From random polynomials to symplectic geometry

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ABSTRACT. We review some recent results on random polynomials and their generalizations in complex and symplectic geometry. The main theme is the universality of statistics of zeros and critical points of (generalized) polynomials of degree N on length scales of order $\frac{D}{\sqrt{N}}$ (complex case), resp. $\frac{D}{N}$ (real case).

1. Introduction

This is a short survey of some results of P. Bleher, J. Neuheisel, B. Shiffman and the author on random polynomials and their generalizations to holomorphic (and almost-holomorphic) sections of ample line bundles, mainly following [**BSZ1, BSZ2, BSZ3, N, ShZe, ShZe2, ShZe3, ShZe4, Ze1, Ze2**]. Motivation to study random polynomials and their generalizations in geometry comes from several sources:

- Classical Analysis: Value distribution theory of polynomials and analytic functions is a classical topic. Computable examples may exhibit non-generic patterns of zeros (or other values) and one would like to understand the typical distribution. One forms ensembles of analytic functions by defining the coefficients to be independent random variables with a given distribution. One can then study expected behaviour, almost sure behaviour and so on (see e.g. [**Kac, LO, O**]).
- PDE: Spherical harmonics of degree N are examples of eigenfunctions of the Laplacian on a compact Riemannian manifold. They are restrictions to the sphere S^m of homogeneous harmonic polynomials of degree N on $\mathbb{R}^{m+1}$. One would like to know about nodal lines, critical points, sup norms (etc.) of general Laplace eigenfunctions. Studying features of random spherical harmonics gives insight into the 'typical' properties of eigenfunctions and avoids pathologies such as occur in [**L, JN**]. Analogues of spherical harmonics of degree N on general compact Riemannian manifolds are linear combinations of eigenfunctions $\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda$ with $\lambda \in [N \log N, (N+1) \log(N+1)]$. References on random spherical harmonics include [**Be, N, V, Ze1**]); for random combinations of eigenfunctions, see [**Ze2**].
- Quantum Chaos: Eigenfunctions of quantum chaotic Hamiltonians are well modeled by random polynomials in regard to distribution of zeros or critical points, to sizes (sup-norms or L^p-norms), to quantum expectation values, and in other respects (see, among others, [**ABST, BBL, HKZ, NoVo, Ze1, Ze2**]). This is analogous to the similarity between eigenvalues of random matrices and eigenvalues of quantum chaotic systems.
- Algebraic Geometry: Holomorphic sections $s \in H^0(M, L^N)$ of the Nth power of an ample line bundle $L \to M$ over a Kähler manifold (M, ω) are quite analogous to homogeneous polynomials of degree N, and coincide with such polynomials when $M = \mathbb{CP}^m, L = \mathcal{O}(1)$ ($\mathbb{CP}^m$ = complex projective m-space, $\mathcal{O}(1)$ is the hyperplane bundle (cf. [**GH**])). The simultaneous zero set $Z_{s_1,\ldots,s_k}$ of k holomorphic sections defines a codimension k algeraic submanifold of M; one would like to know the 'almost sure' properties of such a submanifold.
- Symplectic Geometry: Almost-holomorphic sections $s \in H_J^0(M, L^N)$ of ample line bundles $L \to M$ over almost-complex symplectic submanifolds (M, J, ω) in the sense of [**Don, BoGu**] are very

similar to holomorphic sections in the complex case. Under (difficult) transversality conditions, they have applications in symplectic geometry analogous to those in algebraic geometry.

In this article, we will refer to the setting of holomorphic (or almost-holomorphic) sections as the 'complex case', and the setting of eigenfunctions of Laplacians as the 'real case'. The complex wave functions live on phase space while the real eigenfunctions live on configuration space. The main theme of our work has been the universality of statistics of zeros and critical points of polynomials of large degree N on small length scales (of order $\frac{D}{\sqrt{N}}$ in the complex case, resp. $\frac{D}{N}$ in the real case). We have only considered compact manifolds, and only Gaussian or spherical measures on their spaces of polynomials. Two universality classes have emerged: (i) the 'Heisenberg class' in the complex case (connected to the Heisenberg group), and (ii) the 'Euclidean class' in real case (connected to the Euclidean motion group). It should be mentioned that the 'real case' has many other meanings in the literature on random polynomials (e.g. real polynomials and their real zeros).

2. Mathematical Tools

We give a quick summary of some basic tools and methods that are used below. We will concentrate on general ideas and refer to [**ShZe, ShZe2, BSZ3**] for detailed expositions.

2.1. Vector spaces of large dimension. In defining our ensembles of polynomials, we will be dealing with a sequence $(\mathcal{H}_N, \langle, \rangle_N)$ of Hilbert spaces of increasing dimension $d_N = dim \mathcal{H}_N$, where N is the 'degree of the polynomial', a large integral (semiclassical) parameter. The dimension is given by a (Hilbert) polynomial of the form $d_N \sim a_0 N^m$ in the complex case (with $m = dim_{\mathbb{C}} M$) and by a function of polynomial growth $d_N \sim a_0 N^{m-1}$ in the real case (with $m = \dim_{\mathbb{R}} M$).

In our applications, the spaces $\mathcal{H}_N$ will be one of the following.

2.1.1. *Spherical harmonics and real eigenfunctions.* We denote by Δ the standard Laplacian on S^m and by $\mathcal{H}_N(S^m)$ the space of spherical harmonics of degree N on S^m. They are the eigenfunctions of Δ of eigenvalue $\lambda_{m,N} = N(N+m-1)$ and form a real vector space of dimension $d_{m,N} = \frac{2N+m-1}{N+m-1} \begin{pmatrix} N+m-1 \\ m-1 \end{pmatrix}$.

More generally, we may consider the Laplacian Δ of any compact Riemannian manifold (M, g). In place of spherical harmonics of degree N, we partition the spectrum of $\sqrt{\Delta}$ into intervals $[N \log N, (N+1)\log(N+1)]$ (the reason for the longer length is given in [**Ze2**]), and let $\mathcal{H}_N$ denote the span of the eigenfunctions with eigenvalues in the Nth interval. Linear combinations of such eigenfunctions are of course not eigenfunctions of Δ, but they behave like polynomials of degree N and their random linear combinations give a replacement for random spherical harmonics. Their use for modelling quantum ergodic and quantum mixing eigenfunctions is discussed [**Ze2, HKZ**].

2.1.2. *Holomorphic sections of positive line bundles.* For any Kähler manifold (M, ω) of complex dimension m there exists a holomorphic hermitian line bundle $(L, h) \to (M, \omega)$ whose Ricci curvature $Ric(h) = \omega$. L is called *positive* since it posseses a metric of positive curvature, i.e. $\omega(X, JY)$ defines a Riemannian metric. We denote by L^N the Nth power of L and by $H^0(M, L^N)$ the space of holomorphic sections. Its dimension $d_N = dim H^0(M, L^N)$ is given by the Hilbert polynomial $d_N = \frac{c_1(L)^m}{m!} N^m + \cdots$ for sufficiently large N, where $\cdots$ represent the lower order terms. We equip M with the volume form $dV_\omega = \frac{\omega^m}{m!}$ and $H^0(M, L^N)$ with the inner product $||s||^2 = \int_M h(s(z), s(z)) dV$. For background we refer to [**GH**].

In the simplest case of Riemann surfaces, examples include:

- $M = \mathbb{CP}^1, L = \mathcal{O}(1), L^2 = T\mathbb{CP}^1, h = h_{FS}, \omega = \omega_{FS}$ (Fubini study hermitian metric and curvature (1,1)-form). That $(\mathbb{CP}^1, \omega_{FS})$ is positively curved in the usual Riemannian sense is equivalent to positivity of $T\mathbb{CP}^1$. $H^0(\mathbb{CP}^1, \mathcal{O}(2N))$ may be interpreted as the space of holomorphic vector fields of type $(\frac{\partial}{\partial z})^N$. More simply put, sections are homogeneous holomorphic polynomials $s = \sum_{j=0}^N c_k z_0^k z_1^{N-k}$ of degree N in two complex variables. Such polynomials are known as the $SU(2)$-ensemble.

- $M = \mathbf{H}^2/\Gamma, L = T^*M, h_{FS} =$ hyperbolic metric. Hyperbolic surfaces are negatively curved in the Riemannian sense, so their tangent bundles are negatively curved, and their co-tangent bundles are positively curved. Holomorphic sections are holomorphic differentials of type $(dz)^N$.
- $M = \mathbb{C}/\mathbb{Z}^2, \omega_0 = dz \wedge d\bar{z}$. The complex torus is flat in the Riemannian sense, so neither its tangent nor cotangent bundles are positively curved. The 'quantizing line bundle' with curvature ω_0 is rather the bundle Θ whose sections are the classical theta-functions. The sections of Θ^N are known as theta-functions of level N.

2.1.3. *Almost holomorphic sections.* Symplectic almost-complex manifolds (M, J, ω) possess a similar but analytically more complicated geometric quantization as spaces $H^0_J(M, L^N)$ of 'almost-holomorphic' sections. They are defined by a $\bar{D}$ complex over the S^1-bundle X due to Boutet de Monvel -Guillemin [**BoGu**]. What we need to know about these spaces is that their orthogonal projectors Π_N have the same scaling asymptotics as in the complex case, if one works in suitable (Heisenberg) coordinates [**ShZe2**].

2.2. Gaussian measures and spherical measures. We will restrict attention to two related ensembles:

- *Gaussian ensembles*: We fix an orthonormal basis $\{f_j\}$ of $\mathcal{H}_N$ and write functions as orthonormal sums $f = \sum_{j=1}^{d_N} c_j f_j$. We then define the (complex) Gaussian measure by
$$\gamma = e^{-||f||^2}\mathcal{D}f, \text{ i.e. } \gamma = e^{-|c|^2}dc.$$
  More generally we fix a symmetric matrix Δ on $\mathbb{C}^{d_N}$ with positive (semi-)definite imaginary part and define:

(1)
$$\gamma_\Delta = \frac{e^{-\langle\Delta^{-1}c,c\rangle}}{(2\pi)^{p/2}\det\Delta}dc,$$

  Gaussian ensembles come in both real and complex flavors. In the real case, the exponents acquire factors of $1/2$ and the denominator acquires a square root.
- *Spherical ensembles*: We denote by $S\mathcal{H}_N = \{f \in \mathcal{H}_N : ||f|| = 1\}$. We then equip $S\mathcal{H}_N$ with the uniform (Haar) probability measure ν_N.

We will denote the expected value of a random variable X with respect to the Gaussian ensemble by $\mathbf{E}_\gamma(X)$ (resp. $\mathbf{E}_\nu(X)$ for the spherical ensemble).

These two ensembles are *equivalent* in the sense that the two large dimension limits give equivalent results when scaled properly. A more precise formulation goes as follows:

Let $T_N : \mathbb{R}^{d_N} \to R^k$, $N = 1, 2, \ldots$, be a sequence of linear maps, where $d_N \to \infty$. Suppose that $\frac{1}{d_N}T_N T_N^* \to \Delta$. Then $T_{N*}\nu_{d_N} \to \gamma_\Delta$.

2.3. Sequences of random polynomials. We are often interested in sequences of polynomials $\{s_N\}$ chosen independently and at random from $\mathcal{H}_N$ from either a Gaussian or spherical ensemble. We therform form the product probability space $(\mathcal{H}_\infty, \mu_\infty)$, defined by
$$\mathcal{H}_\infty = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_N \times \cdots, \quad \mu_\infty = \times_{N=1}^\infty \mu_N, \quad (\mu_N = \gamma_N \text{ or } \nu_N).$$

When we say that a sequence of polynomials $\{s_N\}$ of increasing degerees does something almost surely, we mean that the set of such sequences has measure one in this product ensemble.

2.4. Szegö kernels. Our results depend on the fact that certain statistical properties of polynomials can be expressed in terms of the reproducing kernels $\Pi_N(x, y)$ (orthogonal projections) of the Hilbert spaces $\mathcal{H}_N$. They are known as Szego kernels, and are essentially the same as the 'coherent states' of the physics literature. The local structure of the Szegö kernel is given by the following scaling asymptotics:

THEOREM 2.1. *As $N \to \infty$, we have:*
- [**BSZ2, ShZe2**] *Complex case:*
$$\Pi_N(z_0 + \frac{u}{\sqrt{N}}, \frac{\theta}{N}, z_0 + \frac{v}{\sqrt{N}}, \frac{\varphi}{N}) \sim \frac{1}{\pi^m}e^{i(\theta-\varphi)}e^{u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)}\{1 + \frac{1}{\sqrt{N}}p_1(u, v; z_0) + \cdots\}.$$

The leading order term is the Szegö kernel for the reduced Heisenberg group, whence the name 'Heisenberg class.'

- *Real case:*

$$N^{-m+1}\Pi_N(x_0 + \frac{u}{N}, x_0 + \frac{v}{N}) \sim \Gamma(\frac{m-1}{2})(\frac{|u-v|}{2})^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(|u-v|) + \cdots,$$

where J_ν is the Bessel function of the first kind of order ν (whence the name 'Euclidean class'.)

The proof of the scaling asymptotics in the complex holomorphic case [**BSZ2**] is based on the Boutet-de-Monvel- Sjostrand parametrix for the Szegö kernel, which is valid for positive line bundles. A similar parametrix was constructed in the symplectic almost-complex case [**ShZe2**], and the scaling asymptotics were derived from it. In the real case of S^m, the scaling asymptotics are closely related to the 'Mehler-Heine formula'. The terms 'Heisenberg class' and 'Euclidean class' suggest infinite dimensional Gaussian ensembles related to representations of the Heisenberg and Euclidean motion groups.

3. Distribution of zeros and critical points

We now state some results on the zeros and critical points of random generalized polynomials. Let $(s_1, \ldots, s_k) \in H^0(M, L^N)^k$ or let $(s_1, \ldots, s_k) \in \mathcal{H}_N^k$ in the real case. When $k = 1$ we omit the subscript. We will use the following notation.

- $Z_s = \{x : s(x) = 0\}$ denotes the zero set of s, and $|Z_s|$ denotes the $m-k$-submanifold Riemannian volume density induced by ω in the complex case or by g in the real case. We further denote by $||Z_s||$ the mass of $|Z_s|$ and define the probability measures $\tilde{Z}_s = \frac{|Z_s|}{||Z_s||}$. When one takes m sections (or functions) in dimension m, then the simultaneous zeros are almost surely a discrete set and the measure $\tilde{Z}_s$ is the normalized sum of delta-functions at the zeros.
- $C_s = \{z : \nabla s(z) = 0\}$ denotes the critical point set of s. In the holomorphic case, ∇ is the holomorphic connection compatible with h. We note that C_s is almost surely a discrete set. We define $|C_f| = \sum_{z_j : \nabla s(z_j) = 0} \delta(z_j)$, $||C_s|| = \#C_s$, and $\tilde{C}_s = \frac{|C_s|}{\#C_s}$.
- By the density of zeros at degree N we mean the coefficient $K_{1,k}^N(z)$ of the measure $K_{1,k}^N(z)dV = \mathbf{E}|Z_s|$, i.e. $\int_M \varphi \mathbf{E}|Z_s| = \mathbf{E} \int_M \varphi |Z_s|$ for $\varphi \in C(M)$. Similarly, we denote by $K_1^{crit,N}(z)$ the density (relative to the given volume form) of $\mathbf{E}\tilde{C}_s$.
- More generally, we define the pair correlation densities of zeros (resp. critical points) by $K_{2,k}^N(z^1, z^2)dV = \mathbf{E}(|Z_s| \times |Z_s|)$, resp. $K_{2,k}^{crit,N}(z^1, z^2)dV = \mathbf{E}(|C_s| \times |C_s|)$. They are densities of measures on $M \times M$. Roughly speaking, the two-point correlation gives the probability density of finding a pair of zeros (or critical points) at (z^1, z^2). More generally, there are n-point correlation functions, but for simplicity we only consider $n = 1, 2$.

3.1. Statement of results. We have results on several levels: expected values, almost sure behaviour, and scaling asymptotics. The following theorems are valid on any complex or almost-complex symplectic manifold (M, ω), equipped with a hermitian complex line bundle of curvature ω.

THEOREM 3.1. *In the complex case, the density of zeros, resp. critical points, satisfies:*

- [**ShZe**] *Complex zeros:* $K_{1,k}^N(z)dV = \omega^k + O(\frac{1}{N})$.
- [**ShZe3**] *Complex critical points: There exists a universal constant γ_m depending only on the dimension such that $K_1^{crit,N}(z)dV = \gamma_m \frac{\omega^m}{m!} N^m + O(N^{m-1})$, In particular, the expected number of critical points is given by $\mathbf{E}\#C_s = \gamma_m Vol_\omega(M)N^m + O(N^{m-1})$. Here, $Vol_\omega(M) = \frac{c_1(L)^m}{m!}$ is the volume of (M, ω). The density of critical points is therefore universal.*

Thus, zeros and critical points tend to concentrate in regions of high curvature. Similar results should hold in the real case. In the case of S^m, such density results are obvious since they must be rotationally invariant.

3.2. Almost sure distribution of zeros. A deeper question on distribution of zeros is whether sequences of individual sections tend to have equidistributed zeros.

THEOREM 3.2. *We have:*
- [**ShZe**] *For a random sequence in the complex codimension k case, $\{s_N\}$, $\tilde{Z}_{s_N} \to \omega^k$ almost surely.*
- [**N**] *For a random sequence of spherical harmonics on $S^m, m \geq 6$, we have $\tilde{Z}_{s_N} \to dvol$ almost surely. The same is true in (Cesaro) mean for dimensions < 6.*

3.3. Universality and scaling of correlations. The next level of results concerns the statistics of zeros and critical points on the length scale $\frac{D}{\sqrt{N}}$ (complex case) or $\frac{D}{N}$ (real case). For brevity we only describe the results in the complex case. Upon magnifying a small ball $B_{\frac{D}{\sqrt{N}}}(z_0)$ around an arbitrary point z_0 of this radius, one loses track of the specifics of the geometrical setting and obtains universal limiting correlations. More precisely, such a universal limit occurs if one chooses the coordinates properly.

THEOREM 3.3. *Scaling limits of correlations of zeros and critical points are universal. That is:*

$$\frac{1}{N^{2k}} K_{2k}^N \left(\frac{z^1}{\sqrt{N}}, \frac{z^2}{\sqrt{N}} \right) \to K_{2km}^\infty(z^1, z^2)$$

in the sense of measures.

The limit correlation function denoted $K_{2km}^\infty(z^1, z^2)$ is unique within a universality class for each statistic under consideration. In the complex and almost-complex cases, there exists a limit correlation function $K_{2km}^{\mathbb{C},zeros,\infty}(z^1, z^2)$ of zeros, and another limit correlation function $K_{2km}^{\mathbb{C},crits,\infty}(z^1, z^2)$ for critical points. There are analogous results in the real case. The limit correlations are explicitly computable. In [**BSZ1**] and elsewhere we give explicit formulae for low values of m and graph the results. The result for zeros in dimension one agrees with the formula of Hanny [**Han**] on $\mathbb{CP}^1$, as it must since the result is universal. We now briefly describe the elements of the proof. The details differ, but the principles are the same, for complex and real cases, and for zeros or critical points. Hence we concentrate on zeros in the complex case.

3.3.1. *Step One: Relating correlations and joint probability distributions.* Following an idea due originally to Kac and Rice in the case of real polynomials of one variable, we express the correlation measures in terms of the joint probability distribution of the random variables $s(z^1), \ldots, s(z^n), \nabla s(z^1), \ldots, \nabla s(z^n)$. This JPD is defined by

$$\widetilde{\mathbf{D}}_z^N := \widetilde{D}_n^N(x, \xi, z) dx d\xi = (J_z)_* \nu_N \, , \quad J_z(s) = (s(z), \nabla s(z)),$$

i.e. it is the push-forward of the Gaussian measure γ_N under the linear jet map $J(z)$.

The desired expression for correlations in terms of the JPD is given by the following generalization of the Kac-Rice formula [**Kac**] to the geometric setting of this article:

THEOREM 3.4. [**BSZ1, BSZ3, ShZe2**] *In the case of correlations of complex zeros, we have:*

$$K_{2k}^N(z) = \int d\xi \, \widetilde{D}_n^N(0, \xi, z) \prod_{p=1}^n \det(\xi^p \xi^{p*}) \, .$$

Analogous formulae exist for critical points, but involve the jet maps $(\nabla s(z), \nabla^2 s(z))$. The real case is similar to the complex case, but is somewhat more complicated because Π_N is oscillatory rather than exponentially decaying [**N**].

3.3.2. *Step two: scaling asymptotics of the JPD.* The JPD is a (generalized) Gaussian measure on the complex vector space of 1-jets: $\widetilde{\mathbf{D}}_z^N = \gamma_{\Delta^N(z)}$, where the covariance matrix $\Delta^N(z) = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is given in terms of the Szegö kernel and its covariant derivatives, as follows:

$$A = \left(A_{p'}^p \right) = \frac{1}{d_N} \Pi_N(z^p, 0; z^{p'}, 0) \, , \qquad B = \left(B_{p'q'}^p \right) = \frac{1}{d_N} \overline{\nabla}_{q'}^2 \Pi_N(z^p, 0; z^{p'}, 0) \, ,$$

$$C = \left(C_{p'q'}^{pq} \right) = \frac{1}{d_N} \nabla_q^1 \overline{\nabla}_{q'}^2 \Pi_N(z^p, 0; z^{p'}, 0) \, , \quad p, p' = 1, \ldots, n, \quad q, q' = 1, \ldots, 2m \, .$$

Here, ∇_q^1, respectively ∇_q^2, denotes the differential operator on $X \times X$ given by applying ∇_q to the first, respectively second, factor. The link between the JPD and the Szego kernel stems ultimately from the fact

that Π_N is the covariance matrix of γ on $\mathcal{H}_N$. Using the scaling asymptotics of the Szegö kernel we obtain (in the complex case):

THEOREM 3.5. ([**ShZe2**], Theorem 5.4) *With the same notations and assumptions, we have:*

$$\widetilde{\mathbf{D}}^N_{(z^1/\sqrt{N},...,z^n/\sqrt{N})} \longrightarrow \mathbf{D}^\infty_{(z^1,...,n^n)} = \gamma_{\Delta^\infty(z)}$$

where $\mathbf{D}^\infty_{(z^1,...,z^n)}$ is a universal Gaussian measure, and $\Delta^N(z/\sqrt{N}) \to \Delta^\infty(z)$.

The covariance matrix Δ^∞ is given in terms of the Szegö kernel for the Heisenberg group:

(2)
$$\Delta^\infty(z) = \frac{m!}{c_1(L)^m} \left(\begin{array}{cc} A^\infty(z) & B^\infty(z) \\ B^\infty(z)^* & C^\infty(z) \end{array} \right),$$

where

$$
\begin{aligned}
A^\infty(z)^p_{p'} &= \Pi^{\mathbf{H}}_1(z^p,0;z^{p'},0), \\
B^\infty(z)^p_{p'q'} &= \begin{cases} (z^p_{q'} - z^{p'}_{q'})\Pi^{\mathbf{H}}_1(z^p,0;z^{p'},0) & \text{for} \quad 1 \le q \le m \\ 0 & \text{for} \quad m+1 \le q \le 2m \end{cases}, \\
C^\infty(z)^{pq}_{p'q'} &= \begin{cases} (\delta_{qq'} + (\bar{z}^{p'}_q - \bar{z}^p_q)(z^p_{q'} - z^{p'}_{q'}))\Pi^{\mathbf{H}}_1(z^p,0;z^{p'},0) & \text{for} \quad 1 \le q, q' \le m \\ 0 & \text{for} \quad \max(q,q') \ge m+1 \end{cases}.
\end{aligned}
$$

The scaling limit thus gives the correlations between zeros in the 'Heisenberg ensemble', an infinite dimensional Gaussian ensemble. In the real case (i.e. S^m), the limit correlations coincide with those in an ensemble related to the Euclidean motion group, and involving the Bessel kernel [**N**].

3.4. Hole probabilities. Another application of the scaling applications is to 'hole probabilities', i.e. the probability that a ball $B_r(z_0)$ of radius r around a point $z_0 \in M$ is zero-free. The following result combines our scaling asymptotics and Sodin's reformulation (and substantial simplification) of Offord's large deviations results on hole probabilities for entire analytic functions in the plane [**O, So**]. It is based on the Poincare-Lelong formula, so at this time of writing it has only been proved for one holomorphic section, and no comparable results have been proved for random spherical harmonics.

THEOREM 3.6. *Let $P_N(D;z_0) = Prob\{s \in H^0(M,L^N) : Z_s \cap B_{\frac{D}{\sqrt{N}}}(z_0) = \emptyset\}$. Then there exists positive constants C_1, C_2 such that, for any N, $P_N(D;z_0) \le C_1 e^{-C_2 D^2}$.*

4. Quantum ergodicity and random waves

We end with a brief discussion of the connection between random polynomials and quantum chaos. The intutitive idea that quantum chaotic eigenfunctions should resemble 'Gaussian random waves' seems to have been first suggested by M. V. Berry [**B**]. A precise formulation of this random wave model and some numerical results are given in [**ABST**] [**HR**].

Random spherical harmonics on S^m, or random combinations of eigenfunctions of Laplacians on general compact Riemannian manifolds as described above, provide a rather different random wave model for quantum chaotic eigenfunctions. To motivate the model, we recall that the diagonal sums of squares

(3)
$$S_p(\lambda) = \sum_{j:\lambda_j \le \lambda} |(A\varphi_j, \varphi_j) - \bar{\sigma_A}|^p, \quad (\text{with } \bar{\sigma_A} = \int_{S^*M} \sigma_A d\mu),$$

and their off-diagonal analogues, are used to characterize eigenfunctions as quantum ergodic, quantum mixing and so on. Here, A is an observable (zeroth order pseudodifferential operator), σ_A is its principal symbol. For quantizations of ergodic systems, $S_p(\lambda) \to 0$ as $\lambda \to \infty$, and it is of some interest to measure the rate and to relate it to the classical dynamics. In the random spherical harmonics model one has the following rate (for similar results see [**Ze2, ShZe**]):

THEOREM 4.1. ([**Ze1**], *Lemma (2.15)) Let $\{\varphi_{Nj}\}$ be a random orthonormal basis of spherical harmonics of S^2. Then: $E(S_2(N)) = \frac{1}{N}[\frac{1}{vol(S^*S^2)} \int_{S^*S^2} |\sigma^{ave}_A(\zeta) - \bar{\sigma_A}|^2 d\mu(\zeta)] + O(1/N^2)$.*

The optimist may conjecture a similar rate for quantum chaotic eigenfunctions. In [**Ze1**] it is further proved that almost all orthonormal bases of spherical harmonics are quantum ergodic, and this was improved to quantum unique ergodicity by VanderKam [**V**]. On the level of quantum mixing, however, random spherical harmonics do not provide a good model, but random combinations of Laplace eigefunctions on generic Riemannian manifolds do; that was the motivation for studying the model in [**Ze2**]. Quantum mixing systems involve off-diagonal sums like (3) with constant gaps between eigenvalues. Conversely, if the eigenfunctions satisfy (3) and the analogous off-diagonal estimates for all gaps, then the system is classically mixing. Hence the random wave model is not a good model for ergodic systems which fail to be mixing (cf. [**HKZ**]).

A further relation between Gaussian random waves and quantum chaotic eigenfunctions does not seem to have been explored, even numerically. Let $\{\varphi_\lambda\}$ be eigenfunctions of a quantum chaotic system, and consider the local rescaling $\varphi_\lambda(x_0 + \frac{u}{\lambda})$. The rescaled eigenfunction is an eigenfunction of the rescaled operator, which is asymptotically Euclidean. Hence $\varphi_\lambda(x_0 + \frac{u}{\lambda})$ can be expanded asymptotically in terms of plane waves, and one might ask how the frequencies are distributed.

References

[ABST]   R. Aurich, A. Backer, R. Schubert, and M. Taglieber, Maximum norms of chaotic quantum eigenstates and random waves, Physica D 129 (1999), 169-89.

[AS]   R. Aurich and F. Steiner, Statistical properties of highly excited quantum eigenstates of a strongly chaotic system, Physica D 64 (1993), 185-214.

[Be]   P. Berard, Volume des ensembles nodaux des fonctions propres du laplacien. (French) [Volume of the nodal sets of eigenfunctions of the Laplacian] Bony-Sjostrand-Meyer seminar, 1984–1985, Exp. No. 14, 10 pp., cole Polytech., Palaiseau, 1985.

[B]   M. V. Berry, Regular and irregular semiclassical wavefunctions, J. Phys. A 10 (1977), 2083-2091.

[BSZ1]   P. Bleher, B. Shiffman and S. Zelditch, Poincaré-Lelong approach to universality and scaling of correlations between zeros, *Comm. Math. Phys.* 208 (2000), 771–785.

[BSZ2]   P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds, *Invent. Math.*, to appear, http://xxx.lanl.gov/abs/math-ph/9904020.

[BSZ3]   P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of zeros on symplectic manifolds, to appear in MSRI volume on Random Matrices.

[BBL]   E. Bogomolny, O. Bohigas, and P. Leboeuf, Quantum chaotic dynamics and random polynomials, *J. Stat. Phys.* 85 (1996), 639–679.

[BoGu]   L. Boutet de Monvel and V. Guillemin, *The Spectral Theory of Toeplitz Operators*, Ann. Math. Studies 99, Princeton Univ. Press, Princeton, 1981.

[Don]   S. Donaldson, Symplectic submanifolds and almost complex geometry, *J. Diff. Geom.* 44 (1996), 666–705.

[GH]   P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[Han]   J. H. Hannay, Chaotic analytic zero points: exact statistics for those of a random spin state, *J. Phys. A: Math. Gen.* 29 (1996), 101–105.

[HR]   D.A. Hejhal and B. N. Rackner, On the topography of Maass wave forms for $PLS(2,\mathbb{Z})$: experiments and heuristics, Exp. Math. 1 (1992), 275-305.

[HKZ]   L. Kaplan, E. J. Heller, Weak quantum ergodicity. Phys. D 121 (1998), no. 1-2, 1–18; S. Zelditch, Appendix to: "Weak quantum ergodicity" [Phys. D 121 (1998), 19–23.

[JN]   D. Jakobson and Nadirashvili, Eigenfunctions with few critical points (preprint, 2000).

[Kac]   M. Kac, On the average number of real roots of a random algebraic equation, *Bull. Amer. Math. Soc.* 49 (1943), 314–320.

[L]   H. Lewy, On the minimum number of domains in which the nodal lines of spherical harmonics divide the sphere, Comm. Partial Differential Equations 2 (1977), no. 12, 1233–1244.

[LO]   J. Littlewood and A. Offord, On the number of real roots of random algebraic equations I, II, III, *J. London Math. Soc.* 13 (1938), 288–295; *Proc. Camb. Phil. Soc.* 35 (1939), 133–148; *Math. Sborn.* 12 (1943), 277–286.

[N]   J. Neuheisel, The asymptotic distribution of nodal sets on spheres, Johns Hopkins University PhD thesis (2000).

[NoVo]   S. Nonnenmacher and A. Voros, Chaotic eigenfunctions in phase space, *J. Stat. Phys.* 92 (1998), 431–518.

[O]   A. C. Offord, The distribution of zeros of power series whose coefficients are independent random variables, Indian J. Math. 9 (1967), 175-196.

[ShZe]   B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles, *Comm. Math. Phys.*, 200 (1999), 661–683.

[ShZe2]   B. Shiffman and S. Zelditch, Random almost holomorphic sections of ample line bundles over symplectic manifolds (preprint, 2000).

[ShZe3]   B. Shiffman and S. Zelditch, Distribution of critical points of random polynomials (in preparation).

[ShZe4]   B. Shiffman and S. Zelditch, Random polynomials with fixed Newton polytope (in preparation).

[So]   M. Sodin, Zeros of Gaussian analytic functions, Math. Res. Lett. (to appear).
[V]   J. M. VanderKam, L^∞ norms and quantum ergodicity on the sphere, *Int. Math. Res. Notices* 7 (1997), 329–347.
[Ze1]   S. Zelditch, Quantum ergodicity on the sphere, Comm. Math. Phys., **146**, 61-71 (1992).
[Ze2]   S. Zelditch, A random matrix model for quantum mixing, Int.Math. Res.Not. 3 (1996), 115-137.

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