! \cdots \alpha_m!(k - |\alpha|)!},$$

where as above, $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

We further have $|s_\alpha(z)|^2_{h_{FS}^k} = |z^\alpha|^2 e^{-k \log(1+|z|^2)}$ and therefore,

$$\mathcal{P}_{h_{FS}^k}(\alpha, z) = \tbinom{k}{\alpha_1, \dots, \alpha_m} |z^\alpha|^2 e^{-k \log(1+|z|^2)},$$

and since

$$e^{-ku_{FS}(\frac{\alpha}{k})} = |s_\alpha(\mu_{FS}^{-1}(\frac{\alpha}{k}))|^2_{h_{FS}^k} = (\frac{\alpha}{k})^\alpha (1 - \frac{|\alpha|}{k})^{k-|\alpha|}$$

we have

$$\mathcal{P}_{h_{FS}^k}(\alpha) = \frac{k!}{\alpha_1! \cdots \alpha_m!(k - |\alpha|)!} (\frac{\alpha}{k})^\alpha (1 - \frac{|\alpha|}{k})^{k-|\alpha|}.$$

4. Szegö kernel of a toric variety

We will use Proposition 3.3 to reduce the joint asymptotics of $\mathcal{P}_{h^k}\alpha, z)$ in (k, α) to asymptotics of the Bergman-Szegö kernel off the diagonal. We now review some general facts about diagonal and off-diagonal expansions of these kernels, for which complete details can be found in [SZ], and we also consider some special properties of toric Bergman-Szegö kernels which are very convenient for calculations; to some extent they derive from [STZ1], but the latter only considered Szegö kernels for powers of Bergman metrics.

The Szegö kernels $\hat{\Pi}_{h^k}(x, y)$ are the Fourier coefficients of the total Szegö projector $\hat{\Pi}_h(x, y) : \mathcal{L}^2(X) \to \mathcal{H}^2(X)$, where $\mathcal{H}^2(X)$ is the Hardy space of boundary values of holomorphic functions on D^* (the kernel of $\bar{\partial}_b$ in $L^2(X)$). Thus,

$$\hat{\Pi}_{h^k}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \hat{\Pi}_h(e^{it}x, y)\, dt.$$

The properties we need of $\hat{\Pi}_{h^k}(x, y)$ are based on the Boutet de Monvel-Sjöstrand construction of an oscillatory integral parametrix for the Szegö kernel ([BSj]):

$$\begin{gathered}\hat{\Pi}(x, y) = S(x, y) + E(x, y)\ , \\ \text{with}\ \ S(x, y) = \int_0^\infty e^{i\lambda\psi(x,y)} s(x, y, \lambda) d\lambda\,, \qquad E(x, y) \in \mathcal{C}^\infty(X \times X)\,.\end{gathered} \tag{81}$$

The phase function ψ is of positive type and is given in the local coordinates above by

$$\psi(z, t; w, t') = \frac{1}{i}\left[1 - e^{\varphi(z,w) - \frac{1}{2}(\varphi(z)+\varphi(w))} e^{i(t-t')}\right]\ . \tag{82}$$

Here, $\varphi(z, w)$ is the almost analytic extension of the local Kähler potential with respect to the frame, i.e., $h = e^{-\varphi(z)}$; see (59) for the notion of almost analytic extension. The amplitude $s(z,t;w,t',\lambda)$ is a semi-classical amplitude as in [BSj] (Theorem 1.5), i.e., it admits a polyhomogeneous expansion $s \sim \sum_{j=0}^{\infty} \lambda^{m-k} s_j(x,y) \in S^m(X \times X \times \mathbb{R}^+)$.

The phase $\psi(z,t;w,t')$ is the generating function for the graph of the identity map along the symplectic cone $\Sigma \subset T^*X$ defined by $\Sigma = \{(x, r\alpha_x) : r > 0\}$ where α_x is the Chern connection one form. Hence the singularity of $\hat{\Pi}(x,y)$ only occurs on the diagonal and the symbol s is understood to be supported in a small neighborhood $(M \times M)_\delta$ of the anti-diagonal. It will be useful to make the cutoff explicit by introducing a smooth cutoff function $\chi(d(z,w))$ where χ is a smooth even function on $\mathbb{R}$ and $d(z,w)$ denotes the distance between z, w in the base Kähler metric.

As above, we denote the k-th Fourier coefficient of these operators relative to the S^1 action by $\hat{\Pi}_{h^k} = S_{h^k} + E_{h^k}$. Since E is smooth, we have $E_{h^k}(x,y) = O(k^{-\infty})$, where $O(k^{-\infty})$ denotes a quantity which is uniformly $O(k^{-n})$ on $X \times X$ for all positive n. Hence $E_{h^k}(z,w)$ is negligible for all the calculations and estimates of this article, and further it is only necessary to use a finite number of terms of the symbol s. For simplicity of notation, we will use the entire symbol.

It follows that (with $x = (z,t), y = (w,0)$ and with $\chi(d(z,w))$ as above),

$$\begin{aligned} \hat{\Pi}_{h^k}(x,y) &= S_{h^k}(x,y) + O(k^{-\infty}) \\ &= k \int_0^\infty \int_0^{2\pi} e^{ik(-t+\lambda\psi(z,t;w,0))} \chi(d(z,w))\, s\big(z,t;w,0,k\lambda\big) dt d\lambda + O(k^{-\infty}) \end{aligned} \tag{83}$$

The integral is a damped complex oscillatory integral since (61) implies that

$$\Im\psi(x,y) \geq C d(x,y)^2, \quad (x, y \in X), \tag{84}$$

for (x,y) sufficiently close to the diagonal, as one sees by Taylor expanding the phase around the diagonal (cf. [BSj], Corollary 1.3). It follows from (83) and from (84) that the Szegö kernel $\Pi_{h^k}(z,w)$ on M is 'Gaussian' in small balls $d(z,w) \leq \frac{\log k}{\sqrt{k}}$, i.e.,

$$|\, \hat{\Pi}_{h^k}(z,\varphi;w,\varphi')| \leq C k^m e^{-k d(z,w)^2} + O(k^{-\infty}), \quad (\text{when } d(z,w) \leq \frac{\log k}{\sqrt{k}}), \tag{85}$$

and on the complement $d(z,w) \geq \frac{\log k}{\sqrt{k}}$ it is rapidly decaying. This rapid decay can be improved to long range (sub-Gaussian) exponential decay off the diagonal given by the global Agmon estimates,

$$|\, \hat{\Pi}_{h^k}(z,\varphi;w,\varphi')| \leq C k^m e^{-\sqrt{k} d(z,w)}. \tag{86}$$

We refer to [Chr, L] for background and references.

It is helpful to eliminate the integrals in (83) by complex stationary phase. Expressed in a local frame and local coordinates on M, the result is

Proposition 4.1. *Let (L,h) be a C^∞ positive hermitian line bundle, and let $h = e^{-\varphi}$ in a local frame. Then in this frame, there exists a semi-classical amplitude $A_k(z,w) \sim k^m a_0(z,w) + k^{m-1} a_1(z,w) + \cdots$ in the parameter k^{-1} such that,*

$$\hat{\Pi}_{h^k}(z,0;w,0) = e^{k(\varphi(z,w) - \frac{1}{2}(\varphi(z)+\varphi(w)))} \chi_k(d(z,w))\, A_k(z,w) + O(k^{-\infty}),$$

where as above, $\chi_k(d(z,w)) = \chi(\frac{k^{1/2}}{\log k} d(z,w))$ is a cutoff to $\frac{\log k}{\sqrt{k}}$ - neighborhood of the diagonal.

Proof. This follows from the scaling asymptotics of [SZ] or from Theorem 3.5 of [**?**]. We refer there for a detailed proof of the scaling asymptotics and only sketch a somewhat intuitive proof.

The integral (83) is a complex oscillatory integral with a positive complex phase. With no loss of generality we may set $\varphi' = 0$. Taking the λ-derivative gives one critical point equation

$$1 - e^{\varphi(z,w) - \frac{1}{2}(\varphi(z)+\varphi(w))} e^{it} = 0$$

and the critical point equation in t implies that $\lambda = 1$. The λ-critical point equation can only be satisfied for complex t with imaginary part equal to the negative of the 'Calabi diastasis function' (60), i.e.,

$$\Im t = D(z,w),$$

and with real part equal to $-\Im\varphi(z,w)$. To obtain asymptotics, we therefore have to deform the integral over S^1 to the circle $|\zeta| = e^{-D(z,w)}$. Since $d(z,w) \leq C\frac{\log k}{\sqrt{k}}$ by assumption, the deformed contour is a slightly re-scaled circle by the amount $\frac{\log k}{\sqrt{k}}$; in the complete proofs, the contour is held fixed and the integrand is rescaled as in [SZ]. The contour deformation is possible modulo an error $O(k^{-M})$ of arbitrarily rapid polynomial decay because the integrand may be replaced by the parametrix (up to any order in λ) which has a holomorphic dependence on the $\mathbb{C}^*$ action on L^*, hence in $e^{i\theta}$ to a neighborhood of S^1 in $\mathbb{C}$. This is immediately visible in the phase and with more work is visible in the amplitude (this is the only incompleteness in the proof; the statement can be derived from [SZ] and also [Chr]). We need to use a cutoff to a neighborhood of the diagonal of $M \times M$, but it may be chosen to be independent of θ.

By deforming the circle of integration from the unit circle to $|\zeta| = e^{D(z,w)}$ and then changing variables $t \to t + iD(z,w)$ to bring it back to the unit circle, we obtain

$$\hat{\Pi}_{h^k}(x,y) \sim k \int_0^\infty \int_0^{2\pi} e^{ik(-t-iD(z,w)-\lambda\psi(z,t+iD(z,w);w,0))} s\big(z, t+iD(z,w); w, 0, k\lambda\big) dt d\lambda \mod k^{-\infty}. \tag{87}$$

The new critical point equations state that $\lambda = 1$ and that $e^{i\Im\varphi(z,w)}e^{it} = 1$. The calculation shows that $\psi = 0$ on the critical set so the phase factor on the critical set equals $e^{\varphi(z,w)-\frac{1}{2}(\varphi(z)+\varphi(w))}$. The Hessian of the phase on the critical set is $\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$ as in the diagonal case and the rest of the calculation proceeds as in [Ze1]. (As mentioned above, a complete proof is contained in [SZ]).

$\square$

4.1. **Toric Bergman-Szegö kernels.** In the toric case, we may simplify the expression for the Szegö kernels in Proposition 4.1 using the almost analytic extension (cf. §2.4); (59)) of the Kähler potential $\varphi(z,w)$ to $M \times M$, which has the form

$$F_{\mathbb{C}}(z \cdot \bar{w}) = \text{ the almost analytic extension of } F(|z|^2) \text{ to } M \times M. \tag{88}$$

The almost analytic extension will be illustrated in some analytic examples below, where it is the analytic continuation.

Thus, we have:

PROPOSITION **4.2.** *For any hermitian toric positive line bundle over a toric variety, the Szegö kernel for the metrics h_φ^k have the asymptotic expansions in a local frame on M,*

$$\Pi_{h^k}(z,w) \sim e^{k\left(F_{\mathbb{C}}(z\cdot\bar{w})-\frac{1}{2}(F(||z||^2)+F(||w||^2))\right)} A_k(z,w) \ \ mod\, k^{-\infty},$$

where $A_k(z,w) \sim k^m \left(a_0(z,w) + \frac{a_1(z,w)}{k} + \cdots\right)$ *is a semi-classical symbol of order* m.

As an example, the Bargmann-Fock(-Heisenberg) Szegö kernel with $k = 1$ and $H = I$ is given (up to a constant C_m depending only on the dimension) by

$$\hat{\Pi}_{h_{BF}}(z,\theta,w,\varphi) = e^{z\cdot\bar{w}-\frac{1}{2}(|z|^2+|w|^2)}e^{i(\theta-\varphi)} = \sum_{\alpha\in\mathbf{N}^n}\frac{z^\alpha\overline{w^\alpha}}{\alpha!}e^{-\frac{1}{2}(|z|^2+|w|^2)}e^{i(\theta-\varphi)}.$$

The higher Szegö kernels are Heisenberg dilates of this kernel:

$$\hat{\Pi}_{h^k_{BF}}(x,y) = \frac{1}{\pi^m}k^m e^{ik(t-s)}e^{k(\zeta\cdot\bar{\eta}-\frac{1}{2}|\zeta|^2-\frac{1}{2}|\eta|^2)}, \tag{89}$$

where $x = (\zeta, t)\,,\ y = (\eta, s)\,.$ In this case, the almost analytic extension is analytic and $F_{BF,\mathbb{C}}(z,w) = z\cdot\bar{w}$.

A second example is the Fubini-Study Szegö kernel on $\mathcal{O}(k)$, which lifts to $S^{2m-1}\times S^{2m-1}$ as

$$\hat{\Pi}_{h^k_{FS}}(x,y) = \sum_J \frac{(k+m)!}{\pi^m j_0!\cdots j_m!}x^J\bar{y}^J = \frac{(k+m)!}{\pi^m k!}\langle x,y\rangle^k\,. \tag{90}$$

Recalling that $x = e^{i\theta}\frac{e(z)}{||e(z)||}$ in a local frame e over an affine chart, the Szegö kernel has the local form on $\mathbb{C}^m\times\mathbb{C}^m$ of

$$\hat{\Pi}_{h^k_{FS}}(z,0;w,0) = \frac{(k+m)!}{\pi^m k!}e^{k\log\frac{(1+z\cdot\bar{w})}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}}\,. \tag{91}$$

Thus, $F_{FS,\mathbb{C}}(z,w) = \log(1+z\cdot\bar{w})$.

4.2. **Asymptotics of derivatives of toric Bergman/Szegö kernels .** One of the key ingredients in of Theorem 1.1 is the asymptotics of derivatives of the contracted Bergman/Szegö kernel

$$\Pi_{h^k_t}(z,z) = F_{h^k_t}(z,z)||e^k_L(z)||^2_{h^k} = \hat{\Pi}_{h^k}(z,0;z,0) \tag{92}$$

in (t,z). (The notation is slightly ambiguous since in (73) it is used for the un-contracted kernel, but it is standard and we hope no confusion will arise since one is scalar-valued and the other is not.) These derivatives allow us to make simple comparisions to derivatives of $\varphi_k(t,z)$. Since we ultimately interested in C^k norms we need asymptotics of derivatives with respect to non-vanishing vector fields. We can use the vector fields $\frac{\partial}{\partial\rho_j}$ away from $\mathcal{D}$ and the vector fields $\frac{\partial}{\partial r_j}$ near $\mathcal{D}$. The calculations are very similar, but we carry them both out in some detail here. Later we will tend to suppress the calculations with $\frac{\partial}{\partial r_j}$ to avoid duplication; the reader can check in this section that the calculations and estimates are valid.

Only the leading coefficient and the order of asymptotics are relevant. The undifferentiated diagonal asymptotics are of the following form: for any $h\in P(M,\omega)$,

$$\Pi_{h^k}(z,z) = \sum_{i=0}^{d_k}||s_i(z)||^2_{h_k} = a_0k^m + a_1(z)k^{m-1} + a_2(z)k^{m-2} + \ldots \tag{93}$$

where a_0 is constant and as above $d_k + 1 = \dim H^0(M, L^k)$.

We first consider derivatives with respect to ρ. Calculating ρ derivatives of $\Pi_{h^k}(e^{\rho/2}, e^{\rho/2})$ is equivalent to calculating θ-derivatives of $\Pi_{h_t^k}(e^{i\theta} z, z)$. Using (62) we have

$$\Pi_{h_t^k}(e^{i\theta} z, z) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{e^{i\langle \alpha, \theta\rangle} |z^\alpha|^2 e^{-kF_t(e^{i\theta}|z|^2)}}{\mathcal{Q}_{h_t^k}(\alpha)}.$$

The results are globally valid but are not useful near $\mathcal{D}$ since on each stratum some of the vector fields generating the $(\mathbb{C}^*)^m$ action vanish.

In the following, we use the tensor product notation $(\frac{\alpha}{k} - \mu_t(e^{\rho/2}))^{\otimes 2}_{ij}$ for $(\frac{\alpha_i}{k} - \mu_t(e^{\rho/2})_i)(\frac{\alpha_j}{k} - \mu_t(e^{\rho/2})_j)$.

Proposition 4.3. *For $i, j = 1, \dots, m$ we have,*

(1) $k^{-m} \sum_{\alpha \in kP \cap \mathbb{Z}^m} (\frac{\alpha}{k} - \mu_t(e^{\rho/2})) \frac{e^{\langle \alpha, \rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)} = O(k^{-2})$;

(2) $\frac{1}{\Pi_{h_t^k}(z,z)} \left(- \sum_{\alpha \in kP \cap \mathbb{Z}^m} (\frac{\partial}{\partial t} \log \mathcal{Q}_{h_t^k}(\alpha)) \ \frac{e^{\langle \alpha, \rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)} \right) - k \frac{\partial}{\partial t} \varphi_t = O(k^{-1})$;

(3) $\frac{1}{\Pi_{h_t^k}(z,z)} \left(k^2 \sum_{\alpha \in kP \cap \mathbb{Z}^m} (\frac{\alpha}{k} - \mu_t(e^{\rho/2}))^{\otimes 2}_{ij} \frac{e^{\langle \alpha, \rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)} \right) - k \frac{\partial^2 \varphi_t}{\partial \rho_i \partial \rho_j} = O(k^{-1})$;

(4) $\frac{1}{\Pi_{h_t^k}(z,z)} \left(k \sum_{\alpha \in kP \cap \mathbb{Z}^m} (\frac{\alpha}{k} - \mu_t(e^{\rho/2}))_i (\frac{\partial}{\partial t} \log \mathcal{Q}_{h_t^k}(\alpha)) \frac{e^{\langle \alpha, \rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)} \right) - k \frac{\partial^2 \varphi_t}{\partial \rho_i \partial t} = O(k^{-1})$.

Proof. To prove (1), we differentiate and use (53)-(62) and (93) to obtain

$$O(k^{m-1}) = \nabla_\rho \Pi_{h_t^k}(e^{\rho/2}, e^{\rho/2}) \quad = \quad k \sum_{\alpha \in kP \cap \mathbb{Z}^m} (\tfrac{\alpha}{k} - \mu_t(e^{\rho/2})) \tfrac{e^{\langle \alpha, \rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)}.$$

To prove (2) we differentiate

$$\log \Pi_{h_t^k}(e^{\rho/2}, e^{\rho/2}) = \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{e^{\langle \alpha, \rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)}$$

with respect t to produce the left side. Since the leading coefficient of (93) is independent of t, the t derivative has the order of magnitude of the right side of (2).

To prove (3), we take a second derivative of (1) in ρ (or θ) to get

$$\begin{aligned} \nabla^2_\rho \Pi_{h_t^k}(e^{\rho/2}, e^{\rho/2}) \quad &= -k \nabla \mu_t(e^{\rho/2}) \Pi_{h_t^k}(e^{\rho/2}, e^{\rho/2}) \\ &\quad + k^2 \sum_{\alpha \in kP \cap \mathbb{Z}^m} (\tfrac{\alpha}{k} - \mu_t(e^{\rho/2}))^{\otimes 2} \tfrac{e^{\langle \alpha, \rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)}. \end{aligned}$$

Then (3) follows from (93) and the fact that $\nabla \mu_t(e^{\rho/2})) = \nabla^2 \varphi$. Similar calculations show (4).

$\square$

In our applications, we actually need asymptotics of logarithmic derivatives. They follow in a straightforward way from Proposition 4.3, using that $\Pi_{h^k}(z,z) \sim k^m$. We record the results for future reference.

Proposition 4.4. *We have:*

- $\frac{1}{k}\nabla_\rho \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \frac{\sum_\alpha (\frac{\alpha}{k}-\mu_t(z)) \ \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\left(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)} = O(\frac{1}{k^2})$
- $\frac{1}{k}\frac{\partial}{\partial t} \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \frac{\sum_\alpha \partial_t \log\left(\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)}\right) \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\left(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)} - \frac{\partial\varphi_t}{\partial t} = O(\frac{1}{k^2})$.

Proposition **4.5.** *We have:*

(1) $\frac{1}{k}\nabla^2_\rho \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \frac{1}{k}\sum_{\alpha,\beta}(\alpha-\beta)^{\otimes 2} \ \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}\left(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^{-2} - \left(\frac{\partial^2\varphi_t}{\partial\rho_i\partial\rho_j}\right) = O(\frac{1}{k^2})$

(2) $\frac{1}{k}\frac{\partial}{\partial t}\nabla_\rho \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \frac{1}{k}\frac{\sum_{\alpha,\beta}(\alpha-\beta) \ \partial_t \log\left(\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}\right) \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} - \left(\frac{\partial^2\varphi_t}{\partial\rho_i\partial t}\right) = O(\frac{1}{k^2})$.

(3)

$$\frac{1}{k}\frac{\partial^2}{\partial t^2} \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}$$

$$= \frac{1}{k}\frac{\sum_{\alpha,\beta}\left(\partial_t^2 \log\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)} + \ (\partial_t \log\frac{1}{\mathcal{Q}_{h_t^k}})(\partial_t \log(\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)})\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} - \left(\frac{\partial^2\varphi_t}{\partial\rho_i\partial t}\right) = O(\frac{1}{k^2}).$$

Finally, we consider the analogous derivatives with respect to the radial coordinates r_j near $\mathcal{D}$. We assume z is close to the component of $\mathcal{D}$ given in local slice orbit coordinates by $z'=0$ and let $r'=(r_j)_{j=1}^p$ denote polar coordinates in this slice as discussed in §2. The Szegö kernel then has the form

$$\Pi_{h_t^k}(z,z) = \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{\Pi_{j=1}^p r_j^{2\alpha_j} e^{\langle\rho'',\alpha''\rangle} e^{-kF_t(r_1^2,\ldots,r_p^2,e^{\rho_{p+1}},\ldots,e^{\rho_m})}}{\mathcal{Q}_{h_t^k}(\alpha)}. \tag{94}$$

The coefficients of the expansion (93) are smooth functions of r_j^2 and the expansion may be differentiated any number of times.

The behavior of $\Pi_{h_t^k}(z,z)$ for $z\in\mathcal{D}$ has the new aspect that many of the terms vanish. The extreme case is where z is a fixed point. We choose the slice coordinates so that it has coordinates $z=0$. We observe that only the term with $\alpha=0$ in (94) is non-zero, and the αth term vanishes to order $|\alpha|$.

Since $\frac{\partial}{\partial r_j} = \frac{2}{r_j}\frac{\partial}{\partial\rho_j}$ where both are defined, the calculations above are only modified by the presence of new factors of $\frac{2}{r_j}$ in each space derivative. Since we are applying the derivative to functions of r_j^2, it is clear that the apparent poles will be cancelled. Indeed, the r_j derivative removes any lattice point α with vanishing α_j component. Comparing these derivatives with derivatives of (94) gives the following:

Proposition **4.6.** *For $n=1,\ldots,p$, we have:*

- $\frac{1}{k}\frac{\partial}{\partial r_n}\log \Pi_{h_t^k}(z,z) = \dfrac{\sum_{\alpha:\alpha_n\neq 0} \frac{2(\frac{\alpha_n}{k}-\mu_{tn}(z))}{r_n} \frac{\Pi_{j=1}^p r_j^{2\alpha_j} e^{\langle \rho'', \alpha''\rangle} e^{-kF_t(r_1^2,\dots,r_p^2,e^{\rho_{p+1}},\dots,e^{\rho_m})}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_\alpha \frac{\Pi_{j=1}^p r_j^{2\alpha_j} e^{\langle \rho'', \alpha''\rangle} e^{-kF_t(r_1^2,\dots,r_p^2,e^{\rho_{p+1}},\dots,e^{\rho_m})}}{\mathcal{Q}_{h_t^k}(\alpha)}} = O(\frac{1}{k^2})$

In effect, the exponent α is taken to $\alpha - (0,\dots,1_n,\dots)$ in the sum or removed if $\alpha_n = 0$, where $(0,\dots,1_n,\dots)$ is the lattice point with only a 1 in the nth coordinate. There are similar formulae for the second derivatives $\frac{\partial^2}{\partial r_n \partial r_i}, \frac{\partial^2}{\partial r_n \partial t}, \frac{\partial^2}{\partial r_n \partial \rho_i}$. The only important point to check is that the modification changing α to $\alpha - (0,\dots,1_n,\dots)$ does not affect the proofs in §7-8.

5. Localization of Sums: Proof of Lemma 1.2

The following Proposition immediately implies Lemma 1.2:

Proposition 5.1. *Given (t,z), and for any $\delta, C > 0$, there exists $C' > 0$ such that*

$$\frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \mathcal{P}_{h_t^k}(\alpha, z) = O(k^{-C}), \quad \textit{if } |\frac{\alpha}{k} - \mu_t(z)| \geq C' k^{-\frac{1}{2}+\delta}.$$

Proof. The proof is based on integration by parts. All of the essential issues occur in the Bargmann-Fock model, so we first illustrate with that case.

5.1. **Bargmann-Fock case.** To analyze the decay of $\mathcal{P}_{h^k_{BF}}(\alpha, z)$ as a function of lattice points α, it seems simplest to use the integral formula (suppressing the factor k^m and normalizing the volume of $\mathbf{T}^m$ to equal one),

$$k^{|\alpha|}\frac{|z^\alpha|^2}{\alpha!}e^{-k|z|^2} = (2\pi)^{-m}\int_{T^m} e^{-k\left(|z|^2(1-e^{i\theta}) - i\langle \frac{\alpha}{k},\theta\rangle\right)}d\theta = e^{-k|z|^2}(2\pi)^{-m}\int_{T^m} e^{k\left(|z|^2 e^{i\theta}) - i\langle \frac{\alpha}{k},\theta\rangle\right)}d\theta. \tag{95}$$

We observe that the rightmost expression in (95) is $e^{-k|z|^2}$ times a complex oscillatory integral with phase

$$\Phi_{z,\frac{\alpha}{k}}(\theta) = |z|^2(e^{i\theta}-1) - i\langle \frac{\alpha}{k}, \theta\rangle.$$

We observe that (consistent with Proposition 2.1),

$$\nabla_\theta \Phi_{z,\frac{\alpha}{k}}(\theta) = i(|z|^2 e^{i\theta} - \frac{\alpha}{k}) = 0 \iff e^{i\theta}|z|^2 = |z|^2 = \frac{\alpha}{k}.$$

Further, we claim that

$$|\nabla_\theta \Phi_{z,\frac{\alpha}{k}}(\theta)| \geq ||z|^2 - \frac{\alpha}{k}|. \tag{96}$$

Indeed, the function

$$f_{z,\alpha}(\theta) := \left|e^{i\theta}|z|^2 - \frac{\alpha}{k}\right|^2 = \sum_{j=1}^m \left(\cos\theta_j\ |z_j|^2 - \frac{\alpha_j}{k}\right)^2 + \left(\sin\theta_j\ |z_j|^2\right)^2$$

on $\mathbf{T}^m$ has a strict global minimum at $\theta = 0$ as long as $|z_j|^2 \neq 0, \frac{\alpha_j}{k} \neq 0$ for all j. It still has a global minimum without these restrictions, but the minimum is no longer strict. We note that this discussion of global minima is possible only because the Kähler potential admits a global analytic continuation in (z,w); in general, one can only analyze critical points near the diagonal.

We integrate by parts with the operator

$$\mathcal{L} = \frac{1}{k} \frac{1}{|\nabla_\theta \Phi_{z,\frac{\alpha}{k}}|^2} \overline{\nabla_\theta \Phi_{z,\frac{\alpha}{k}}} \cdot \nabla_\theta, \tag{97}$$

i.e., we apply its transpose

$$\mathcal{L}^t = -\frac{1}{k} \frac{1}{|\nabla_\theta \Phi_{z,\frac{\alpha}{k}}|^2} \nabla_\theta \Phi_{z,\frac{\alpha}{k}} \cdot \nabla_\theta - \frac{1}{k} \nabla_\theta \cdot \frac{1}{|\nabla_\theta \Phi_{z,\frac{\alpha}{k}}|^2} \nabla_\theta \Phi_{z,\frac{\alpha}{k}} \tag{98}$$

to the amplitude. The second (divergence) term is -1 times

$$\frac{1}{k} \frac{\nabla \cdot \nabla \Phi_{z,\frac{\alpha}{k}}}{|\nabla \Phi_{z,\frac{\alpha}{k}}|^2} + \frac{1}{k} \frac{\langle \nabla^2 \Phi_{z,\frac{\alpha}{k}} \cdot \nabla \Phi_{z,\frac{\alpha}{k}}, \nabla \Phi_{z,\frac{\alpha}{k}} \rangle}{|\nabla \Phi_{z,\frac{\alpha}{k}}|^4}. \tag{99}$$

We will need to take into account the k-dependence of the coefficients, and therefore introduce some standard spaces of semi-classical symbols. We denote by $S^n_\delta(\mathbf{T}^m)$ the class of smooth functions $a_k(\theta)$ on $\mathbf{T}^m \times \mathbb{N}$ satisfying

$$\sup_{e^{i\theta} \in \mathbf{T}^m} |D^\gamma_\theta a_k(\theta)| \leq C k^{n + |\gamma| \delta}. \tag{100}$$

Here we use multi-index notation $D^\gamma_\theta = \prod_{j=1}^m (\frac{\partial}{i \partial \theta_j})^{\gamma_j}$. Thus, each D_{θ_j} derivative gives rise to an extra order of k^δ in estimates of a_k. We note that products of symbols satisfy

$$S^{n_1}_\delta \times S^{n_2}_\delta \subset S^{n_1+n_2}_\delta. \tag{101}$$

We now claim that (with δ the same as in the statement of the Proposition),

(1) $\frac{\nabla_\theta \Phi_{z,\frac{\alpha}{k}}}{|\nabla_\theta \Phi_{z,\frac{\alpha}{k}}|^2} \in S^{\frac{1}{2}-\delta}_{\frac{1}{2}-\delta}$
(2) (99) lies in $S^{-2\delta}_{\frac{1}{2}-\delta}$.

In (2), we note the pre-factor $\frac{1}{k}$. To prove the claim, we first observe that the sup norm estimates are correct by (96) and from the fact that $\frac{\nabla \Phi_{z,\frac{\alpha}{k}}}{|\nabla \Phi_{z,\frac{\alpha}{k}}|}$ is a unit vector. We further consider derivatives of (1)-(2). Each θ derivative essentially introduces one more factor of $k|\nabla_\theta \Phi_{z,\frac{\alpha}{k}}|$ and hence raises the order by $k^{\frac{1}{2}-\delta}$. This continues to be true for iterated derivatives, proving the claim.

Now we observe that

$$\mathcal{L}^t : S^n_{\frac{1}{2}-\delta} \to S^{n-2\delta}_{\frac{1}{2}-\delta}. \tag{102}$$

Indeed, the first term of $\mathcal{L}^t$ is the composition of (i) ∇_θ, which raises the order by $\frac{1}{2} - \delta$, (ii) multiplication by an element of $S^{\frac{1}{2}-\delta}_{\frac{1}{2}-\delta}$ which again raises the order by $\frac{1}{2} - \delta$ (iii) times $\frac{1}{k}$ which lowers the order by 1. The second term is a multiplication by $\frac{1}{k}$ times an element of $S^{1-2\delta}_{\frac{1}{2}-\delta}$ and thus also lowers the order by 2δ.

It follows that each partial integration by $\mathcal{L}$ introduces decay of $k^{-2\delta}$, hence for any $M > 0$,

$$\begin{aligned} (95) \quad &= \quad e^{-k|z|^2} (2\pi)^{-m} \int_{T^m} e^{k\left(|z|^2 e^{i\theta}) - i \langle \frac{\alpha}{k}, \theta \rangle\right)} ((\mathcal{L}^t)^M 1) d\theta \\ &= O\left(k^{-2\delta}\right)^M e^{-k|z|^2} \int_{T^m} e^{k \Re (|z|^2 e^{i\theta})} d\theta = O(k^{-2\delta M}) \end{aligned}$$

in this region.

5.2. **General case.** We now generalize this argument from the model case to the general one. With no loss of generality we may choose coordinates so that z lies in a fixed compact subset of $\mathbb{C}^m$, where the open orbit is identified with $(\mathbb{C}^*)^m$. In the open orbit we continue to write $|z|^2 = e^\rho$. The first step is to obtain a useful oscillatory integral formula for $\mathcal{P}_{h^k}(\alpha, z)$. By Proposition 3.3 and Proposition 4.2, we have

$$\mathcal{P}_{h^k}(\alpha, z) = (2\pi)^{-m} \int_{T^m} e^{k(F_{\mathbb{C}}(e^{i\theta}|z|^2) - F(|z|^2))}\, \chi(d(z, e^{i\theta}z))\, A_k\big(z, e^{i\theta}z, 0\big) e^{i\langle \alpha, \theta\rangle} d\theta \; + O(k^{-\infty}). \tag{103}$$

The phase is given by

$$\Phi_{z, \frac{\alpha}{k}}(\theta) = F_{\mathbb{C}}(e^{i\theta}|z|^2) - F(|z|^2) - i\langle \frac{\alpha}{k}, \theta\rangle \tag{104}$$

where as above, $F_{\mathbb{C}}(e^{i\theta}|z|^2)$ is the almost analytic continuation of the Kähler potential $F(|z|^2)$ to $M \times M$. By (84) and (61), it satisfies

$$\Re(F_{\mathbb{C}}(e^{i\theta}|z|^2) - F(|z|^2))) \leq -C d(z, e^{i\theta}z)^2, \;\; (\text{for some } \; C > 0). \tag{105}$$

Hence, the integrand (103) is rapidly decaying on the set of θ where $d(z, e^{i\theta}z)^2 \geq C \frac{\log k}{k}$ (see also (86)), and we may replace $\chi(d(z, e^{i\theta}z))$ by $\chi(k^{\frac{1}{2}-\delta'} d(z, e^{i\theta}z)) \in S^0_{\frac{1}{2}-\delta'}$, since the contribution from $1 - \chi(k^{\frac{1}{2}-\delta'} d(z, e^{i\theta}z))$ is rapidly decaying. Here, δ' is an arbitrarily small constant and we may choose it so that $\delta' < \delta$ in the Proposition. (We did not use such cutoffs in the Bargmann-Fock case since the real analytic potential had a global analytic extension with obvious properties, but as in §2.4, it is necessary for almost analytic extensions).

The set $d(z, e^{i\theta}z) \leq C \frac{k^{\delta'}}{\sqrt{k}}$ depends strongly on the position of z relative to $\mathcal{D}$, or equivalently on the position of $\mu_h(z)$ relative to ∂P. For instance, if z is a fixed point then $d(z, e^{i\theta}z) = 0$ for all θ. However, we will not need to analyze these sets until the next section.

We now generalize the integration by parts argument. Our goal is to prove that $\mathcal{P}_{h_t^k}(\alpha, z) = O(k^{-C})$ if $|\frac{\alpha}{k} - \mu_t(z)| \geq C k^{-\frac{1}{2}+\delta}$. Now, the gradient in θ of the phase of (103) is given by

$$\nabla_\theta \Phi(z, \frac{\alpha}{k})(\theta) = \nabla_\theta F_{\mathbb{C}}(e^{i\theta}|z|^2) - i\frac{\alpha}{k} = i(\mu_{\mathbb{C}}(z, e^{i\theta}z) - \frac{\alpha}{k}), \tag{106}$$

where $\mu_{\mathbb{C}}(z, e^{i\theta}z)$ is the almost analytic extension of the moment map (see §2.4). The following Lemma is obvious, but we display it to highlight the relations between the small parameters δ of the Proposition and δ' in our choice of cutoffs.

Lemma **5.2.** *If* $|\frac{\alpha}{k} - \mu_t(z)| \geq C k^{-\frac{1}{2}+\delta}$, *and if* $d(z, e^{i\theta}z) \leq C k^{-\frac{1}{2}+\delta'}$ *with* $\delta' < \delta$, *then* $|(\mu(z, e^{i\theta}z) - \frac{\alpha}{k})| \geq C' k^{-\frac{1}{2}+\delta}$.

Proof. By Proposition 2.1,

$$\begin{aligned} |(\mu(z, e^{i\theta}z) - \tfrac{\alpha}{k})|^2 &= |(\Re\mu(z, e^{i\theta}z) - \tfrac{\alpha}{k})|^2 + |\tfrac{1}{2}\nabla_\theta D(z, e^{i\theta}z)|^2 \\ &\geq |(\mu(z) - \tfrac{\alpha}{k})|^2 + O(d(e^{i\theta}z, z)). \end{aligned}$$

□

It follows that under the assumption $|\frac{\alpha_k}{k} - \mu_t(z)| \geq C k^{-\frac{1}{2}+\delta}$ of the Proposition, we may integrate by parts with the operator

$$\mathcal{L} \;=\; \tfrac{1}{k} |\nabla_\theta \Phi_{z, \frac{\alpha}{k}}|^{-2} \, \nabla_\theta \Phi_{z, \frac{\alpha}{k}} \cdot \nabla_\theta \; . \tag{107}$$

The transpose $\mathcal{L}^t$ has the same form (98) as for the Bargmann-Fock example, the only significant change being that it is now applied to a non-constant amplitude A_k and to the cutoff $\chi(k^{\frac{1}{2}-\delta'}d(z,e^{i\theta}z)) \in S^0_{\frac{1}{2}-\delta'}$ as well as to its own coefficients. Differentiations of A_k do preserve the orders of terms; the only significant change in the symbol analysis in the Bargmann-Fock case is that differentiations of $\chi(k^{\frac{1}{2}-\delta'}d(z,e^{i\theta}z))$ bring only improvements of order $k^{-\delta'}$ rather than $k^{-\delta}$. However, the order still decreases by at least $2\delta'$ on each partial integration, and therefore repeated integration by parts again gives the estimate

$$|\mathcal{P}_{h^k}(\alpha,z)| = O\left((k^{-\delta'})^M \int_{\mathbf{T}^m} e^{k(\Re F(e^{i\theta}|z|^2)-F(|z|^2)} d\theta\right) = O\left((k^{-\delta'})^M\right). \tag{108}$$

□

Remark: It is natural to use integration by parts in this estimate since the decay in $\mu_t(z)-\frac{\alpha}{k}$ must use the imaginary part of the phase and is not a matter of being far from the center of the Gaussian.

5.3. **Further details on the phase.** For future reference (see Lemma 6.2), we Taylor expand the phase (104) in the θ variable to obtain

$$\Phi_{z,\frac{\alpha}{k}}(\theta) = i\langle\mu(z)-\frac{\alpha}{k},\theta\rangle + \langle H_{\frac{\alpha}{k}}\theta,\theta\rangle + R_3(k,e^{i\theta}\mu^{-1}(\frac{\alpha}{k})), \tag{109}$$

where $R_3 = O(|\theta|^3)$. Here, $H_{\frac{\alpha}{k}} = \nabla^2 F(\mu^{-1}(\frac{\alpha}{k}))$ denotes the Hessian of φ at $\frac{\alpha}{k}$ (see (57) of §2.2) Indeed, we have

$$\begin{aligned} F_{\mathbb{C}}(e^{i\theta}|z|^2) - F(|z|^2) &= \int_0^1 \tfrac{d}{dt} F_{\mathbb{C}}(e^{it\theta}|z|^2)dt \\ &= \int_0^1 \langle\nabla_\theta F(e^{it\theta}|z|^2), i\theta\rangle dt \\ &= \langle\nabla_\rho F(e^\rho)),(i\theta)\rangle + \int_0^1 (t-1)\nabla^2_\rho(F(e^{it\theta+\rho}))(i\theta)^2/2dt \\ &= i\langle\mu(z),\theta\rangle + \nabla^2_\rho(F(e^\rho))(i\theta)^2 + R_3(k,e^{i\theta}\mu^{-1}(\tfrac{\alpha}{k}),\theta) \\ &= i\langle\mu(z),\theta\rangle + \langle H_z\theta,\theta\rangle + R_3(k,\theta,\mu^{-1}(\tfrac{\alpha}{k})), \end{aligned} \tag{110}$$

in the notation (57), where $H_z = \nabla^2_\rho F(|z|^2)$ and where

$$R_3(k,\theta,\rho) := \int_0^1 (t-1)^2 \langle\nabla^3_\rho(F(e^{it\theta+\rho})),(i\theta)^3/3!\rangle dt. \tag{111}$$

6. Proof of Regularity Lemma 1.3 and joint asymptotics of $\mathcal{P}_{h^k}(\alpha)$

The first statement that $\mathcal{R}_\infty(t,x)$ is C^∞ up to the boundary follows from (58),

$$\begin{aligned} \mathcal{R}_\infty(t,x) &= \left(\frac{\delta_{\varphi_t}(x)\cdot\prod_{r=1}^d \ell_r(x)}{\left(\delta_{\varphi_0}(x)\cdot\prod_{r=1}^d \ell_r(x)\right)^{1-t}\left(\delta_{\varphi_1}(x)\cdot\prod_{r=1}^d \ell_r(x)\right)^t}\right)^{1/2} \\ &= \left(\frac{\delta_{\varphi_t}(x)}{\delta_{\varphi_0}(x)^{1-t}\delta_{\varphi_1}(x)^t}\right)^{1/2}, \end{aligned} \tag{112}$$

where the functions δ_φ are positive, bounded below by strictly positive constants and C^∞ up to ∂P.

We now consider the asymptotics of $\mathcal{R}_k(t,\alpha)$. We determine the asymptotics of the ratio by first determining the asymptotics of the factors of the ratio. We could use either the expression (30) in terms of norming constants $\mathcal{Q}_h^k(\alpha)$ for the dual expression in terms of $\mathcal{P}_{h^k}(\alpha)$ in Corollary 3.2. Each approach has its advantages and each seems of interest in the geometry of toric varieties, but for the sake of simplicity we only consider $\mathcal{P}_{h^k}(\alpha)$ here. In [SoZ1] we take the opposite approach of focusing on the norming constants. The advantage of using $\mathcal{P}_{h^k}(\alpha)$ is that it may be represented by a smooth complex oscillatory integral up to the boundary, while $\mathcal{Q}_h^k(\alpha)$ are singular oscillatory integrals over P. A disadvantage of $\mathcal{P}_{h^k}(\alpha)$ is that it does not extend to a smooth function on $\bar{P}$ and has singularities on ∂P.

The asymptotics of $\mathcal{P}_{h^k}(\alpha)$ are straightforward applications of steepest descent in compact subsets of $M\backslash\mathcal{D}$ but become non-uniform at $\mathcal{D}$. To gain insight into the general problem we again consider first the Bargmann-Fock model, where by (79) we have

$$\mathcal{P}_{h_{BF}^k}(\alpha) = k^m e^{-|\alpha|}\frac{\alpha^\alpha}{\alpha!} = (2\pi)^{-m} k^m \int_{\mathbf{T}^m} e^{k\langle e^{i\theta}-1-i\theta,\frac{\alpha}{k}\rangle} d\theta. \tag{113}$$

As observed before, the factors of k cancel so 'asymptotics' means asymptotics as $\alpha\to\infty$. This indicates that we do not have asymptotics when α ranges over a bounded set, or equivalently when $\frac{\alpha}{k}$ is $\frac{C}{k}$-close to a corner. On the other hand, steepest descent asymptotics applies in a coordinate α_j as long as $\alpha_j\to\infty$. Our aim in general is to obtain steepest descent asymptotics of $\mathcal{P}_{h^k}(\alpha)$ in directions far from facets and Bargmann-Fock asymptotics in directions near a facet.

6.1. **Asymptotics of $\mathcal{P}_{h^k}(\alpha)$.**

The analysis of $\mathcal{P}_{h^k}(\alpha)$ is closely related to the analysis of $\mathcal{P}_{h^k}(\alpha,z)$ in §5.2, and in a sense is a continuation of it. But the arguments are now more than integrations-by-parts. We obtain the asymptotics of $\mathcal{P}_{h^k}(\alpha)$ from the integral representation analogous to (103) (see also Proposition 4.2 and Corollary 3.4). Modulo rapidly decaying functions in k, we have (in the notation of Proposition 4.2),

$$\mathcal{P}_{h^k}(\alpha) \sim (2\pi)^{-m}\int_{\mathbf{T}^m} e^{-k(F_{\mathbb{C}}(e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k}))-F(\mu_h^{-1}(\frac{\alpha}{k})))} A_k\big(e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k}),\mu_h^{-1}(\tfrac{\alpha}{k}),0,k\big)e^{i\langle\alpha,\theta\rangle}d\theta. \tag{114}$$

This largely reduces the asymptotic calculation of $\mathcal{P}_{h^k}(\alpha)$ to facts about the off-diagonal asymptotics of the Szegö kernel (cf. Proposition 4.2).

The integral (114) is the oscillatory integral (103) but with $z=\mu^{-1}(\frac{\alpha}{k})$. Hence, as in (104), its phase is

$$\Phi_{\frac{\alpha}{k}}(\theta) = F_{\mathbb{C}}(e^{i\theta}\mu^{-1}(\frac{\alpha}{k})) - F(\mu^{-1}(\frac{\alpha}{k})) - i\langle\frac{\alpha}{k},\theta\rangle. \tag{115}$$

As in (84) and (105) (but with i included in as part of the phase),

$$\Re\Phi_{\frac{\alpha}{k}}(\theta) \le -Cd(\mu^{-1}(\frac{\alpha}{k}), e^{i\theta}\mu^{-1}(\frac{\alpha}{k}))^2, \text{ (for some } C>0). \tag{116}$$

Specializing (106) to our $z=\mu^{-1}(\frac{\alpha}{k})$, we get

$$\nabla_\theta\Phi_{\frac{\alpha}{k}}(\theta) = \nabla_\theta F_{\mathbb{C}}(e^{i\theta}\mu^{-1}(\frac{\alpha}{k})) - i\frac{\alpha}{k} = i(\mu_{\mathbb{C}}(\mu^{-1}(\frac{\alpha}{k}), e^{i\theta}\mu^{-1}(\frac{\alpha}{k})) - \frac{\alpha}{k}). \tag{117}$$

By Proposition 2.1, the complex phase has a critical point at values of θ such that $d(z, e^{i\theta} z) \leq \delta$, and $e^{i\theta}\mu^{-1}(\frac{\alpha}{k}) = \mu^{-1}(\frac{\alpha}{k})$. For $\frac{\alpha}{k} \notin \partial P$, the only critical point is therefore $\theta = 0$. The phase then equals zero, and hence at the critical point the real part of the phase is at its maximum of zero.

For $\frac{\alpha}{k} \notin \partial P$, the critical point $\theta = 0$ is non-degenerate. Specializing (109) to $z = \mu^{-1}(\frac{\alpha}{k})$, we have

$$\begin{aligned} F_{\mathbb{C}}(e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k})) - F(\mu_h^{-1}(\tfrac{\alpha}{k})) &= \int_0^1 \tfrac{d}{dt} F_{\mathbb{C}}(e^{it\theta}\mu_h^{-1}(\tfrac{\alpha}{k}))dt \\ &= i\langle \tfrac{\alpha}{k}, \theta\rangle + i\langle H_{\frac{\alpha}{k}}\theta, \theta\rangle + R_3(k, \theta, \mu^{-1}(\tfrac{\alpha}{k}), \end{aligned} \tag{118}$$

where R_3 is defined in (111). Hence,

$$\Phi_{\frac{\alpha}{k}}(\theta) = \langle H_{\frac{\alpha}{k}}\theta, \theta\rangle + R_3(\theta, k, \mu^{-1}(\frac{\alpha}{k})), \tag{119}$$

and finally,

$$\mathcal{P}_{h^k}(\alpha) \sim (2\pi)^{-m} \int_{\mathbf{T}^m} e^{-k\langle H_{\frac{\alpha}{k}}\theta, \theta\rangle} e^{kR_3(\theta, k, \mu^{-1}(\frac{\alpha}{k}))} A_k\big(\mu_h^{-1}(\tfrac{\alpha}{k}), e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k}), 0, k\big) d\theta \ . \tag{120}$$

Non-degeneracy of the phase is the statement that $H_{\frac{\alpha}{k}}$ is a non-degenerate symmetric matrix, and this follows from strict convexity of the Kähler potential or symplectic potential, see (57). But as discussed in §2.2, the $H_{\frac{\alpha}{k}}$ has a kernel when $\frac{\alpha}{k} \in \partial P$. Hence the stationary phase expansion is non-uniform for $\frac{\alpha}{k} \in P$ and is not possible when $\frac{\alpha}{k} \in \partial P$. This accounts for the fact that we need to break up the analysis into several cases, and that we cannot rely on the complex stationary phase method for all of them.

Specializing (85) and (86), we have

$$| \Pi_{h^k}(e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k}), \mu_h^{-1}(\frac{\alpha}{k}))| \leq Ck^m e^{-Ckd(\frac{\alpha}{k}), e^{i\theta}\frac{\alpha}{k}))^2} + O(e^{-C\sqrt{k}d(z, e^{i\theta}z)}). \tag{121}$$

Hence, the integrand of (114) is negligible off the set of θ where $d(\mu^{-1}(\frac{\alpha}{k}), e^{i\theta}\mu^{-1}(\frac{\alpha}{k})) \leq C\frac{\log k}{\sqrt{k}}$. We now observe that for $d(z, e^{i\theta}z) \leq C\frac{k^\delta}{\sqrt{k}}$,

$$d(e^{i\theta}z, z)^2 \sim \sum_j (1 - \cos\theta_j)\ell_j(\mu(z)), \tag{122}$$

where we sum over j such that $|\ell_j(\mu(z))| << 1$ (we will make this precise in Definition 6.1). In particular,

$$d(e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k}), \mu_h^{-1}(\frac{\alpha}{k}))^2 \sim \sum_j (1 - \cos\theta_j)\ell_j(\frac{\alpha}{k}). \tag{123}$$

Indeed, both in small balls in the interior and near the boundary, the calculation is universal and hence is accurately reflected in the Bargmann-Fock model with all $H_j = 1$, where the distance squared equals

$$\sum_{j=1}^m |e^{i\theta_j}z_j - z_j|^2 = 2\sum_{j=1}^m |z_j|^2(1 - \cos\theta_j) = 2\sum_{j=1}^m \ell_j(\mu(z))(1 - \cos\theta_j). \tag{124}$$

This motivates the following terminology:

Definition: Let $0 < \delta_k << 1$. We say:

- $x \in P$ is δ_k-close to (resp. δ_k-far from) the facet $F_j = \{\ell_j = 0\}$ if $\ell_j(x) \leq \delta_k$ (resp. $\geq \delta_k$).

- x is a δ_k-interior point if it is δ_k-far from all facets.

There are m possible cases according to the number of facets to which x is δ_k-close. Of course, x can be δ_k-close to at most m facets, in which case it is δ_k-close to the corner defined by the intersection of these facets. We thus define

$$\mathcal{F}_{\delta_k}(x) = \{r : |\ell_r(x)| < \delta_k\}. \tag{125}$$

We also let

$$\delta_k^{\#}(x) = \#\mathcal{F}_{\delta_k}(x) \tag{126}$$

denote the number of δ_k-close facets to x. Dual to the sets $\mathcal{F}_{\delta_k}$ above are the sets

$$\mathcal{F}_{F_{i_1},\dots,F_{i_r}} = \{x : \mathcal{F}_{\delta_k}(x) = \{i_1, \dots, i_r\}\}. \tag{127}$$

The asymptotics of $\mathcal{P}_{h^k}(\alpha)$ depend to the leading order on the determinant of the inverse of the Hessian of the phase of (114) (see also (103)) at $\theta = 0$. This Hessian is the same as the Hessian of the Kähler potential discussed in §2.2, and we recall that its inverse is the Hessian G of the symplectic potential. Hence, the asymptotics are in terms of the determinant of G, which has first order poles on ∂P. This indicates that the asymptotics are not uniform up to ∂P. We saw this as well in the explicit example of the Bargmann-Fock case. We define

$$\mathcal{G}_{\varphi,\delta_k}(x) = \left(\delta_\varphi(x) \cdot \prod_{j \notin \mathcal{F}_{\delta_k}(x)} \ell_j(x)\right)^{-1}, \tag{128}$$

where the functions δ_φ are defined in §2.2. When x is δ_k-far from all facets, then $\mathcal{G}_\varphi(x) = \det G_\varphi$ (cf. 58). We also define $\mathcal{P}_{h^k_{BF}}(k\ell_j(x))$ to be the unique real analytic extension of (79) to all $x \in [0, \infty)$. We then consider Bargmann-Fock type functions of type (79) which are adapted to the corners of our polytope P:

$$\mathcal{P}_{P,k,\delta_k}(x) = \prod_{j \in \mathcal{F}_{\delta_k}(x)} \mathcal{P}_{h^k_{BF}}(k\ell_j(\frac{\alpha}{k})) \tag{129}$$

and

$$\tilde{\mathcal{P}}_{P,k}(x) = \prod_{j=1}^{d} k^{-1}(2\pi\ell_j(x))^{1/2}\mathcal{P}_{h^k_{BF}}(k\ell_j(x)), \tag{130}$$

When we straighten out the corners by affine maps to be standard octants and separate variables $x = (x', x'')$ into directions near and far from ∂P, then $\mathcal{P}_{P,k,\delta_k}(x)$ is by definition a function of the near variables x' and $\mathcal{G}_{\varphi,\delta_k}(x)$ is by definition a function of the far variables x''.

The main result of this section is:

Proposition 6.1. *We have*

$$\mathcal{P}_{h^k}(\alpha) \;=\; C_m k^{\frac{m}{2}}\sqrt{\det G_\varphi(\frac{\alpha}{k})}\tilde{\mathcal{P}}_{P,k}(\frac{\alpha}{k})\;\left(1 + R_k(\frac{\alpha}{k}, h)\right), \tag{131}$$

where $R_k = O(k^{-\frac{1}{3}})$ and C_m is a positive constant depending only on m. The expansion is uniform in the metric h and may be differentiated in the metric parameter h twice with a remainder of the same order.

Equivalently, with $\delta_k^\#$ defined in (126) and by letting $\delta_k = k^{-\frac{2}{3}}$,

$$\mathcal{P}_{h^k}(\alpha) \;=\; C_m k^{\frac{1}{2}\,(m-\delta_k^\#(\frac{\alpha}{k}))}\sqrt{\mathcal{G}_{\varphi,\delta_k}(\frac{\alpha}{k})}\,\mathcal{P}_{P,k,\delta_k}(\frac{\alpha}{k})\;\Big(1+R_k(\frac{\alpha}{k},h)\Big), \tag{132}$$

where again $R_k = O(k^{-\frac{1}{3}})$.

The factor $k^{\frac{1}{2}\,(m-\delta_k^\#(\frac{\alpha}{k}))}$ is due to the fact that we apply complex stationary phase in $m-\delta_k^\#(\frac{\alpha}{k})$ variables to a complex oscillatory integral with symbol of order $k^{(m-\delta_k^\#(\frac{\alpha}{k}))}$.

As a check, let us consider the m-dimensional Bargmann-Fock case where $\delta_k^\#(\frac{\alpha}{k}) = r$, and with no loss of generality we will assume that the first r facets are the close ones. The factor k^m in the symbol of the Szegö kernel is then split into k^r (absorbed in $\mathcal{P}_{P,k,\delta_k}$) and k^{m-r} in the far factor. As discussed in §3.0.1, the far factor should have the form

$$k^{m-r}\prod_{j=r+1}^{m} e^{-\alpha_j}\frac{\alpha_j^{\alpha_j}}{\alpha_j!}\sim k^{m-r}\prod_{j=r+1}^{m}\alpha_j^{-\frac{1}{2}}.$$

The asymptotic factor in Proposition 6.1,

$$k^{\frac{1}{2}\,(m-\delta_k^\#(\frac{\alpha}{k}))}\left(\prod_{j=r+1}^{m}\frac{k}{\alpha_j}\right)^{\frac{1}{2}},$$

matches this expression. Here, and throughout the proof, we always straighten out the corner to a standard octant when doing calculations in coordinates.

Secondly, as a check on the remainder, we note that it arises from two sources. As will be seen in the proof, in 'far directions' the stationary phase remainder has the form $O(\frac{1}{kd(\frac{\alpha}{k},\partial P)})$ while in the near directions it has the form $O(k(d(\frac{\alpha}{k},\partial P))^2)$. When $d(\frac{\alpha}{k},\partial P)\sim k^{-\frac{2}{3}}$ the remainders match.

We break up the proof into cases according to the distance of $\frac{\alpha}{k}$ to the various facets as $k\to\infty$. Since we are studying joint asymptotics in (α,k), α may change with k.

6.2. **Interior asymptotics.**

- **$\frac{\alpha}{k}$ is δ-far from all facets**

We first consider the case where $\frac{\alpha}{k}$ is δ-far from all facets as an introduction to the problems we face. In this case, we obtain asymptotics of the integral (114) by a complex stationary phase argument. But it is not quite standard even in this interior case. In the next section, we goo on to consider the same expansion when δ depends on k.

Lemma **6.2.** *Assume that there exists $\delta > 0$ such that $\ell_j(\frac{\alpha}{k}) \geq \delta$ for all j, i.e., that $\frac{\alpha}{k}$ is δ-far from all facets. Then there exist bounded smooth functions $A_{-j}(x)$ on $\bar{P}$ such that*

$$\mathcal{P}_{h^k}(\alpha)\sim\; C_m k^{\frac{m}{2}}\,\sqrt{\det G_\varphi(\frac{\alpha}{k})}\;\left(1+\frac{A_{-1}(\frac{\alpha}{k})}{k}+\frac{A_{-2}(\frac{\alpha}{k})}{k^2}+\cdots+O_\delta(k^{-M})\right).$$

Here, $G_\varphi = \nabla^2 u$ (§2.2) and $G_\varphi(\frac{\alpha}{k})$ is its value at $\frac{\alpha}{k}$; its norm is $O(\delta^{-1})$ and its determinant is $O(\delta^{-m})$.

Before going into the proof, we note that the only assumption on the limit points of $\frac{\alpha}{k}$ is that they are δ-far from facets. The lattice points α are implicitly allowed to vary with k. Asymptotics of the left side clearly depend on the asymptotics of the points $\frac{\alpha}{k}$, and the Lemma states how they do so.

Proof. We now apply the complex stationary phase method, or more precisely its proof. The usual complex stationary phase theorem applies to exponents $k\Phi(\theta)$ where $\Phi(\theta)$ is a positive phase function with a non-degenerate critical point at $\theta = 0$. In our case, the phase is also k-dependent since it depends on $\frac{\alpha}{k}$ and the asymptotics of (120) therefore depend on the asymptotics of $\frac{\alpha}{k}$ in the domain $d(\frac{\alpha}{k}, \partial P) \geq \delta$. Our stated asymptotics also depend on the behavior of $\frac{\alpha}{k}$ in the same way.

Although the exact statement of complex stationary phase [Hö] (Theorem 7.7.5) does not apply, the proof applies without difficulty in this region. Namely, we introduce a cutoff $\chi_\delta(\theta) = \chi(\delta^{-1}\theta) \in C^\infty(\mathbf{T}^m)$ which $\equiv 1$ in a δ-neighborhood of $\theta = 0$ and which vanishes outside a 2δ-neighborhood of $\theta = 0$. We decompose the integral into its χ_δ and $1 - \chi_\delta$ parts. A standard integration by parts argument, essentially the same as in Lemma 1.2 shows that the $1 - \chi_\delta$ term is $= O(\delta^{-M} k^M)$ for all $M > 0$. In the χ_δ part the integral may be viewed as an integral over $\mathbb{R}^m$ and we may apply the Plancherel theorem as in the standard stationary phase argument to obtain

$$\mathcal{P}_{h^k}(\alpha) \sim \frac{C_m}{\sqrt{\det(kH_{\frac{\alpha}{k}})}} \int_{\mathbb{R}^m} e^{-\langle (kH_{\frac{\alpha}{k}})^{-1}\xi, \xi\rangle} \mathcal{F}_{\theta\to\xi}\left(e^{kR_3(\theta,k,\mu^{-1}(\frac{\alpha}{k})))} A_k\big(\mu_h^{-1}(\tfrac{\alpha}{k}), e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k}), 0\big)\right)(\xi) d\xi \ , \tag{133}$$

where $\mathcal{F}_{\theta\to\xi}$ is the Fourier transform.

The stationary phase expansion (see [Hö], Theorem 7.7.5) is the following:

$$\sim (\frac{2\pi}{k})^{m/2} \frac{e^{\frac{i\pi}{4}\mathrm{sgn} H_{\frac{\alpha}{4}}}}{\sqrt{|\det H_{\frac{\alpha}{k}}|}} \sum_j^\infty k^{-j} \mathcal{P}_{\frac{\alpha}{k},j} A_k\big(\mu_h^{-1}(\frac{\alpha}{k}), e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k}), 0\big)|_{\theta=0} \tag{134}$$

where

$$\mathcal{P}_{\frac{\alpha}{k},j} A_k(0) = \sum_{\nu-\mu=j} \sum_{2\nu\geq 3\mu} \frac{i^{-j} 2^{-\nu}}{\mu!\nu!} \langle H_{\frac{\alpha}{k}}^{-1} D_\theta, D_\theta\rangle^\nu (A_k R_3^\mu)|_{\theta=0} \tag{135}$$

The only change in the standard argument is that we have a family of quadratic forms $H_{\frac{\alpha}{k}}$ depending on parameters (α, k) rather than a fixed one. But the standard proof is valid for this modification. As in the standard proof, we expand the exponential in (133) and evaluate the terms and the remainder of the exponential factor just as in [Hö] Theorem 7.7.5, to obtain (134), which becomes

$$\begin{aligned} &\big(\det(k^{-1} G_\varphi(\tfrac{\alpha}{k}))\big)^{1/2} \textstyle\sum_{j=0}^M k^{-j} \left(\langle G_\varphi(\tfrac{\alpha}{k})(D_\theta, D_\theta\rangle\right)^j \chi_\delta e^{kR_3(k,\theta,\mu^{-1}(\frac{\alpha}{k}))} A_k\big(\mu_h^{-1}(\tfrac{\alpha}{k}), e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k}), 0, k\big)|_{\theta=0} \\ &+ O(k^{-M} \sup_{\theta\in supp\chi_\delta} \left| \langle G_\varphi(\tfrac{\alpha}{k})(D_\theta, D_\theta\rangle^M \chi_\delta e^{kR_3(k,\theta,\mu^{-1}(\frac{\alpha}{k}))} A_k\big(\mu_h^{-1}(\tfrac{\alpha}{k}), e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k}), 0, k\big)\right| . \end{aligned} \tag{136}$$

Here, G_x is the Hessian of the symplectic potential, i.e., the inverse of $H_{\mu^{-1}(x)}$. (cf. 2.2). We recall that G_x has poles x_j^{-1} of order one when $x \in \partial P$. When $d(\frac{\alpha}{k}, \partial P) \geq \delta$, its norm is therefore $O(\delta^{-1})$ and its determinant is $O(\delta^{-m})$. Since R_3 vanishes to order 3 at the critical point, the terms of the expansion can be arranged into terms of descending order as in the standard proof. If we recall that the leading term of S is k^m, we obtain the statement of Proposition 6.1 in the δ-interior case.

□

- **$\frac{\alpha}{k}$ is δ_k-far from facets with $k\delta_k \to \infty$**

We continue to study the complex oscillatory integral (114) but now allow $\frac{\alpha}{k}$ to become δ_k-close to some facet, and obtain a stationary phase expansion (with very possibly slow decrease in the steps) under the condition that $k\delta_k \to \infty$. This should be feasible since the phase $k\Phi_{\frac{\alpha}{k}}$ is still rapidly oscillating in this region, albeit at different rates in different directions according to the proximity of $\frac{\alpha}{k}$ to a particular facet. The principal complication is as as follows:

- The Hessian $G_\varphi(\frac{\alpha}{k})$ now has components which blow up like δ_k^{-1} near the close facets. In the stationary phase expansion, we get factors of
$$k^{-j}\langle G_\varphi(\frac{\alpha}{k})D_\theta, D_\theta\rangle^j A_k\big(\mu_h^{-1}(\frac{\alpha}{k}), e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k}), 0, k\big) R_3(k,\theta,\mu^{-1}(\frac{\alpha}{k}))^\mu$$
both in the expansion and remainder. We must verify that these terms still are of descending order.

As a guide, we note that by (95), the Bargmann-Fock phase with $\mu(z) = \frac{\alpha}{k}$ is given by
$$\Phi_{BF,\frac{\alpha}{k}}(\theta) = \langle \frac{\alpha}{k}, e^{i\theta} - i\theta\rangle = \langle \cos\theta + i(\sin\theta - \theta), \frac{\alpha}{k}\rangle,$$
while the amplitude is constant. In this case, the phase factors into single-variable factors and one can employ the complex stationary phase method separately to each. In the general case, we will roughly split the variables θ into two groups (θ', θ''), depending on $\frac{\alpha}{k}$, so that the θ' variables are paired with the small components of $\frac{\alpha}{k}$ while the θ'' variables are paired with its large components. The complex stationary phase method applies equally to either $d\theta'$ or $d\theta''$ integral, but the orders of the terms are determined by the proximity of $\frac{\alpha}{k}$ to the facets.

Lemma **6.3.** *Let $\{\delta_k\}$ be a sequence such that $k\delta_k \to \infty$. Assume that $\ell_j(\frac{\alpha}{k}) \geq \delta_k$ for all j, i.e.,, that $\frac{\alpha}{k}$ is δ_k far from all facets. Then in the notation of Lemma 6.2, we have*
$$\mathcal{P}_{h^k}(\alpha) \sim C_m k^{\frac{m}{2}} \sqrt{\det G_\varphi(\frac{\alpha}{k})}\, \left(1 + \frac{A_{-1}(\frac{\alpha}{k})}{k} + \frac{A_{-2}(\frac{\alpha}{k})}{k^2} + \cdots + \frac{A_{-M}(\frac{\alpha}{k})}{k^2} + O(k\delta_k)^{-M}\right),$$
where now
$$A_{-j}(\frac{\alpha}{k}) \leq D\delta_k^{-1} = Cd(\frac{\alpha}{k}, \partial P)^{-j}.$$

Remark: One may regard this as an expansion in the semi-classical parameter $(k\delta_k)^{-1} = (kd(\frac{\alpha}{k}, \partial P))^{-1}$.

Proof. We need to prove that the expansion (157) may be re-arranged into terms of decreasing order and that the remainder can be made to have an arbitrarily small order k^{-M} by taking sufficiently many terms.

To analyze the expansion (157), we begin with a decomposition of the inverse Hessian $G_{\frac{\alpha}{k}}$, which is the Hessian of the symplectic potential, which has the form u_0+g where $g \in C^\infty(\bar{P})$ and where u_0 is the canonical symplectic potential (56). We continue to fix a small $\delta > 0$ as in the previous section, and consider the facets to which $\frac{\alpha}{k}$ is δ-close. We use the affine transformation to map these δ-close facets to the hyperplanes $x'_j = 0$. In these coordinates, we may write the symplectic potential as

$$u_\varphi(x) = \sum_{j \in \mathcal{F}_\delta} x'_j \log x'_j + g(x), \tag{137}$$

where the Hessian of g is bounded with bounded derivatives near $\frac{\alpha}{k}$. The Hessian $G_{\frac{\alpha}{k}}$ then decomposes into the sum,

$$G_\varphi(x) = \sum_{j \in \mathcal{F}_\delta(\frac{\alpha}{k})} \frac{1}{x'_j}\, \delta_{jj} + \nabla^2 g := G^s_x + \nabla^2 g, \tag{138}$$

where $\nabla^2 g$ is smooth up to the boundary in a neighborhood of $\mathcal{F}_{\delta_k}(\frac{\alpha}{k})$. The notation $G_\varphi(x)^s$ refers to the 'singular part' of G_x. The choice of δ is not important; we are allowing $\frac{\alpha}{k}$ to become δ_k close to some facets, and for any choice of δ, the sum will include such facets.

The decomposition (138) of the inverse Hessian induces a block decomposition of the Hessian operator $\langle G_{\frac{\alpha}{k}} D_\theta, D_\theta\rangle$. The chage of variables to x above induces an affine change of the θ variables, as follows: We are using the coordinates (x', x'') on P with x' denoting the linear coordinates in the directions of the normals to the facets $\mathcal{F}_\delta(\frac{\alpha}{k})$. The normals corresponding to $\mathcal{F}_\delta(\frac{\alpha}{k})$ generate the isotropy algebra of the sub-torus $(\mathbf{T}^m)'$ fixing the near facets. We have $\mathbf{T}^m = (\mathbf{T}^m)' \times (\mathbf{T}^m)''$, and denote the corresponding coordinates by (θ', θ'').

The Hessian operator in these coordinates has the form

$$\langle G_\varphi(\frac{\alpha}{k}) D_\theta, D_\theta\rangle = \sum_{j \in \mathcal{F}_\delta(\frac{\alpha}{k})} \frac{k}{\alpha'_j}\, D^2_{\theta'_j \theta'_j} + \langle G_\varphi(\frac{\alpha}{k})'' D_\theta, D_\theta\rangle, \tag{139}$$

where the second term has bounded coefficients. Evidently, the change to the interior stationary phase expansion is entirely due to the singular part of the Hessian operator,

$$\langle G^s \frac{\alpha}{k} D_\theta, D_\theta\rangle := \sum_{j \in \mathcal{F}_\delta(\frac{\alpha}{k})} \frac{k}{\alpha'_j}\, D^2_{\theta'_j \theta'_j}. \tag{140}$$

We now consider the order of magnitude of the terms in the jth term (135), which has the form

$$k^{-\nu}\, (\langle G_\varphi(\frac{\alpha}{k}) D_\theta, D_\theta\rangle^\nu A_k\big(\mu_h^{-1}(\frac{\alpha}{k}), e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k}), 0, k\big) R_3(k, \theta, \mu^{-1}(\frac{\alpha}{k}))^\mu)|_{\theta=0} \tag{141}$$

with $\nu - \mu = j$ and with $2\nu \geq 3\mu$. The latter constraint is evident from the fact that R_3 vanishes to order 3.

Using (139), $\langle G_\varphi(\frac{\alpha}{k}) D_\theta, D_\theta\rangle^\nu$ becomes a sum of terms of which the most singular is

$$\langle G^s_\varphi(\frac{\alpha}{k}) D_\theta, D_\theta\rangle^\nu := (\sum_{j \in \mathcal{F}_\delta(\frac{\alpha}{k})} \frac{k}{\alpha'_j}\, D^2_{\theta'_j \theta'_j})^\nu.$$

We will only discuss the terms generated by this operator; the discussion is similar but simpler for the other terms. In the extreme case of $\langle G''_{\frac{\alpha}{k}} D_\theta, D_\theta\rangle^\nu$, the discussion is essentially the same as in the previous section; in particular, (141) has order k^{-j}.

The problem with each application of $\langle G^s_\varphi(\frac{\alpha}{k}) D_\theta, D_\theta\rangle$ is that it raises the order by the maximum of $\frac{k}{\alpha'_j}$, which may be as large as $k\delta_k$. Although we have an overall k^{-j} and constraints $\nu - \mu = j, 2\nu \geq 3\mu$, it is not hard to check that these are not sufficient to produce negative exponents of k.

The key fact which saves the situation is that the phase $\Phi_{\frac{\alpha}{k}}$ and amplitude S depend on θ as functions of $e^{i\theta}|\mu^{-1}(\frac{\alpha}{k})|^2$. Although R_3 has a more complicated θ-dependence, its third and higher derivatives are the same as those of $\Phi_{\frac{\alpha}{k}}$, and it is obvious that only these contibute to (141). Hence derivatives in θ bring in factors of $|\mu^{-1}(\frac{\alpha}{k})|^2$ by the chain rule. Due to the behavior of the moment map near a facet, these chain rule factors cancel a square root of the blowing up factor in $G_\varphi(\frac{\alpha}{k})$. This turns out to be sufficient for a descending series due to the power k^{-j} and constraint $2\nu \geq 3\nu$.

Before giving all the details, let us consider what should be the 'worst' terms of (141), i.e., the ones with the least decay in k. Each factor of R_3 comes with a factor of k, so one would expect terms with large μ to be 'worst'. The 'worst' term will be one with a maximum μ and where a maximum number of applications on operator $\langle G^s_{\frac{\alpha}{k}} D_\theta, D_\theta\rangle^\nu$ is applied to the 'chain-rule' factors $(e^{i\theta}|\mu^{-1}(\frac{\alpha}{k}))_j|^2$ (the jth component of this vector), obtained from an application of some $D_{\theta'_j}$ to S or to R_3. If instead we differentiate S or R_3 again, we pull out another chain rule factor, which cancels more of the bad coefficient $\frac{k}{\alpha'_j}$.

We now give the rigorous argument. The terms of (141) have the form,

$$k^{-\nu+\mu}\, G_\varphi(\frac{\alpha}{k})^{i_1 j_1} \cdots G_\varphi(\frac{\alpha}{k})^{i_\nu j_\nu} D^{\beta_1} R_3 \cdots D^{\beta_\mu} R_3 D^{\beta_{\mu+1}} S, \tag{142}$$

where $|\beta| = 2\nu$ and where D^{β_q} denote universal constant multiples of the multinomial differential operators $\frac{\partial^{\beta_q}}{\partial\theta^{n_1}\cdots\partial\theta^{n_{\beta_q}}}$ where the union of the indices agrees with $\{i_1, j_1, \dots, i_\nu, j_\nu\}$. We need each $|\beta_q| \geq 3$ for $q \leq \mu$ to remove the zero of R_3. If we only consider the most singular term, then we need $i_q = j_q \in \mathcal{F}_\delta(\frac{\alpha}{k})$. In this case our term becomes

$$k^{-\nu+\mu} \left(\prod_{j=1; q_j\in\mathcal{F}_\delta(\frac{\alpha}{k})}^{\nu} \frac{k}{\alpha'_{q_j}}\right) D^{\beta_1} R_3 \cdots D^{\beta_\mu} R_3 D^{\beta_{\mu+1}} S, \tag{143}$$

For each factor $\frac{k}{\alpha'_{q_j}}$, there exist two factors of the associated differential operator $\frac{\partial}{\partial\theta_{q_j}}$. When one is applied to either R_3 or S it pulls out a chain rule factor $e^{i\theta_{q_j}}|\mu^{-1}(\frac{\alpha}{k}))_j|^2$. The second could be applied to this factor, hence need not introduce any new factors of $|\mu^{-1}(\frac{\alpha}{k}))_j|^2$. We now estimate (143) by

$$|(143)| \leq k^{-\nu+\mu} \left(\prod_{j=1; q_j\in\mathcal{F}_\delta(\frac{\alpha}{k})}^{\nu} \frac{k}{\alpha'_{q_j}}\right) \prod_{j=1}^{\mu} |\mu^{-1}(\frac{\alpha}{k})_{q_j}|^2. \tag{144}$$

Now $\mu^{-1}(x) = \nabla u_\varphi(x)$ in ρ coordinates. So the square of the q_jth component of $\mu^{-1}(\frac{\alpha}{k})$ equals $\log\frac{\alpha_{q_j}}{k}$ plus a bounded remainder in ρ coordinates; here as above we are using the x_j

coordinates adapted to $\frac{\alpha}{k}$. It follows that in the z coordinates adapted to the facets of $\mathcal{D}$ corresponding to the hyperplanes $x'_j = 0$, with $|z_j|^2 = e^{\rho_j}$, $|\mu^{-1}(\frac{\alpha}{k})_{q_j}|^2 \leq C\frac{\alpha_j}{k}$. The constant C comes from the smooth part of the symplectic potential and has a uniform bound. As a check on the square root, we note that for the approximating Bargmann-Fock model we have $|z_j|^2 = \frac{\alpha_j}{k}$. It follows from (144) and $\frac{k}{\alpha'_j} \leq Cd(\frac{\alpha}{k}, \partial P)^{-1}$ that

$$\begin{aligned} |(143)| \leq Ck^{-\nu+\mu} \left(\prod_{j=1; q_j \in \mathcal{F}_\delta(\frac{\alpha}{k})}^{\nu} \frac{k}{\alpha'_{q_j}}\right) \prod_{j=1}^{\mu} \frac{k}{\alpha'_{q_j}} &\leq Ck^{-\nu+\mu} d(\tfrac{\alpha}{k}, \partial P)^{-\nu+\mu} \\ &= C(kd(\tfrac{\alpha}{k}, \partial P))^{-j}, \end{aligned} \tag{145}$$

Effectively, the 'semi-classical parameter' has changed from k^{-1} to $k^{-1}d(\frac{\alpha}{k}, \partial P)^{-1}$, a natural parameter in boundary problems. As long as $kd(\frac{\alpha}{k}, \partial P) \to \infty$ at some fixed rate, we obtain a descending expansion.

□

6.3. **Boundary zones: Corner zone .** Having dealt with the case where $|\frac{\alpha_j}{k}| \geq \delta_k$, we now turn to the complementary cases where $d(\mu(z), \partial P) \leq \delta_k$, i.e., at least for one j, $|\frac{\alpha_j}{k}| \leq \delta_k$ or equivalently, $\frac{\alpha}{k}$ is δ_k-close to at least one facet. The choice of the scale δ_k is so that it it is small enough to justify the Bargmann-Fock approximation in the 'near' variables.

In this section, we consider the extreme 'corner' case where $\mu(z)$ lies in a δ_k-corner, i.e., where there exists a vertex $v \in \partial P$ so that $d(\mu(z), v) \leq \delta_k$. Putting $v = 0$, the assumption becomes that $|\mu(z)| \leq C\delta_k$. Our main object is to determine the scale δ_k so that the Bargmann-Fock approximation is valid. That is, for $z = \mu^{-1}(\frac{\alpha}{k})$ we should have in the multi-index notation of §2.6 (see (113)),

$$\mathcal{P}_{h^k}(\alpha) \sim \mathcal{P}_{h^k_{BF}}(\alpha) = k^m (2\pi)^{-m} \int_{\mathbf{T}^m} e^{-k\left(\sum_{j=1}^m H_{j\bar{j}}(e^{i\theta_j} - 1 + i\theta_j, \frac{\alpha_j}{k}\rangle\right)} d\theta. \tag{146}$$

LEMMA **6.4.** *If $\mu(z)$ lies in a δ_k-corner, then*

$$\mathcal{P}_{h^k}(\alpha) = C_m \mathcal{P}_{h^k_{BF}}(\alpha)(1 + O(\delta_k) + O(k\delta_k^2)) = C_m \mathcal{P}_{h^k_{BF}}(\alpha)(1 + O(k\delta_k^2)).$$

Proof. We may assume that $v = 0$ and that the corner is a standard octant. The phase is

$$k\left(F_{\mathbb{C}}(|z|^2 e^{i\theta}) - F(|z|^2) - \langle\frac{\alpha}{k}, \theta\rangle\right). \tag{147}$$

We Taylor expand $F(w)$ at $w = 0$:

$$F_{\mathbb{C}}(e^{i\theta}|z|^2) = F(0) + F'(0)e^{i\theta}|z|^2 + O(|z|^4),$$

so that

$$F_{\mathbb{C}}(|z|^2 e^{i\theta}) - F(|z|^2) = F'(0)|z|^2(e^{i\theta} - 1)) + O(|z|^4).$$

Since $|z|^2 = O(\delta_k)$, it follows that k times the quartic remainder is $O(k\delta_k^2) = o(1)$ as long as $\delta_k = o(\frac{1}{\sqrt{k}})$. Hence this part of the exponential is a symbol of order zero and may be absorbed into the amplitude. Further we note that $F'(0)|z|^2 = \mu(z) + O(|z|^4)$ and therefore we have

$$k\left(F_{\mathbb{C}}(|z|^2 e^{i\theta}) - F(|z|^2) - i\langle\frac{\alpha}{k}, \theta\rangle\right) = k\mu(z)((1 - \cos\theta) + i(\sin\theta - \theta)) + O(|z|^4)).$$

It follows that when $\mu(z) = \frac{\alpha}{k} = O(\delta_k)$, the phase equals

$$\alpha((1-\cos\theta) + i(\sin\theta - \theta)) + O(k\delta_k^2).$$

Absorbing the $e^{O(k\delta_k^2)} = 1 + O(k\delta_k^2)$ term into the amplitude produces an oscillatory integral with the same phase function as for the Bargmann-Fock kernel.

Now let us consider the amplitude of the integral. We continue to use the notation of Proposition 4.2. The amplitude has a semi-classical expansion $A_k(z,w) \sim k^m a_0(z,w) + k^{m-1}a_1(z,w) + \cdots$. Further, the $\mathbf{T}^m$-invariance implies that $A_k(e^{i\theta}z, e^{i\theta}w) = A_k(z,w)$. The leading order amplitude equals 1 when $z = w$ and thus

$$a_0(z, e^{i\theta}w) = 1 + Ce^{i\theta}|z|^2 + O(|z|^4),$$

hence the full symbol satisfies

$$A_k(z, e^{i\theta}z) = k^m(1 + Ce^{i\theta}|z|^2 + \cdots) + O(\delta_k^2).$$

When $\mu(z) = \frac{\alpha}{k} = O(\delta_k)$ we thus have

$$A_k(z, e^{i\theta}z) = k^m\left(1 + Ce^{i\theta}\frac{\alpha}{k} + O(\delta_k^2)\right).$$

Therefore, $\mathcal{P}_{h^k}(\alpha) = \mathcal{P}_{h^k_{BF}}(\alpha)(1 + O(\delta_k) + O(k\delta_k^2))$ in the corner region.

$\square$

6.4. **Boundary zones: Mixed boundary zone .** Now let us consider the general case where $d(\mu(z), \partial P) \leq \delta_k$, but where $\mu(z)$ is not necessarily in a corner. Thus, at least one component $\frac{\alpha_j}{k} = O(\delta_k)$ but not all components need to satisfy this condition. We refer to this case as 'mixed' since some components are small and some are not.

The basic idea to handle this case is to split the components into 'near' and 'far' parts, to use Taylor expansions and Bargmann-Fock approximations in the near components, and to use complex stationary phase in the far components. By §6.2, complex stationary phase works for any sequence δ_k satisfying $k\delta_k \to \infty$, and by §6.3 the Taylor-Bargmann-Fock approximation works whenever $\delta_k = o(\frac{1}{\sqrt{k}})$, so we have some flexibility in choosing δ_k.

Remark: In fact, we see that both the complex stationary phase and the Bargmann-Fock approximations are valid for k satisfying (for instance) $\frac{C\log k}{k} \leq \delta_k \leq C'\frac{1}{\sqrt{k}\log k}$, although the remainder estimates will not be equally sharp by both methods. In fact, the stationary phase remainder is of order $(k\delta_k)^{-1}$ while the Bargmann-Fock remainder is of order $k\delta_k^2$; the two remainders agree when $\delta_k = k^{-\frac{2}{3}}$ and for small δ_k the Bargmann-Fock remainder is smaller.

We first choose linear coordinates so that $\mu(z) = \frac{\alpha}{k}$ is δ_k' close to the first r facets and δ_k' far from the $p := m - r$ remaining facets, and by an affine map we position the first r facets as the first r coordinate hyperplanes at $x = 0$, and the remaining facets as the remaining coordinate hyperplanes. We use coordinates (x', x'') relative to this splitting. We also write the z variables as (z', z'') in the corresponding slice-orbit coordinates and (θ', θ'') as the associated coordinates on $\mathbf{T}^m$.

We now introduce two small scales, a smaller one δ_k' to define the nearest facets, and a larger one δ_k''. The Bargmann-Fock approximation will be used in x' variables which are δ_k' close to a facet. It is sometimes advantageous to use the Bargmann-Fock approximation also x'' which are δ_k'' small, but the complex stationary phased method is also applicable. In the following, we continue to use the notation above Proposition 6.1.

LEMMA **6.5.** *Assume $\mu(z)$ lies in the mixed boundary zone $\{|x'| \leq \delta'_k, |x''| \leq \delta''_k\}$. If*

$$\eta_k = k^{-1}(\delta''_k)^{-1} + k(\delta'_k)^2 + k(\delta'_k)^2\delta''_k + \delta'_k \to 0,$$

then $\mathcal{P}_{h^k}(\alpha)$ has an asymptotic expansion

$$\mathcal{P}_{h^k}(\alpha) = C_m k^{m-\frac{p}{2}} \sqrt{\mathcal{G}_{\varphi,\delta_k}(\frac{\alpha}{k})}\mathcal{P}_{P,k,\delta'_k}(\alpha)(1 + O(\eta_k)).$$

Our strategy for obtaining asymptotics of $\mathcal{P}_{h^k}(\alpha)$ in this case is as follows:

- We employ steepest descent in the p directions which are δ''_k-far from all facets, i.e., in the x'' variables. This removes the x'' variables and produces an expansion analogous to that of Lemma 6.2.
- In the remaining x' variables, we Taylor expand the phase and amplitude in the directions δ_k-close to ∂P as in §6.3.
- We thus obtain universal asymptotics to leading order depending only on the number of facets to which $\frac{\alpha}{k}$ is δ_k-close.

Proof. We are still working on the oscillatory integral with phase (114), but we now treat it as an iterated complex oscillatory integral in the variables (θ', θ'') defined above. We first consider the $d\theta''$ integral,

$$I_k(\theta', \tfrac{\alpha}{k}) := (2\pi)^{-p} \int_{\mathbf{T}^p} e^{k(F_{\mathbb{C}}(e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k})) - F(\mu_h^{-1}(\frac{\alpha}{k})))} A_k\big(e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k}), \mu_h^{-1}(\tfrac{\alpha}{k}), 0, k\big) e^{-i\langle\alpha,\theta\rangle} d\theta'', \tag{148}$$

where p is the number of θ'' variables. We also let $r = m - p$ be the number of θ' variables. We now verify that we may apply the complex stationary phase method to the $d\theta''$ integral for fixed θ'. Throughout this section, we put $z = \mu^{-1}(\frac{\alpha}{k})$ and often write $|z'|^2, |z''|^2$ for the modulus square of the associated complex coordinate components of this point in the open orbit.

The first step is to simplify the complex phase. As in §6.3, we Taylor expand $F_{\mathbb{C}}(e^{i\theta'}|z'|^2, e^{i\theta''}|z''|^2)$ in the z' variable (and only in the z' variable) to obtain

$$F_{\mathbb{C}}(e^{i\theta'}|z'|^2, e^{i\theta''}|z''|^2) = F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) + F'_1(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2 + O(|z'|^4),$$

where F_1 is the z'-derivative of F. The phase is then

$$\begin{aligned} &k\left(F_{\mathbb{C}}(e^{i\theta'}|z'|^2, e^{i\theta''}|z''|^2) - F(|z'|^2, |z''|^2) - i\langle\tfrac{\alpha'}{k}, \theta'\rangle) - i\langle\tfrac{\alpha''}{k}, \theta''\rangle\right) \\ &= k\left(F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) - F(0, |z''|^2)\right) + k\left(F'_1(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2 - F'_1(0, |z''|^2)|z'|^2\right) \\ &-k\left(i\langle\tfrac{\alpha'}{k}, \theta'\rangle + i\langle\tfrac{\alpha''}{k}, \theta''\rangle\right) + O(k|z'|^4). \end{aligned} \tag{149}$$

We now absorb the exponentials of the terms $kO(|z'|^4)$, $ki\langle\frac{\alpha'}{k}, \theta'\rangle$ of the phase (149) into the amplitude, i.e., we take the new amplitude A''_k to be the old one A_k multiplied by this factor. The term $kO(|z'|^4)$ is $o(1)$, while $ki\langle\frac{\alpha'}{k}, \theta'\rangle$ is constant in θ'', so their exponentials are symbols in θ'' and may be absorbed into the amplitude. Morevoer, the term $-F'_1(0, |z''|^2)|z'|^2$ is independent of θ'' so its exponential may also be absorbed into the amplitude

The phase function for the $d\theta''$ integral thus simplifies to

$$k\left(F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) - F(0, |z''|^2)\right) + k\left(F'_1(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2\right) - ki\langle\frac{\alpha''}{k}, \theta''\rangle \tag{150}$$

Due to the presence of $|z'|^2$, the terms $k(F_1'(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2 - F_1'(0, |z_2|^2)|z'|^2)$ are $O(k\delta')$, hence of much lower order than the remaining terms. To simplify the phase further, we now argue that their exponentials can also be absorbed into the amplitude, albeit as exponentially growing rather than polynomially growing factors in k. Since $F_1'(0, |z_2|^2)|z'|^2)$ is independent of θ'', it can be factored out of the θ'' integral, so the key factor is

$$E_k(\theta'') := e^{k(F_1'(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2)}, \tag{151}$$

where in the notation for E_k we omit its dependence on the parameters $|z''|^2, |z'|^2, \theta'$. Thus we would like to show that complex stationary phase method applies to the complex oscillatory integral with phase

$$\Phi''(\theta'') := \Big(F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) - F(0, |z''|^2\Big) - i\langle \frac{\alpha''}{k}, \theta''\rangle \tag{152}$$

and with the amplitude $A_k''(\theta'')$ given by the original amplitude A_k multiplied by

$$\exp k\left(F_1'(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2 - F_1'(0, |z''|^2)|z'|^2 + i\langle \frac{\alpha''}{k}, \theta''\rangle + O(k|z'|^4\right).$$

The 'amplitude' is of exponential growth but its growth is of strictly lower exponential growth than the 'phase' factor.

The next (not very important) observation is that by (116), the real part of complex phase damps the integral so that the integrand is negligible on the complement of the set

$$|\theta''| \leq C \frac{\delta'}{d''(\mu(z), \partial P)} \tag{153}$$

modulo rapidly decaying errors. This follows by splitting up the sum in (122)-(123) into the close facets to z and the far facets. The integrand is negligible unless $|\Re\Phi| \leq C\frac{\log k}{k}$; hence it is negligible unless

$$\begin{aligned} d(e^{i\theta}z, z)^2 \;&\sim\; \textstyle\sum_{j \in \mathcal{F}_{\delta_k}(\mu(z))} (1 - \cos\theta_j'')\ell_j''(\mu(z)) + O\left(|z'|^2\right) \\ &\sim\; \textstyle\sum_{j \in \mathcal{F}_{\delta_k}(\mu(z))} (\theta_j'')^2 \ell_j''(\mu(z)) + O\left(\delta_k'\right) \\ &\leq\; C\frac{\log k}{k} \iff \theta_j^2 \leq \frac{O(\delta_k') + O(\frac{\log k}{k})}{d''(\mu(z), \partial P)},\ \forall j \in \mathcal{F}_{\delta_k}(\mu(z)). \end{aligned} \tag{154}$$

Under the assumption that $d''(\mu(z), \partial P) \geq \delta_k''$, the integrand is rapidly decaying unless $\theta_j^2 \leq C\frac{\delta_k'}{\delta_k''}$. We could introduce a cutoff of the form $\chi(\sqrt{\frac{\delta_k''}{\delta_k'}}\theta)$, but for our purposes, it suffices to use a smooth cutoff $\chi_\delta(\theta'')$ around $\theta'' = 0$ with a fixed small δ so that we may use local θ'' coordinates. We then break up the integral using $1 = \chi_\delta + (1 - \chi_\delta)$. The $(1 - \chi_\delta)$ term is rapidly decaying and may be neglected.

We observe that $\nabla_{\theta''} F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) = i\mu_{\mathbb{C}}''(|z''|, e^{i\theta''}|z''|)$ is the complexified moment map for the subtoric variety $z' = 0$, and we can use Proposition 2.1 to see that its only critical point in the domain of integration is at $\theta'' = 0$. We denote the Hessian of the phase (152) at $\theta'' = 0$ by

$$H''_{|z''|^2} = \nabla^2_{\theta''}\Phi''(\theta'')|_{\theta''=0} = \nabla^2_{\theta''} F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2)|_{\theta''=0}, \tag{155}$$

and observe that it equals $iD\mu''_{\mathbb{C}}(|z''|, e^{i\theta''}|z''|)$, the derivative of the moment map from the subtoric variety to its polytope. By the same calculation that led to (138), the $\theta'' - \theta''$ block of the inverse Hessian operator has the form

$$G''_{\varphi}(x'') = \sum_{j=1}^{p} \frac{1}{x''_j} \delta_{jj} + \nabla^2 g := G^s_{x''} + \nabla^2 g, \tag{156}$$

where $|x''| \geq \delta''_k$.

We now must verify that the complex stationary phase expansion

$$\left(\det k^{-1} G''_{\varphi}(|z''|^2)\right)^{1/2} \sum_{j=1}^{M} k^{-j} \left(\langle G''_{\varphi}(|z''|^2)(D_{\theta''}, D_{\theta''}\rangle\right)^j \chi_\delta A''_k(\theta'')|_{\theta''=0} \tag{157}$$

is a descending expansion in well-defined steps and that the remainder

$$k^{-M} \sup_{\theta'' \in supp \chi_\delta} \left| \langle G''_{\varphi}(|z''|^2)(D_{\theta''}, D_{\theta''}\rangle^M A''_k(\theta'')^M \; \chi_\delta A''_k(\theta'') \right|. \tag{158}$$

is of arbitrarily small order as M increases.

We first note that the Hessian operator $k^{-1}\langle G''_{\varphi}(|z''|^2)(D_{\theta''}, D_{\theta''}\rangle$ brings in a net order of $k^{-1}(\delta''_k)^{-1}$, since the coefficients $\frac{1}{x''}$ in the singular part are bounded by $(\delta''_k)^{-1}$. The maximal order terms arise from applying the Hessian operator to the factor E_k. Each derivative can bring down a factor of $kF'_1(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2) = O(k\delta'_k\delta''_k)$. Since there are two θ'' derivatives for each $k^{-1}(\delta''_k)^{-1}$ the maximum order in k from a single factor of $k^{-1}\langle G''_{\varphi}(|z''|^2)(D_{\theta''}, D_{\theta''}\rangle$ applied to A''_k is of order

$$\eta_k = k^{-1}(\delta''_k)^{-1}((k\delta'_k)^2(\delta''_k)^2 + k\delta'_k\delta''_k) = k(\delta'_k)^2\delta''_k + \delta'_k.$$

In particular this is the order of magnitude of the sub-dominant term. Therefore, to obtain a descending expansion in steps of at least $k^{-\epsilon_0}$, we obtain the following necessary and sufficient condition on (δ'_k, δ''_k):

$$\eta_k \leq Ck^{-\epsilon_0}. \tag{159}$$

Under this condition, the series and remainder will go down in steps of $k^{-\epsilon_0}$.

With these choices of (δ'_k, δ''_k), the complex stationary phase expansion gives an asymptotic expansion in powers of $k^{-\epsilon_0}$. Recalling that the unique critical point occurs at $\theta'' = 0$, the remaining $d\theta'$ integral is given by the dimensional constant $C_m(2\pi)^{-r}$ times

$$\mathcal{P}_{h^k}(\alpha) \sim \left(\det(k^{-1}G''_{\varphi}(|z''|^2)\right)^{1/2} \int_{\mathbf{T}^r} e^{ik\langle \frac{\alpha'}{k}, \theta' \rangle} \sum_{j=1}^{M} k^{-j} \left(\langle G''_{\varphi}(|z''|^2)(D_{\theta''}, D_{\theta''}\rangle\right)^j \chi_\delta A''_k(\theta', 0) d\theta', \tag{160}$$

plus the integral of the remainder (158), which is uniform in θ' and integrates to a remainder of the same order. Here we wrote the amplitude as $A''_k(\theta', \theta'')$ and set $\theta'' = 0$ after the differentiations.

The differentiations leave the factor E_k (151) while bringing down polynomials in the derivatives of its phase. The same is true of the factor $e^{kO(||z'||^4)}$ that we absorbed into the amplitude. We now collect these factors and note that the exponent is simply the original phase (149) evaluated at $\theta'' = 0$:

$$\Phi'(\theta'; |z'|^2, |z''|^2) := F_{\mathbb{C}}(e^{i\theta'}|z'|^2, |z''|^2) - F(|z'|^2, |z''|^2) - i\langle \frac{\alpha'}{k}, \theta' \rangle) \tag{161}$$

We also collect the derivatives of this phase and the other factors of A_k and find that

$$\sum_{j=1}^{M} k^{-j} \left(\langle G''_\varphi(|z''|^2)(D_{\theta''}, D_{\theta''}\rangle\right)^j \chi_\delta A''_k(\theta', 0) = e^{k\Phi'(\theta';|z'|^2,|z''|^2)} \tilde{A}_k(\theta'), \tag{162}$$

where $\tilde{A}_k(\theta')$ is a classical symbol in k whose order is the order m of the original symbol A_k. The integral (160) then takes the form

$$\mathcal{P}_{h^k}(\alpha) \quad \sim \quad C_m \left(\det(k^{-1}G''_\varphi(|z''|^2))\right)^{1/2} \int_{\mathbf{T}^r} e^{k\Phi'(\theta';|z'|^2,|z''|^2)} \tilde{A}_k(\theta') d\theta'. \tag{163}$$

This is a corner type integral as studied in §6.3, with $|z''|^2$ as an additional parameter. The asymptotics of (163) are given by Lemma 6.4. It is only necessary to keep track of the powers of $|z'|^2, |z''|^2$ and of the parameter $k^{-1}(\delta''_k)^{-1}(k\delta'_k)^2$ in the analysis of $\tilde{A}_k$.

To do so, we first observe that

$$\nabla_{\theta'} F_{\mathbb{C}}(e^{i\theta'}|z'|^2, |z''|^2) = i\mu'_{\mathbb{C}}((z', z''), (e^{i\theta'}z', z'')), \tag{164}$$

i.e., it is the $'$ component of the complexified moment map. By definition of (z', z'') it equals $\frac{\alpha'}{k}$ when $\theta' = 0$. It follows that $F'_1(0, |z''|^2)|z'|^2 = \frac{\alpha'}{k}$, and the almost analytic extension satisfies

$$F'_1(0, |z''|^2)e^{i\theta'}|z'|^2 = e^{i\theta'}\frac{\alpha'}{k}, \tag{165}$$

where (as previously) the multiplication is componentwise. If we then Taylor expand the phase, we obtain

$$\Phi'(\theta'; |z'|^2, |z''|^2) = F'_1(0, |z''|^2)|z'|^2(1 - e^{i\theta'}) + O(|z'|^4) = \frac{\alpha'}{k}(1 - e^{i\theta'}) + O(|z'|^4). \tag{166}$$

If we absorb the $e^{kO(|z|^4)}$ factor into the amplitude, he integral has now been converted to the form (146) with a more complicated amplitude.

We next observe that

$$\tilde{A}_k = k^m(1 + O(|z'|^2)). \tag{167}$$

Hence, the assumption $|z'|^2 = O(\delta'_k)$ implies that to leading order

$$\begin{aligned} \mathcal{P}_{h^k}(\alpha) \quad &\sim \quad \sqrt{\det k^{-1}\, G''_\varphi(|z''|^2)} k^m \int_{T^r} e^{-k\left((e^{i\theta'}-1-i\theta)\right)\frac{\alpha'}{k}} d\theta'(1 + O(\delta'_k)) \\ &= \quad k^{m-\frac{p}{2}}\sqrt{\det G''_\varphi(|z''|^2))} \mathcal{P}_{h^k_{BF}}(\alpha')(1 + O(\delta'_k)). \end{aligned} \tag{168}$$

This completes the proof of the Lemma.

□

6.5. **Completion of proof of Proposition 6.1.** We now complete the proof of Proposition 6.1.

6.5.1. **Asymptotic expansion for $\mathcal{P}_{h^k}(\alpha)$.** The error terms for the asymptotics of $\mathcal{P}_{h^k}(\alpha)$, in the corner zone, the interior zone and the mixed zone, are given by $k^{-1}(\delta_k'')^{-1}$, $k(\delta_k')^2$ and $\eta_k = k^{-1}(\delta_k'')^{-1} + k(\delta_k')^2 + k(\delta_k')^2\delta_k'' + \delta_k'$ respectively. In order to minimize these terms, we let

$$k^{-1}(\delta_k'')^{-1} = k(\delta_k')^2 \text{ and } 0 < \delta_k' \leq \delta_k''.$$

By elementary calculation, the optimal choice for δ_k' and δ_k'' is given by

$$\delta_k' = \delta_k'' = k^{-\frac{2}{3}} \text{ and } k^{-1}(\delta_k'')^{-1} = k(\delta_k')^2$$

and

$$k^{-1}(\delta_k'')^{-1} = k(\delta_k')^2 = k^{-\frac{1}{3}},\ \eta_k \sim O(k^{-\frac{1}{3}}).$$

We let $\delta_k = k^{-\frac{2}{3}}$ and break up the estimate into four cases.

(1) $|x'|, |x''| \leq \delta_k$: this is the corner case handled in Lemma 6.4 if $k(\delta_k)^2 \to 0$.

$$\mathcal{P}_{h^k}(\alpha) = C_m \mathcal{P}_{h^k_{BF}}(\alpha)(1 + O(k^{-\frac{1}{3}})).$$

(2) $|x'|, |x''| \geq \delta_k$. By Lemma 6.3, stationary phase is valid and

$$\mathcal{P}_{h^k}(\alpha) \sim C_m k^{\frac{m}{2}} \sqrt{\det G_\varphi(\frac{\alpha}{k})} \left(1 + O(k^{-\frac{1}{3}})\right).$$

(3) $|x'| \leq \delta_k$ and $|x''| \geq \delta_k$. By Lemma 6.5,

$$\mathcal{P}_{h^k}(\alpha) = C_m k^{m-\frac{p}{2}} \sqrt{\det G_\varphi''(\tfrac{\alpha}{k})} \mathcal{P}_{P,k,\delta_k'}(\alpha')(1 + O(k^{-\frac{1}{3}})).$$

(4) $|x''| \leq \delta_k$ and $|x'| \geq \delta_k$. This case is the same as case (3) by switching x' and x''.

Combining the formulas above, the asymptotics for $\mathcal{P}_{h^k}(\alpha)$ is given by (132)

$$\mathcal{P}_{h^k}(\alpha) \ = \ C_m k^{\frac{1}{2}\,(m-\delta_k^{\#}(\frac{\alpha}{k}))} \sqrt{\mathcal{G}_{\varphi,\delta_k}(\frac{\alpha}{k})}\, \mathcal{P}_{P,k,\delta_k}(\frac{\alpha}{k}) \ \left(1 + R_k(\frac{\alpha}{k}, h)\right),$$

where $R_k(\frac{\alpha}{k}, h) = O(k^{-\frac{1}{3}})$.

On the other hand, equation (131) is derived by the following calculation.

$$\begin{aligned} k^{\frac{1}{2}(m-\delta_k^{\#})} \sqrt{\mathcal{G}_{\varphi,\delta_k}(\frac{\alpha}{k})}\, \mathcal{P}_{P,k,\delta_k}(\frac{\alpha}{k}) \ &= \ k^{\frac{m}{2}} \sqrt{\det G_\varphi(\frac{\alpha}{k})} \tilde{\mathcal{P}}_{P,k} \prod_{j \notin \mathcal{F}_{\delta_k}(x)} (2\pi k \ell_j(\frac{\alpha}{k}))^{-\frac{1}{2}} e^{|k\ell_j(\frac{\alpha}{k})|} \frac{k\ell_j(\frac{\alpha}{k})}{k\ell_j(\frac{\alpha}{k})^{k\ell_j(\frac{\alpha}{k})}} \\ &= \ k^{\frac{m}{2}} \sqrt{\det G_\varphi(\frac{\alpha}{k})} \tilde{\mathcal{P}}_{P,k}(1 + O(k^{-\frac{1}{3}})), \end{aligned}$$

where the last equality follows from the Stirling approximation.

6.5.2. **Derivatives with respect to metric parameters .** Now suppose that $h = h_t$ is a smooth one-parameter family of metrics. We would like to obtain asymptotics $(\frac{\partial}{\partial t})^j \mathcal{P}_{h_t^k}(\alpha)$ for $j = 1, 2$.

PROPOSITION **6.6.** *For $j = 1, 2$, there exist amplitudes S_j of order zero such that*

$$(\frac{\partial}{\partial t})^j \mathcal{P}_{h_t^k}(\alpha) \;=\; C_m k^{\frac{1}{2}(m-\;\delta_k^{\#}(\frac{\alpha}{k}))} \sqrt{\mathcal{G}_{\varphi_t, \delta_k}(\frac{\alpha}{k})}\, \mathcal{P}_{P,k,\delta_k}(\frac{\alpha}{k}) \;\left(S_j(t, \alpha, k) + R_k(\frac{\alpha}{k}, h) \right),$$

where $R_k = O(k^{-\frac{1}{3}})$. The expansion is uniform in h and may be differentiated in h twice with a remainder of the same order.

Proof. Such time derivatives may also be represented in the form (114)

$$(\frac{\partial}{\partial t})^j \mathcal{P}_{h_t^k}(\alpha) = (2\pi)^{-m} \int_{\mathbf{T}^m} e^{-k(F_t(e^{i\theta}\mu_{h_t}^{-1}(\frac{\alpha}{k})) - F_t(\mu_{h_t}^{-1}(\frac{\alpha}{k})))} A_{k,j}\big(k, t, \alpha, \theta\big) e^{i\langle \alpha, \theta\rangle} d\theta, \tag{169}$$

with a new amplitude $A_{k,j}$ that is obtained by a combination of differentiations of the original amplitude in t and of multiplications by t derivatives of the phase. It is easy to see that t derivatives of the amplitude do not change the estimates above since they do not change the order in growth in k of the amplitude. However, t derivatives of the phase bring down factors $k(\frac{\partial}{\partial t})^j (F_t(e^{i\theta}\mu_{h_t}^{-1}(\frac{\alpha}{k})) - F_t(\mu_{h_t}^{-1}(\frac{\alpha}{k}))$. The second derivative can bring down two factors with $j = 1$ or one factor with $j = 2$. We now verify that, despite the extra factor of k, the new oscillatory integral still satisfies the same estimates as before.

The key point is that, by the calculation (118), the phase $F_t(e^{i\theta}\mu_{h_t}^{-1}(\frac{\alpha}{k})) - F_t(\mu_{h_t}^{-1}(\frac{\alpha}{k})) - i\langle \frac{\alpha}{k}, \theta\rangle$ for any metric h vanishes to order two at the critical point $\theta = 0$; the first derivative vanishes because $\nabla_\theta F(e^{i\theta} z)|_{\theta=0} = i\mu_h(z)$. Hence, the t derivative of the h_t-dependent Taylor expansion (118) for a one-parameter family h_t of metrics also vanishes to order 2, i.e.,

$$(\frac{\partial}{\partial t})^j \left(F_t(e^{i\theta}\mu_{h_t}^{-1}(\frac{\alpha}{k})) - F_t(\mu_{h_t}^{-1}(\frac{\alpha}{k}) \right) = O(|\theta|^2). \tag{170}$$

Thus, for each new power of k one obtains by differentiating the phase factor in t one obtains a factor which vanishes to order two at $\theta = 0$. As a check, we note that in the Bargmann-Fock model, the phase has the form $\sum_j (e^{i\theta_j} - 1 - i\theta_j)\frac{\alpha_j}{k}$.

Let us first consider the first derivative. We repeat the asymptotic analysis but with the new amplitude S_1. In the 'interior region' the stationary phase calculation in Proposition 6.2 proceeds as before, but the leading term (now of one higher order than before) vanishes since it contains the value of (170) at the critical point as a factor. Therefore the asymptotics start at the same order as before but with the value of the second θ-derivative of the amplitude at $\theta = 0$.

In the corner, resp. mixed boundary, zone we obtain an integral of the same type as the ones studied in Lemma 6.4, resp. Lemma 6.5, but again with an amplitude of one higher order given by the t-derivative of the phase. The only change in the calculation is in the Taylor expansion of the amplitude in (167) in the z' variable, which now has the form

$$\tilde{A}_{k,1} = k(\frac{\partial}{\partial t}) \left(F_t(e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k})) - F_t(\mu_h^{-1}(\frac{\alpha}{k})) \right) + O(|z'|^2), \tag{171}$$

so that the final integral now has the form

$$(2\pi)^{-m} k^m \int_{T^r} e^{-k\left((e^{i\theta'}-1-i\theta')\right)\frac{\alpha'}{k}} \left(k(\frac{\partial}{\partial t})\left(F_t(e^{i\theta}\mu_h^{-1}(\frac{\alpha}{k})) - F_t(\mu_h^{-1}(\frac{\alpha}{k}))\right)\right)_{\theta''=0} d\theta'.$$

As noted in (170)

$$\begin{aligned} k(\tfrac{\partial}{\partial t})\left(F_t(e^{i\theta}\mu_t^{-1}(\tfrac{\alpha}{k})) - F_t(\mu_t^{-1}(\tfrac{\alpha}{k}))\right) &= k(\tfrac{\partial}{\partial t})\left(F_t(e^{i\theta}\mu_t^{-1}(\tfrac{\alpha}{k})) - F_t(\mu_t^{-1}(\tfrac{\alpha}{k})) - i\langle\tfrac{\alpha}{k},\theta\rangle\right) \\ &= k\tfrac{\partial}{\partial t}\int_0^1 (1-s)\,\tfrac{\partial^2}{\partial s^2}\left(F_t(e^{is\theta}\mu_t^{-1}(\tfrac{\alpha}{k})\right) ds \\ &= O(k|\theta|^2\tfrac{\alpha}{k}). \end{aligned}$$

Since the stationary phase method applies as long as $|\alpha| \to \infty$ we may assume that $|\alpha| \leq C$ and we see that the factor is then bounded. Here, we have suppressed the subscript $\mathbb{C}$ for the almost-analytic extension to simplify the writing.

As an independent check, we use integration by parts in θ'. We use a cutoff function χ supported near $\theta' = 0$ to decompose the integral into a term supported near $\theta' = 0$ and one supported away from $\theta' = 0$. We use the integration by parts operator

$$\mathcal{L} = \frac{1}{((e^{i\theta'}-1)\alpha')^2}\left(e^{i\theta'}-1\right)\alpha'\cdot\nabla_\theta$$

where we note that the factors of k cancel. The operator is well defined for $\theta' \neq 0$ and repeated partial integration gives decay in α' in case $|\alpha'| \to \infty$. On the support of χ the denominator is not well defined but the vanishing of the phase to order two shows that $\mathcal{L}^t(S_1)$ is bounded.

Now we consider second time derivatives. The second $\frac{\partial}{\partial t}$ could be applied to the phase factor $e^{k\Phi_t}$ again or it could be applied again to (171), and then we have

$$\begin{aligned} \tilde{A}_{k,2} &= (k(\tfrac{\partial}{\partial t})(F_t(e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k})) - F_t(\mu_h^{-1}(\tfrac{\alpha}{k})))^2 \\ &\quad + (k(\tfrac{\partial^2}{\partial t^2})(F_t(e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k})) - F_t(\mu_h^{-1}(\tfrac{\alpha}{k})))^2 + O(|z'|^2). \end{aligned} \tag{172}$$

The first term contains the factor k^2 and after cancellation it induces a term of order $|\alpha'|^2$. In addition this term vanishes to order four at $\theta = 0$. Hence the stationary phase calculation in the case of the first derivative equally shows that the first two terms vanish and thus the factors of k^2 are cancelled. In the regime where stationary phase is not applicable, $|\alpha'|^2$ may be assumed bounded, and additionally one can integrate by parts twice. Thus again this term is bounded.

□

6.6. **Completion of the proof of Lemma 1.3.** So far we have only considered the asymptotics of $\mathcal{P}_{h^k}(t,z)$. We now take the ratios to complete the proof of Lemma 1.3.

LEMMA **6.7.** *With δ_φ defined by (58), we have*

$$\mathcal{R}_k(t,\alpha) = \left(\frac{\det\nabla^2 u_t(\frac{\alpha}{k})}{(\det\nabla^2 u_0(\frac{\alpha}{k}))^{1-t}(\det\nabla^2 u_1(\frac{\alpha}{k}))^t}\right)^{1/2}\left(1 + O(k^{-\frac{1}{3}})\right).$$

The asymptotic in may be differentiated twice with the same order of remainder.

Proof. Combining Corollary 3.2 and Proposition 6.1,we have

$$\mathcal{R}_k(t,\alpha) = \frac{\sqrt{\det G_{\varphi_t}(\frac{\alpha}{k})}\,\tilde{\mathcal{P}}_{P,k}(\frac{\alpha}{k})}{(\sqrt{\det G_{\varphi_0}(\frac{\alpha}{k})}\,\tilde{\mathcal{P}}_{P,k}(\frac{\alpha}{k}))^{1-t}(\sqrt{\det G_{\varphi_1}(\frac{\alpha}{k})}\,\tilde{\mathcal{P}}_{P,k}(\frac{\alpha}{k}))^{t}}(1+O(k^{-\frac{1}{3}})). \tag{173}$$

We observe that the factors of $\tilde{\mathcal{P}}_{P,k}$ cancel out, leaving

$$\mathcal{R}_k(t,\alpha) = \frac{\sqrt{\det G_{\varphi_t}(\frac{\alpha}{k})}}{(\sqrt{\det G_{\varphi_0}(\frac{\alpha}{k})})^{1-t}(\sqrt{\det G_{\varphi_1}(\frac{\alpha}{k})})^{t}}(1+O(k^{-\frac{1}{3}})). \tag{174}$$

By Proposition 6.6, the asymptotic in (173) may be differentiated twice with the same order of remainder, completing the proof.

□

Remark: By (58), we also have

$$\mathcal{R}_k(t,\alpha) = \left(\frac{\delta_{\varphi_0}^{1-t}\delta_{\varphi_1}^{t}}{\delta_{\varphi_t}}\right)^{-\frac{1}{2}}(1+O(k^{-\frac{1}{3}})).$$

Indeed, the factors of $\ell_j(\frac{\alpha}{k})$ are independent of the metrics and cancel out. Also $\left(\frac{\delta_{\varphi_0}^{1-t}\delta_{\varphi_1}^{t}}{\delta_{\varphi_t}}\right)^{-\frac{1}{2}}$ is smooth on P.

The following simpler estimate on logarithmic derivatives is sufficient for much of the proof of the main results:

Lemma **6.8.** *We have:*

1. (1) $\partial_t \log \mathcal{R}_k(t,\alpha)$ *is uniformly bounded.*
2. (2) $\partial_t^2 \log \mathcal{R}_k(t,\alpha)$ *is uniformly bounded.*

Proof. We first note that

$$\partial_t \log \mathcal{R}_k(t,\alpha) = \log \mathcal{P}_{h_1^k}(\alpha) - \log \mathcal{P}_{h_0^k}(\alpha) - \partial_t \log \mathcal{P}_{h_t^k}(\alpha). \tag{175}$$

We note that by Proposition 6.1,

$$\log \mathcal{P}_{h^k}(\alpha) \;=\; \frac{1}{2}\log\det(k^{-1}G_{\varphi}(\frac{\alpha}{k})) \;+ \log \tilde{\mathcal{P}}_{P,k}(\frac{\alpha}{k}) \;+ \log C_m + O(k^{-\frac{1}{3}}). \tag{176}$$

As in Lemma 6.7, the Bargmann-Fock terms cancel between the h_0 and h_1 terms, while the metric factors simplify asymptotically to $\frac{1}{2}\log(\delta_{\varphi_1}\delta_{\varphi_0})$, and this is clearly bounded. To complete the proof of (1), we need that the final ratio is bounded. By Lemma 6.6, we see that in the 'interior' region both numerator and denominator have asymptotics which differ only in the value of a zeroth order amplitude at $\theta = 0$ and that it equals 1 in the case of the denominator. Hence, the ratio is bounded in the interior. Towards the boundary, the denominator is comparable with the Bargmann-Fock model and is bounded below by one. The numerator is also bounded by Lemma 6.6, and therefore the ratio is everywhere bounded.

Now we consider (2), which simplifies to

$$\partial_t^2 \log \mathcal{R}_k(t,\alpha) = -\frac{\partial_t^2 \mathcal{P}_{h_t^k}(\alpha)}{\mathcal{P}_{h_t^k}(\alpha)} + \left(\frac{\partial_t \mathcal{P}_{h_t^k}(\alpha)}{\mathcal{P}_{h_t^k}(\alpha)}\right)^2. \tag{177}$$

As we have just argued, the second factor is bounded. The same argument applies to the first term by Lemma 6.6.

□

7. C^0 and C^1-convergence

We begin with the rather simple proof of C^0-convergence with remainder bounds.

7.1. C^0-convergence.

Proposition **7.1.** $\frac{1}{k}\log Z_k(t,z)e^{-k\varphi_t(z)} = O(\frac{\log k}{k})$ *uniformly for* $(t,z)\in[0,1]\times M$

The Proposition follows from the following:

Lemma **7.2.** *(Upper/Lower bound Lemma) There exist* $C, c > 0$ *so that*

$$c \le \mathcal{R}_k(t,\alpha) \le C.$$

Proof. This follows immediately from Lemma 6.7. □

C^0-convergence is an immediate consequence of the upper and lower bound lemma:

Proof. By the upper/lower bound lemma, there exist positive constants $c, C > 0$ so that

$$c\Pi_{h_t^k}(z,z) \le \sum_{\alpha\in kP\cap\mathbb{Z}^m} \mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} \le C\Pi_{h_t^k}(z,z). \tag{178}$$

Hence,

$$\begin{aligned}\frac{1}{k}\log\Pi_{h_t^k}(z,z) \le \frac{1}{k}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} &\le \frac{1}{k}\log\Pi_{h_t^k}(z,z) + O(\frac{1}{k})\\ &= O(\frac{\log k}{k}),\end{aligned} \tag{179}$$

where the last estimate follows from (93).

□

7.2. C^1-convergence.

We now discuss first derivatives in (t,z). In the z variable the vector fields $\frac{\partial}{\partial\rho_j}$ vanish on $\mathcal{D}$, so can only use them to estimate C^1 norms in directions δ_k far from the boundary. In directions close to the boundary we may choose coordinates so that derivatives in z' near $z'=0$ define the C^1 norm.

The estimates in the ρ and z' derivatives are similar. We carry out the calculations in detail in the ρ variables and then indicate how to carry out the analogous estimates in the z variable.

We also consider t derivative. The key distinction between t and z derivatives is the following:

- z or ρ derivatives bring down derivatives of the phase, which have the form $k(\mu_t(z)-\frac{\alpha}{k})$. The factor of k raises the order of asymptotics while the factor $(\mu_t(z)-\frac{\alpha}{k})$ lowers it by the Localization Lemma.
- t derivatives do not apply to the phase and only differentiate $\mathcal{R}_k(t,\alpha)$ and $\mathcal{Q}_{h_t^k}(\alpha)$.

Proposition **7.3.** *Uniformly for* $(t,z)\in[0,1]\times M$*, we have:*

(1) $\frac{1}{k}\left|\frac{\partial}{\partial\rho_i}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-\frac{1}{2}+\delta})$;

(2) *The same estimate is valid if we differentiate in $\frac{\partial}{\partial r_n}$ in directions near $\mathcal{D}$ as in Proposition 4.6.*

(3) $\frac{1}{k}\left|\frac{\partial}{\partial t}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-\frac{1}{3}})$.

Proof. We first prove (1).

$$\begin{aligned}
&\frac{1}{k}\left|\nabla_\rho\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| \\
= &\left|\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\left(\frac{\alpha}{k}-\mu_t(z)\right)\mathcal{R}_k(t,\alpha)\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}\right| \\
= &\left|\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^m:\,|\frac{\alpha}{k}-\mu_t(z)|\le k^{-\frac{1}{2}+\delta}}\left(\frac{\alpha}{k}-\mu_t(z)\right)\mathcal{R}_k(t,\alpha)\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}\right| + O(k^{-M}) \\
\le &\; Ck^{-\frac{1}{2}+\delta}\left|\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^m:\,|\frac{\alpha}{k}-\mu_t(z)|\le k^{-\frac{1}{2}+\delta}}\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}\right| + O(k^{-M}) \\
\le &\; Ck^{-\frac{1}{2}+\delta},
\end{aligned}$$

proving (1). In this estimate, we use the Localization Lemma 1.2 and the upper/lower bound Lemma 7.2 on $\mathcal{R}_k$.

Regarding $\frac{\partial}{\partial r_n}$ derivatives in (2), the only change to the argument is in summing only α with $\alpha_n \neq 0$ and then changing $\alpha \to \alpha - (0,\dots,1_n,\dots,0)$ as explained in Proposition 4.6. Clearly the localization and the estimates only change by $\frac{1}{k}$.

We now consider the ∂_t derivative. By Proposition 4.4, we have

$$
\begin{aligned}
&\frac{1}{k}\frac{\partial}{\partial t}\log \textstyle\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\\
= \ & \frac{1}{k}\frac{\sum_\alpha \mathcal{R}_k(t,\alpha)\partial_t\log\left(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_t^k(\alpha)}}{\left(\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)} - \frac{\partial}{\partial t}\varphi_t\\
= \ & \frac{1}{k}\frac{\sum_\alpha \mathcal{R}_k(t,\alpha)\partial_t\log\left(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\left(\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)} - \frac{1}{k}\frac{\sum_\alpha \partial_t\log\left(\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)}\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\left(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)} + O(k^{-1})\\
= \ & \frac{1}{k}\frac{\sum_\alpha \mathcal{R}_k(t,\alpha)\partial_t\log\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\left(\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)} + \frac{1}{k}\left(\frac{\sum_\alpha \partial_t\log\mathcal{Q}_{h_t^k}(\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\left(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)} - \frac{\sum_\alpha \mathcal{R}_k(t,\alpha)\partial_t\log\mathcal{Q}_{h_t^k}(\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\left(\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)}\right) + O(k^{-1})
\end{aligned}
\tag{180}
$$

Notice that $\mathcal{Q}_{h_t^k} = \mathcal{R}_k(t,\alpha)(\mathcal{Q}_{h_0^k}(\alpha))^{1-t}(\mathcal{Q}_{h_1^k}(\alpha))^t$ and so $\partial_t\log\mathcal{Q}_k(t,\alpha) = \partial_t\log\mathcal{R}_k(t,\alpha) + \log\left(\frac{\mathcal{Q}_{h_1^k}(t,\alpha)}{\mathcal{Q}_{h_0^k}(t,\alpha)}\right)$. It follows easily from the fact proved in Lemma 1.3 (or more precisely the simpler Lemma 6.8) that $\mathcal{R}_k(t,\alpha) = O(1)$ and $\partial_t\log(\mathcal{R}_k(t,\alpha)) = O(1)$. Also $\log\frac{\mathcal{Q}_{h_1^k}(t,\alpha)}{\mathcal{Q}_{h_0^k}(t,\alpha)} = O(k)$ uniformly in α. Replacing $\mathcal{R}_k$ by $\mathcal{R}_\infty$ plus an error of order $k^{-\frac{1}{3}}$,

$$\frac{1}{k}\frac{\partial}{\partial t}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = O(k^{-\frac{1}{3}}).$$

□

8. C^2-convergence

We now consider second derivatives in ρ, t. Again we must separately consider derivatives in the interior and near the boundary. The following Proposition completes the proof of Theorem 1.1.

Proposition 8.1. *Uniformly for $(t,z)\in[0,1]\times M$, we have, for any $\delta>0$,*

(1) $\frac{1}{k}\left|\frac{\partial^2}{\partial\rho_i\rho_j}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-\frac{1}{3}+2\delta})$;

(2) $\frac{1}{k}\left|\frac{\partial^2}{\partial t\partial\rho_j}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-\frac{1}{3}+2\delta})$;

(3) $\frac{1}{k}\left|\frac{\partial^2}{\partial t^2}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-\frac{1}{3}+2\delta})$

(4) *The same estimates are valid if we replace $\frac{\partial}{\partial r_n}$ in directions near $\mathcal{D}$ as in Proposition 4.6.*

We break up the proof into the four cases. To simplify the exposition, we introduce some new notation for localizing sums over lattice points. By the Localization Lemma 1.2, sums over lattice points can be localized to a ball of radius $O(k^{-\frac{1}{2}+\delta})$ around $\mu_t(z)$. We emphasize that although there are three metrics at play, it is the metric h_t along the Monge-Ampère geodesic that is used to localize the sum. We introduce a notation for localized sums over pairs of lattice points: let

$$\widetilde{\Sigma}_{\alpha,\beta}\ F(\alpha,\beta) := \sum_{|\frac{\alpha}{k}-\mu_t(z)|,|\frac{\beta}{k}-\mu_t(z)|\leq k^{-\frac{1}{2}+\delta}} F(\alpha,\beta). \tag{181}$$

8.1. **Second space derivatives in the interior.** In this section we prove case (1). We have,

$$\begin{aligned} &\frac{1}{k}\left|\frac{\partial^2}{\partial\rho_i\partial\rho_j}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| \qquad (182)\\ =\ &\frac{1}{k}\left|\frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha-\beta)^2\ \ \mathcal{R}_k(t,\alpha)\mathcal{R}_k(t,\beta)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} - k\ \frac{\partial^2}{\partial\rho_i\partial\rho_j}\varphi_t\right|\\ \equiv\ &\frac{1}{k}\left|\frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha-\beta)^2\ \ \mathcal{R}_k(t,\alpha)\mathcal{R}_k(t,\beta)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} - \frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha-\beta)^2\ \ \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}\right|, \end{aligned}$$

modulo $O(\frac{1}{k})$ by Proposition 4.3. We also completed the square and used that the sum over α is a probability measure to replace $\alpha^2-\alpha\beta$ by $\frac{1}{2}(\alpha-\beta)^2$. We also use Lemma 4.5 to write $\frac{\partial^2}{\partial\rho_i\partial\rho_j}\varphi_t$ as a sum over lattice points.

By the Localization Lemma 1.2, each sum over lattice points can be localized to a ball of radius $O(k^{-\frac{1}{2}+\delta})$ around $\mu_t(z)$. Then, by Lemma 1.3 each occurrence of $\mathcal{R}_k(t,\alpha)$ or $\mathcal{R}_k(t,\beta)$ may be replaced by $\mathcal{R}_\infty(t,\frac{\alpha}{k})$ plus an error of order $k^{-\frac{1}{3}}$. Since $\frac{1}{k}(\alpha-\beta)^2 = O(k^{2\delta})$ the total error is of order $k^{2\delta-\frac{1}{3}}$. Since δ is arbitrarily small, this term is decaying. Further, after replacing $\mathcal{R}_k(t,\beta)$ by $\mathcal{R}_\infty(t,\frac{\alpha}{k})$ we may then replace $\frac{\alpha}{k},\frac{\beta}{k}$ by $\mu_t(z)$ at the expense of another error of order $k^{-\frac{1}{2}+\delta}$. By modifying (182) accordingly, we have

$$\begin{aligned} &\frac{1}{k}\left|\frac{\partial^2}{\partial\rho_i\partial\rho_j}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| + O(k^{-\frac{1}{3}+2\delta}) \qquad (183)\\ \equiv\ &\frac{1}{k}\left|\frac{\frac{1}{2}\widetilde{\sum}_{\alpha,\beta}(\alpha-\beta)^2\ \ \mathcal{R}_\infty(t,\mu_t(e^{\rho/2}))^2\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_\infty(t,\mu_t(e^{\rho/2}))\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} - \frac{\frac{1}{2}\widetilde{\sum}_{\alpha,\beta}(\alpha-\beta)^2\ \ \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}\right| \equiv 0, \end{aligned}$$

where $\equiv$ means that the lines agree modulo errors of order $O(k^{-\frac{1}{3}+2\delta})$. In the last estimate, we use that $\mathcal{R}_\infty(t,\mu_t(e^{\rho/2}))^2$ cancels out in the first term. This completes the proof in the spatial interior case.

The modifications when z is close to ∂P are just as in the case of the first derivatives.

8.2. **Mixed space-time derivatives.** The mixed space-time derivative is given by

$$\begin{aligned} & \frac{1}{k}\left|\frac{\partial^2}{\partial\rho_i\partial t}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| \\ = \quad & \frac{1}{k}\left|\frac{1}{2}\frac{\sum_{\alpha,\beta}(\alpha-\beta)\ \ \mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\partial_t\log\left(\frac{\mathcal{R}_k(t,\alpha)\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{R}_k(t,\beta)\mathcal{Q}_{h_t^k}(\alpha)}\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}-k\ \frac{\partial^2}{\partial\rho_i\partial t}\varphi_t\right| \end{aligned}$$

It suffices to prove that

$$\frac{1}{k}\left|\frac{\sum_{\alpha,\beta}(\alpha-\beta)\ \ \partial_t\log\left(\mathcal{R}_k(t,\alpha)\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}\right|=O(k^{-\frac{1}{2}+\delta})$$

and

$$\frac{1}{k}\left|\frac{1}{2}\frac{\sum_{\alpha,\beta}(\alpha-\beta)\ \ \mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\partial_t\log\left(\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}-k\ \frac{\partial^2}{\partial\rho_i\partial t}\varphi_t\right|=O(k^{-\frac{1}{3}+2\delta}).$$

The first estimate follows by the Localization Lemma 1.2 and from Lemma 6.8, i.e., that $\partial_t\log\left(\mathcal{R}_k(t,\alpha)\right)=O(1)$. The second estimate is very similar to that in §8.1, specifically in (183), so we do not write it out in full. In outline, we first apply the Localization Lemma and replace each $\mathcal{R}_k(t,\alpha)$ by $\mathcal{R}_\infty(\mu_t(z))$ with $z=e^{\rho/2}$. The errors in making these replacements are of order $k^{-1/3+\delta}$ because $\partial_t\log\left(\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}\right)=O(k|u_t(\alpha)-u_t(\beta)|)=O(k^{\frac{1}{2}+\delta})$ and because $(\alpha-\beta)=O(k^{\frac{1}{2}+\delta})$ in the localized sum. We then express $\frac{\partial^2}{\partial\rho_i\partial t}\varphi_t$ in terms of the Szegö kernel, i.e., as a sum over lattice points, using Proposition 4.5, and cancel the $\frac{\partial^2}{\partial\rho_i\partial t}\varphi_t$ term. The sum of the remainders is then of order $k^{-1/3+\delta}$, completing the proof in this mixed case.

8.3. **Second time derivatives.** The proof in this case follows the same pattern, although the estimates are somewhat more involved. The main steps are to localize the sums over lattice points, to replace each $\mathcal{R}_k$ by $\mathcal{R}_\infty$, then to cancel out $\mathcal{R}_\infty$ after all replacements, and to see that the resulting lattice point sum cancels $k\ \frac{\partial^2}{\partial t^2}\varphi_t$. The complications are only due to the number of estimates that are required to justify the replacements.

The second time derivative equals

$$\frac{1}{k}\frac{\partial^2}{\partial t^2}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}$$

$$= \frac{1}{k}\frac{\sum_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\left(\partial_t\log\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}(\frac{\mathcal{R}_k(t,\beta)}{\mathcal{Q}_{h_t^k}(\beta)})^{-1}\right)^2\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} \tag{184}$$

$$+\frac{1}{k}\left(\frac{\sum_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\partial_t^2\log\left(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}-k\ \frac{\partial^2}{\partial t^2}\varphi_t\right)$$

Here, we have simplified the numerator of the first term by replacing

$$\left(\partial_t\log\left(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}(\frac{\mathcal{R}_k(t,\beta)}{\mathcal{Q}_{h_t^k}(\beta)})^{-1}\right)\right)\left(\partial_t\log\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\right)\to\frac{1}{2}\left(\partial_t\log\left(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}(\frac{\mathcal{R}_k(t,\beta)}{\mathcal{Q}_{h_t^k}(\beta)})^{-1}\right)\right)^2,$$

which is valid since the expression is anti-symmetric in (α,β) and since we are summing in (α,β).

To simplify the notation, we now abbreviate $\mathcal{R}(\alpha)=\mathcal{R}_k(t,\alpha)$, $\mathcal{T}(\alpha)=\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)}$, $f'=\frac{\partial f}{\partial t}$, and write (184) $=\frac{N}{D}$ where the numerator has the schematic form

$$N = \sum_{\alpha,\beta}\left(\left(\frac{\mathcal{R}'}{\mathcal{R}}(\alpha)+\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)\right)'+\frac{1}{2}\left(\frac{\mathcal{R}'}{\mathcal{R}}(\alpha)+\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)-(\frac{\mathcal{R}'}{\mathcal{R}}(\beta)+\frac{\mathcal{T}'}{\mathcal{T}}(\beta))\right)^2\right)\mathcal{R}(\alpha)\mathcal{T}(\alpha)\mathcal{R}(\beta)\mathcal{T}(\beta)e^{\langle\alpha,\rho\rangle}e^{\langle\beta,\rho\rangle} \tag{185}$$

and where the denominator is $D=(\sum_\alpha\mathcal{R}(\alpha)\mathcal{T}(\alpha))^2$. We omit the factors $\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}$ from the notation since they are always present.

We now compare N and D to the corresponding expressions in the second time derivative of the Szegö kernel in Proposition 4.5. In the latter case, $\mathcal{R}\equiv 1$ so any terms with t-derivatives of $\mathcal{R}$ above do not occur in the third comparison expression of Proposition 4.5. Terms with no t derivatives of $\mathcal{R}$ will be precisely as in the comparison except that $\mathcal{R}$ is replaced by 1. So we consider the sub-sum of N,

$$N_1 = \sum_{\alpha,\beta}\left(\left(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)\right)'+\frac{1}{2}\left(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)-\frac{\mathcal{T}'}{\mathcal{T}}(\beta)\right)\right)^2\right)\mathcal{R}(\alpha)\mathcal{T}(\alpha)\mathcal{R}(\beta)\mathcal{T}(\beta) \tag{186}$$

If we now replace all occurrences of $\mathcal{R}_k(t,\alpha)$ by $\mathcal{R}_\infty(\mu_t(z))$ in both numerator and denominator we get the Szegö kernel expression (the third comparison expression of Proposition 4.5) of order $\frac{1}{k^2}$. (This is verified in more detail at the end of the proof). So we are left with estimating two remainder terms: First, the difference $N_1-\tilde{N}_1$ where $\tilde{N}_1$ is a sum of terms in which we replace at least one $\mathcal{R}(\alpha)$ by $\mathcal{R}_\infty(\mu_t(z))$ (or with β). Second, we must estimate $N-N_1$.

We first consider $N_1-\tilde{N}_1$. It arises by substituting at least one $\mathcal{R}(\alpha)-\mathcal{R}_\infty(\mu_t(z))=O(k^{-\frac{1}{3}})$ for one of the $\mathcal{R}(\alpha)$'s in N_1. We apply the localization argument Lemma 1.2 to replace N_1 (and D) by sums over $\frac{\alpha}{k},\frac{\beta}{k}\in B(\mu_t(z),k^{-\frac{1}{2}+\delta})$. We thus need to estimate the following expression, when at least one $\mathcal{R}(\alpha)$ is replaced by $\mathcal{R}(\alpha)-\mathcal{R}_\infty(\mu_t(z))$:

$$\frac{1}{k}\frac{\widetilde{\sum}_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\left(\partial_t \log \frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}\right)\left(-\partial_t \log \mathcal{Q}_{h_t^k}(\alpha)\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_{\alpha\in B(\mu_t(z),k^{-\frac12+\delta})}\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}$$

$$+\quad \frac{1}{k}\left(\frac{\widetilde{\sum}_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\partial_t^2\log\left(\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)}\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_{\alpha\in B(\mu_t(z),k^{-\frac12+\delta})}\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{(\mathcal{Q}_{h_t^k}(\alpha))}\right)^2}-k\ \frac{\partial^2}{\partial t^2}\varphi_t\right)$$

Due to the factor $\frac{1}{k}$ outside the sum, it suffices to prove that

$$\left(\left(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)\right)'+\frac12\left(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)-\frac{\mathcal{T}'}{\mathcal{T}}(\beta)\right)^2\right)=O(k^{1+2\delta}).$$

By Proposition 3.1, we have

$$\frac{\mathcal{T}'}{\mathcal{T}}=-\frac{\mathcal{P}'}{\mathcal{P}}+ku_t'(\frac{\alpha}{k}).$$

Since $u_t=(1-t)u_0+tu_1$, we have

$$\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)=-\frac{\mathcal{P}'}{\mathcal{P}}+k(u_1-u_0)(\frac{\alpha}{k})=-\frac{\mathcal{P}'}{\mathcal{P}}+k(f_1-f_0)(\frac{\alpha}{k}),$$

where we recall from §2.2 that $u_\varphi=u_0+f_\varphi$ with f_φ smooth up to the boundary of P.

It follows that,

$$\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)-\frac{\mathcal{T}'}{\mathcal{T}}(\beta)=-\frac{\mathcal{P}'}{\mathcal{P}}(\alpha)+\frac{\mathcal{P}'}{\mathcal{P}}(\beta)+k(f_1-f_0)(\frac{\alpha}{k})-k(f_1-f_0)(\frac{\beta}{k}), \tag{187}$$

$$(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha))'=-(\frac{\mathcal{P}'}{\mathcal{P}})'=O(1). \tag{188}$$

with

$$k(f_1-f_0)(\frac{\alpha}{k})-k(f_1-f_0)(\frac{\beta}{k})=kO(|\frac{\alpha}{k}-\frac{\beta}{k}|)\ =\ O(k^{\frac12+\delta}).$$

Further, by Lemma 6.7 (using Lemma 6.6), the factors of

$$\frac{(\frac{\partial}{\partial t})\mathcal{P}_{h_t^k}(\alpha)}{\mathcal{P}_{h_t^k}(\alpha)}\ =\ \frac{\left(S_1(t,\alpha,k)+R_k(\frac{\alpha}{k},h)\right)}{S_0(t,\alpha,k)}=O(1),$$

and similarly $(\frac{\mathcal{P}'}{\mathcal{P}})'=O(1)$. Since (187) is squared, it has terms as large as $O(k^{1+2\delta})$. Taking into account the overall factor of $\frac{1}{k}$ and the presence of at least one factor of size $k^{-\frac13}$ coming from the replacement of at least one $\mathcal{R}_k(t,\alpha)$ by $\mathcal{R}_\infty(\mu_t(z))$, we see that $N_1-\tilde{N}_1$ has order $k^{-\frac13+2\delta}$ and again this decays for sufficiently small δ.

Now we estimate $N - N_1$, which consists of terms with at least one t-derivative of $\mathcal{R}$. By Lemma 6.7, the terms with no t derivatives on $\mathcal{T}$ give the terms

$$\frac{1}{k}\frac{\widetilde{\sum}_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\left(\partial_t \log\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{R}_k(t,\beta)}\right)\left(\partial_t \log \mathcal{R}_k(t,\alpha)\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{(\mathcal{Q}_{h_t^k}(\alpha))}\right)^2}$$

$$+\quad \frac{1}{k}\frac{\widetilde{\sum}_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\partial_t^2\log\left(\mathcal{R}_k(t,\alpha)\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{(\mathcal{Q}_{h_t^k}(\alpha))}\right)^2} = O(k^{-1}),$$

by Lemma 1.3.

This leaves us with the terms

$$\left(\frac{\mathcal{R}'}{\mathcal{R}}(\alpha) - \frac{\mathcal{R}'}{\mathcal{R}}(\beta)\right)\left(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha) - \frac{\mathcal{T}'}{\mathcal{T}}(\beta)\right).$$

Again by Lemma 6.8, the first term is $O(1)$ while the second factor is (187) and has size $kk^{-\frac{1}{2}+\delta}$. Here, we again use Propositions 3.1 and 6.6. Due to the overall factor of $\frac{1}{k}$ this term has size $k^{-\frac{1}{2}+\delta}$.

Therefore, as stated above, up to errors of order $k^{-1/3+\delta}$, (184) is simplified to $-\frac{\partial^2}{\partial t^2}\varphi_t$ plus

$$\frac{1}{k}\left(\frac{\widetilde{\sum}_{\alpha,\beta}\mathcal{R}_\infty(\mu_t(e^{\rho/2}))\mathcal{R}_\infty(\mu_t(e^{\rho/2}))\left(\partial_t^2\log\left(\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)}\right) + (\partial_t \log\frac{1}{\mathcal{Q}_{h_t^k}})(\partial_t\log(\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)})\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_{\alpha\in B(\mu_t(z),k^{-\frac{1}{2}+\delta})}\mathcal{R}_\infty(\mu_t(e^{\rho/2}))\frac{e^{\langle\alpha,\rho\rangle}}{(\mathcal{Q}_{h_t^k}(\alpha))}\right)^2}\right) \tag{189}$$

As before, we cancel the factors of $\mathcal{R}_\infty(\mu_t(e^{\rho/2}))$. The resulting difference then cancels to order $k^{-1/2+\delta}$ by Lemma 4.5 (3).

This completes the proof of the second time derivative estimate, and hence of the main theorem.

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