

MACDONALD'S IDENTITIES AND THE LARGE N LIMIT OF YM_2 ON THE CYLINDER

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ABSTRACT. The purpose of this paper is to determine the large N asymptotics of the free energy $F_N(a, U|A)$ of YM_2 (two-dimensional Yang Mills theory) with gauge group $G_N = SU(N)$ on a cylinder where a is a so-called principal element of type ρ . Mathematically,

$$F_N(U_1, U_2|A) = \frac{1}{N^2} \log H_{G_N}(A/2N, U_1, U_2)$$

is the central heat kernel of G_N . We find that

$$F_N(a_N, U_N|A) \sim \frac{N}{A} \Xi(d\theta, d\sigma)$$

where Ξ is an explicit quadratic functional in the limit distribution $d\sigma$ of eigenvalues of U_N , which depends only on the integral geometry of $SU(2)$. The factor of N appears to contradict some predictions in the physics literature on the large N limit of YM_2 on the cylinder (due to Gross-Matytsin, Kazakov-Wynter others).

1. INTRODUCTION

The purpose of this note is to determine the large N asymptotics of many cases of the free energy $F_N(U_{C_1}, U_{C_2}|A)$ of YM_2 (two-dimensional Yang Mills theory) with gauge group $SU(N)$ on a cylinder. Interest in this large N limit problem was raised around ten years ago in a series of physics papers by Douglas, Kazakov, Wynter, Gross, Matytsin and others [DK, KW, GM1, GM2]. They predicted, on the basis of physical and formal mathematical arguments, that the large N limit of $F_N(U_{C_1}, U_{C_2}|A)$ should be well-defined and related to the action along a path of the complex Burgers equation with boundary densities determined by the limiting eigenvalue densities of U_{C_1}, U_{C_2} . In certain cases they predict a phase transition between a weak coupling and a strong coupling regime.

Our results give the asymptotics of $F_N(a_N, U_N|A)$ where a_N is a *principal element of type* ρ of $SU(N)$ in the sense of [Ko] (see Section 2), and where the second sequence $\{U_N\}$ can be any sequence of elements of $SU(N)$ which possesses limiting eigenvalue density $d\sigma$. The limiting eigenvalue density of a_N is uniform measure $d\theta$ on S^1 . Our asymptotics reveal some surprising results:

- $F_N(U_{C_1}, U_{C_2}|A) \sim \frac{N}{A} \Xi(d\theta, d\sigma)$ for a certain (explicit) functional Ξ , unlike the predictions that it tend to a limit functional as $N \rightarrow \infty$. The extra factor of N signals an error in some of the heuristic physics arguments, and (as emphasized to the author by M. Douglas and C. Vafa) calls into question whether the large N of YM_2 on the cylinder is actually a string theory as discussed in [GM1, GM2, KW] and elsewhere.

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At least, it indicates that extra hypotheses on the matrix pairs U_{C_1}, U_{C_2} are necessary for the conjectured picture to be correct.

- The principal asymptotic term is visibly analytic in A , i.e. it never exhibits a phase transition to leading order, even though $d\sigma$ could be any probability measure on S^1 . This aspect of the results is consistent with the physics predictions, in that the boundary condition $d\theta$ puts the system in its strong coupling regime and it should therefore not exhibit a phase transition between weak and strong coupling [D].

The calculation presented here appears to be the first rigorous calculation of the large N limit of the partition function for YM_2 on the cylinder. Hence, it is unclear at this time whether the anomalous factor of N only occurs in the special case where $U_{C_1} = a_N$ is a principal element of type ρ or whether it holds more generally in this context.

We should emphasize that the other rigorous mathematical results (of which we are aware) have largely confirmed the large N limit picture developed in [DK, GM1, GM2, KW] and elsewhere. The Douglas-Kazakov phase transition of the genus 0 partition function at $A = \pi^2$ has been proved by Boutet de Monvel - Shcherbina in [BS]. A large deviations analysis of spherical integrals by Guionnet-Zeitouni in [GZ] justifies some of the predictions by Matytsin [M] and Gross-Matytsin [GM1] on the asymptotics of characters $\chi_R(U)$. However, the results of [GZ] pertain to the analytic continuation of χ_R to positive elements of $GL(N, \mathbb{C})$ and it is possible that Matytsin's predictions fail when $U = a_N$. We intend to explore this question in a future article.

To state our results, we introduce some notation. The partition function of YM_2 on a cylinder, with gauge group equal to G , is given by Migdal's formula (see [W, W2]):

$$\mathcal{Z}_G(U_1, U_2|A) = \sum_{R \in \widehat{G}} \chi_R(U_1) \chi_R(U_2^*) e^{-\frac{A}{2N} C_2(R)} \quad (1)$$

Here, $A \geq 0$ is the area of the cylinder, and the sum runs over the irreps (irreducible representations) of G , with χ_R the character of R and with $C_2(R)$ equal to the eigenvalue of the Casimir Δ in the irrep R . Thus, \mathcal{Z}_G is the value at time $t = \frac{A}{2N}$ of the *central* heat kernel of G :

$$H_G(t, U_1, U_2) = \sum_{R \in \widehat{G}} \chi_R(U_1) \chi_R(U_2^*) e^{-t C_2(R)}, \quad (2)$$

i.e. the kernel of the heat operator acting on the space of central functions on $SU(N)$. It is obtained from the usual heat kernel by averaging both variables over conjugacy classes. Since $\mathcal{Z}_G(U_1, U_2|A)$ is conjugacy invariant, one may assume that U_1, U_2 are diagonal and we write $U_j = D(e^{i\theta_1^j}, \dots, e^{i\theta_\ell^j})$, where ℓ is the rank of G (i.e. the dimension of its maximal torus). The main quantity of interest is the free energy, defined by

$$F_G(U_{C_1}, U_{C_2}|A) = \frac{1}{N^2} \ln \mathcal{Z}_G(U_{C_1}, U_{C_2}|A). \quad (3)$$

We now consider the large N limit of the partition function and free energy. The large N limit refers to an increasing sequence G_N of groups, e.g. the classical groups $G_N = U(N), SU(N), SO(N), Spin(N)$. For the sake of simplicity we restrict attention to $SU(N)$ and we abbreviate $F_{SU(N)}$ by F_N (etc.) The limit we are interested in is a pointwise limit of the central heat kernel, which obviously requires some discussion since the space on which the heat kernel is defined changes with N .

The large N limit of F_N is defined as follows: take a pair of sequences $\{U_{Nj}\}, U_{Nj} \in SU(N)$ ($j = 1, 2$) of elements whose eigenvalue distributions

$$d\sigma_{Nj} := \frac{1}{\ell_N} \sum_{k=1}^{\ell_N} \delta(e^{i\theta_k^{Nj}}) \in \mathcal{M}(S^1)$$

tend to a limit measures σ_j . Here, $\ell_N = N - 1$ denotes the rank of the group, and $\mathcal{M}(S^1)$ denotes the probability measures on the unit circle, and convergence is in the weak sense of measures. We will denote this situation by $U_{Nj} \rightarrow \sigma_j$. That is,

$$U_{Nj} \rightarrow \sigma_j \iff \frac{1}{\ell_N} \sum_{k=1}^{\ell_N} \delta(e^{i\theta_k^{Nj}}) \rightarrow \sigma_j \in \mathcal{M}(S^1), \quad (j = 1, 2). \quad (4)$$

CONJECTURE 1.1. [GM1] (pages 8-9) *Assume that $U_1 \rightarrow \sigma_1, U_2 \rightarrow \sigma_2$. Then*

$$F_N(U_1, U_2|A) \rightarrow F(\sigma_1(\theta), \sigma_2(\theta)|A) = S(\sigma_1(\theta), \sigma_2(\theta)|A)$$

$$-\frac{1}{2} \int_{S^1} \int_{S^1} \sigma_1(\theta) \sigma_1(\phi) \log |\sin \frac{\theta-\phi}{2}| d\theta d\phi - \frac{1}{2} \int_{S^1} \int_{S^1} \sigma_2(\theta) \sigma_2(\phi) \log |\sin \frac{\theta-\phi}{2}| d\theta d\phi,$$

where the functional S is a solution of the Hamilton Jacobi equation

$$\frac{\partial S}{\partial A} = \frac{1}{2} \int_0^{2\pi} \sigma_1(\theta) \left[\left(\frac{\partial}{\partial \theta} \frac{\delta S}{\delta \sigma_1(\theta)} \right)^2 - \frac{\pi^2}{3} \sigma_1^2(\theta) \right].$$

It is often assumed in the physics articles that the limit measures have densities. The function S can be identified as the action along the solution of the boundary value problem for Hopf-Burgers equation

$$\begin{cases} \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial x} = 0, \\ \Im f(0, x) = \sigma_1, \quad \Im f(A, x) = \sigma_2. \end{cases}$$

Under certain assumptions on the limit densities, it is further conjectured that a third order phase transition between the weak and strong coupling regimes should occur. In the case of the disc (the cylinder with $U_1 = I$), Kazakov-Wynter and Gross-Matytsin have predicted a phase transition point at

$$A = \frac{\pi}{2\Theta(\pi)}, \quad \Theta(\theta) := \int \frac{d\sigma(\theta')}{\theta - \theta'},$$

Here, $d\sigma$ is the limit distribution for U_2 and the integral is presumably to be understood as the Hilbert transform on S^1 (although it is written as the Hilbert transform on \mathbb{R}). The weak coupling regime is characterized by a gap in the support of the limit density (i.e. an interval of S^1 on which $d\sigma_1$ vanishes).

The mathematical evidence for these conjectures is largely based on approximating the discrete sum (1) over $R \in \hat{G}$ by a continuous (and only partially defined) integral over a space of “densities of Young tableaux”, to which the saddle point method is applied (cf. §3). These arguments are not rigorous and also make some implicit assumptions (see the discussion after Theorem 1.3).

The main observation of the present article is that there exists a simple, rigorous alternative for calculating the large N limit when one of the arguments is a so-called principal element

of type ρ . This alternative method is based on the use MacDonalD's identities (in Kostant's formulation) to factor the partition function as a product over positive roots. Background on MacDonalD's identities will be provided in §2 (see also [MAC, Ko, F, PS]).

The first result one obtains this way is a limit formula in which the time variable in the partition function is *not* rescaled. We use the notation $d\sigma * \bar{d}\sigma(x)$ for the measure defined by

$$\int_{S^1} f(e^{ix}) d\sigma * \bar{d}\sigma(x) := \int_{S^1} \int_{S^1} f(e^{i(x-y)}) d\sigma(x) d\sigma(y). \quad (5)$$

Thus, $*$ denotes convolution of measures and $\bar{d}\sigma$ is short for $d\sigma(\bar{x})$. For notational simplicity we sometimes denote $e^{ix} \in S^1$ more simply by x .

THEOREM 1.2. *Let $k_N \in \mathfrak{su}(N)$ be diagonal matrices with entries θ_j^N , and assume $d\sigma_N := \frac{1}{\ell_N} \sum_{j=1}^{\ell_N} \delta(e^{i\theta_j^N}) \rightarrow \sigma$. Then, as $N \rightarrow \infty$,*

$$\frac{1}{N^2} \log H_{SU(N)}(t, a_N, e^{k_N}) \rightarrow -\frac{1}{2} \log \eta(it) + \frac{1}{2} \int_{\mathbb{R}} \log H_{SU(2)}(t, e^{ix}, e^{\frac{i\pi}{2}}) d\sigma * \bar{d}\sigma(x).$$

Here, $H_{SU(2)}$ is the central heat kernel of $SU(2)$ and e^{ix} is short for the diagonal matrix $D(e^{ix}, e^{-ix})$. The element a is the principal element of type ρ of $SU(2)$, namely $a = D(e^{\frac{i\pi}{2}}, e^{-\frac{i\pi}{2}})$.

As discussed in [F] (Proposition 1.3), the central heat kernel of $SU(2)$ at these special values is given by

$$H_{SU(2)}(t, e^{ix}, e^{\frac{i\pi}{2}}) = \frac{\theta_1(e^{i\pi x}, it)}{2e^{-\pi t/4} \sin \pi x} \quad (6)$$

where

$$\theta_1(z, t) = 2 \sum_{n=0}^{\infty} (-1)^n \sin\{(2n+1)z\} e^{-\pi(n+1/2)^2 t} \quad (7)$$

is Jacobi's theta function. MacDonalD's identities are more often stated in terms of Jacobi theta functions, but we find it easier to work directly with heat kernels. We note that the factor of π in the exponent $\pi(n+1/2)^2 t$ is responsible for the appearance of πt in many expressions to follow in the heat kernel.

At first sight, this result seems to explain the normalization $\frac{1}{N^2} \log \mathcal{Z}_N$. However, the large N limit conjectures concern the scaling limit of the central heat kernel under $t \rightarrow \frac{A}{2N}$. This puts into play a simultaneous limit process in $d\sigma_N * \bar{d}\sigma_N \rightarrow d\sigma * \bar{d}\sigma$ and in the asymptotics of theta functions. We find that for our cases of the problem, this rescaling changes the growth rate of the free energy.

THEOREM 1.3. *Suppose that e^{k_N} is a sequence such that $d\sigma_N \rightarrow d\sigma$. Then,*

$$\begin{aligned} F_N(a_N, e^{k_N} | \frac{A}{2N}) &= \frac{1}{2} \int_{S^1} \log H_{SU(2)}(A/2N, e^{ix}, e^{\frac{i\pi}{2}}) d\sigma_N * \bar{d}\sigma_N(e^{ix}) - \frac{1}{2} \log \eta(\frac{iA}{2N}) + O(1/N) \\ &\sim -\frac{N}{A} \left\{ \int_{S^1} \frac{1}{\pi} \min\{d(e^{ix}, e^{i\pi/2}), d(e^{ix}, e^{-i\pi/2})\}^2 d\sigma * \bar{d}\sigma(e^{ix}) - \frac{\pi}{12} \right\}, \end{aligned}$$

where $d(e^{ix}, e^{iy})$ is the distance along S^1 .

The minimal distance above arises as the distance $d(C(a), e^{ix})$ in $SU(2)$ of the diagonal element $D(e^{ix}, e^{-ix})$ to the conjugacy class $C(a)$ of a of the principal element of type ρ , which is conjugate to $D(e^{i\pi/2}, e^{-i\pi/2})$.

We note that there are two terms of opposite sign in the leading order term. We note that the terms cancel when $U_1 = a = U_2$, since then $d\sigma * \overline{d\sigma} = \frac{d\theta}{2\pi}$ and the first term reduces to $-\frac{N}{A} \left\{ \frac{1}{\pi} \frac{4}{2\pi} \frac{1}{3} \left(\frac{\pi}{2}\right)^3 - \frac{\pi}{12} \right\} = 0$. This appears to be the kind of case studied in [GM1]. But the two terms cannot cancel in all cases, and indeed, the leading term does not cancel in the simplest case, where $k_N = 0$ for all N , i.e. where $U_2 = Id$. We then have (see [F1] or [F], Proposition 1.2)

$$H_{SU(N)}(t, a_N, I) = e^{-\dim SU(N) t/24} \eta(it)^{\dim SU(N)}, \quad (8)$$

where $\eta(t)$ is Dedekind's η -function. As is easy to see (and will be verified below), the eigenvalue distribution of a_N tends to $\frac{d\theta}{2\pi}$, while that of I is obviously δ_1 . In this case, the asymptotic mass equals 1 and the continuous term equals zero. We separate out this special case since the result is most easily checked on this example:

PROPOSITION 1.4. *When $U_1 = a_N$ and $U_2 = I$, so that $\sigma_1 = d\theta, \sigma_2 = \delta_1$, then*

$$\frac{1}{N^2} \log \mathcal{Z}_{SU(N)}(e^{4\pi i \rho}, 1|A) = -\frac{2N}{A} \left\{ \frac{\pi}{12} \right\} - \frac{1}{2} \log\left(\frac{A}{2N}\right) - A/48N + O(e^{-cN}).$$

Note that Theorem 1.3 reduces to Proposition 1.4 when $d\sigma = \delta_0$.

Some final comments on the anomalous extra factor of N . The prediction that $\frac{1}{N^2} \log \mathcal{Z}_N(A)$ should have a limit determined by a variational problem is one of many predictions of this kind in field theory and statistical mechanics and it seems very strange that an extra factor of N should appear in our calculations. It is a mystery how it would appear in the graphical or diagrammatic calculations which initially suggested that the large N limit of gauge theory is a string theory, i.e. without using (1). It seems best to leave it to the string theorists to decide how much the anomalous factor of N affects the picture of the large N limit of gauge theory as a string theory.

We are on somewhat safer ground in trying to account for the extra factor of N in the formal calculations based on (1). From discussions with M. Douglas and V. Kazakov, it seems plausible that the anomaly is due to the unusual behavior of the character values $\chi_R(a)$ at the principal element of type ρ . It was proved by Kostant [Ko] that the only character values at this element are $\chi_R(a) = -1, 0, 1$ and it is reasonable to expect that the value oscillates regularly between these values. It appears that the physics predictions implicitly assumed a less oscillatory behaviour in character values, and in particular less oscillation in the signs of character values. The heuristic calculations in [GM1, GM2, KW] of the large N limit of the free energy on the cylinder were based on special cases (such as $U_2 = U_1^*$) which do not have such oscillations in sign. The sign oscillation causes much more cancellation than expected, and this could explain why our asymptotics have the form $e^{-N^3 \Xi}$ rather than $e^{-N^2 \Xi}$. These special values $\chi_R(a)$ might also be inconsistent with Matytsin's character asymptotics, and we plan to check this in the future.

This paper is organized as follows. In Section 2, we review MacDonal's identity and associated objects which we will use in the proof of the main result. The main results are proved in Section 3. In Section 4, we discuss some aspects of the proof and mention some other results which could be proved by the methods of this paper.

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and the predictions in the physics papers. This work was partially supported by the Clay Mathematics Insitute.

2. BACKGROUND ON MACDONALD'S IDENTITIES

In this section, we review the η function and MacDonal's identity. Our main references are the articles [Fr] of I. Frenkel and those [F, F1] of H. Fegan. Since the articles employ very different, possibly confusing, notational conventions, we highlight the main ones:

- In [Fr], the η -function is defined in a non-standard way in the lower half-plane (see §4.4) and consequently the heat kernel is evaluated at time $t = 4\pi ib$ where $\Im b < 0$. We will quote results of [Fr] in terms of t and we will use the usual definition of η as a modular form on the upper half plane.
- In [Fr], the special element of type ρ is denoted $e^{4\pi i\rho}$, but the exponent refers to the dual element of the Cartan subalgebra rather than the linear functional ρ . To avoid confusion, we denote it simply by a as in [F].
- Fegan works with the Schrodinger equation rather [F1] (1.1) rather than the heat equation, so our formulae differ from his in that his it is our t .

2.1. **η function.** The Dedekind eta-function [I, A] is defined by

$$\eta(z) = e^{2\pi iz/24} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}), \quad \Im z > 0. \quad (9)$$

It is a modular cusp form of weight $1/2$, i.e. it satisfies

$$\eta(\gamma z) = \theta(\gamma) j_{\gamma}(z)^{1/2} \eta(z), \quad \text{if } \gamma \in SL(2, \mathbb{Z}),$$

where $\theta(\gamma)$ is a certain multiplier which we will not need to know in detail (see [I], §2.8). When $\gamma z = -1/z$, we have

$$\eta\left(-\frac{1}{z}\right) = (iz)^{1/2} \eta(z).$$

In studying the free energy, we are particularly interested in $\log \eta(iy)$ as $y \rightarrow 0+$. The transformation law for $\log \eta(iy)$ goes as follows [A]: For real numbers $y > 0$ we have:

$$\log \eta(iy) = -\frac{\pi}{12y} - \frac{1}{2} \log y + \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{2\pi m/y}}. \quad (10)$$

2.1.1. *Jacobi's theta function.* Although we mainly use the heat kernel parametrix on $SU(2)$ to obtain asymptotic formulae, the alternative in terms of Jacobi's theta function (6) can be used to check the details. The small time expansion of the heat kernel can be obtained from Jacobi's imaginary transformation law:

$$\theta_1(-x/\tau, -1/\tau) = i(-i\tau)^{1/2} e^{-ix^2/\tau} \theta(x, \tau).$$

Thus, we have:

$$\theta_1\left(x, \frac{iA}{2N}\right) = i\left(\frac{2N}{A}\right)^{1/2} e^{-2x^2 N/A\pi} \theta_1\left(\frac{2iN}{A}x, \frac{2iN}{A}\right) \quad (11)$$

Here, we use that

$$\tau = \frac{iA}{2N} \implies -1/\tau = i\frac{2N}{A} \implies e^{-\pi\frac{A}{2N}} \rightarrow e^{-\pi\frac{2N}{A}}.$$

The large N or small time asymptotics now follow from (7).

2.2. Central heat kernels. Let G be any compact, connected Lie group. We denote by R the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, where \mathfrak{h} is its Cartan subalgebra. We denote by R_+ the positive roots, by P the lattice of weights and by $P_{++} \subset P$ the dominant weights.

We recall that the eigenvalue of the Casimir operator (bi-invariant Laplacian) Δ of G in the representation with highest weight λ is equal to

$$\Delta|_{V_\lambda} = (||\lambda + \rho||^2 - ||\rho||^2)Id|_{V_\lambda}, \quad \rho = 1/2 \sum_{\alpha \in R_+} \alpha.$$

The fundamental solution of the heat equation is given by $k(t, x, y) = \nu(xy^{-1}, t)$ where

$$\nu(g, t) = \sum_{\lambda \in P_{++}} (\dim V_\lambda) \chi_\lambda(g) e^{-t/2(||\lambda + \rho||^2 - ||\rho||^2)}, \quad (12)$$

As in [Fr], §4.3 it then follows that the central heat kernel is given by

$$H_G(t, h, k) = \sum_{\lambda \in P_{++}} \chi_\lambda(e^h) \chi_\lambda(e^{-k}) e^{-t/2(||\lambda + \rho||^2 - ||\rho||^2)}. \quad (13)$$

This is simply a different notation for (2).

2.3. $SU(2)$. As was recognized clearly by H. Fegan and others, the central heat kernel of $SU(2)$ plays an important role in MacDonalD's identities. There is one positive root α , which can be identified with 1 if we choose it as the basis of the Cartan dual subalgebra. Then $\rho = \frac{1}{2}$ and the Killing form is $B(x, y) = \frac{1}{2}xy$. In this case, the principal element a of type ρ has eigenvalues $e^{\pm \frac{i\pi}{2}}$. The weight lattice is $\frac{1}{2}\mathbb{Z}$. The character of the irreducible representation of highest weight λ is $\chi_\lambda(x) = \frac{\sin(2\lambda+1)x}{\sin \pi x}$. We have:

$$\chi_\lambda(a) = \begin{cases} -1, & \lambda \text{ is an odd integer} \\ 0, & \text{is not an integer} \\ 1, & \text{is an even integer} \end{cases}$$

The eigenvalues of the Casimir are $c(\lambda) = \frac{1}{2}\lambda(\lambda + 1)$.

The central heat kernel $H(t, a, y)$ at the special point a may be expressed in terms of Jacobi's theta function as in (6). As a special case of MacDonalD's identities, we further have (cf. [F1], (3.12)):

$$H_{SU(2)}(t, a, 1) = (e^{-\pi t/12} \eta(it))^3. \quad (14)$$

2.4. MacDonalD identities. We briefly review MacDonalD's identities for a compact, semi-simple, simply connected Lie group G and its relation (proved by Kostant [Ko]) to the central heat kernel. They are thus valid for $SU(N)$. We follow [F, F1, Fr].

MacDonalD's identity involves a conjugacy class C_a of special elements of G , which we will denote by a , which are conjugate to e^{2x_ρ} where x_ρ is dual under the Killing form to ρ . Such elements are called 'principal elements of type ρ '. A principal element of type ρ is characterized in [Ko] (see 1.3 p. 181) as a regular element such that the order of $Ad(a)$

equals h (the order of the Coxeter element). One has $h = N$ for $SU(N)$, so the eigenvalues of $Ad(a)$ must be distinct N th roots of unity. Hence

$$a_N \rightarrow \frac{d\theta}{2\pi}, \quad (15)$$

i.e. the eigenvalues of a_N become uniformly distributed in the large N limit. A detailed description of such elements can be found in [Ko, F1, AF].

In Kostant's formulation, as described by Frenkel, the MacDonald's identities take the form ([Fr], Proposition (4.4.5))

$$\theta(k, 4\pi it) = \sigma(k) \sum_{\lambda \in P_{++}} \chi_\lambda(a) \chi_\lambda(e^{-k}) e^{-t/2 \|\lambda + \rho\|^2}, \quad (16)$$

where (see [Fr], (1.1.8))

$$\sigma(k) = \prod_{\alpha \in R_+} (e^{\langle \alpha, k \rangle / 2} - e^{-\langle \alpha, k \rangle / 2}),$$

where $\theta(k, it)$ is the theta-function defined in Definition (4.4.1) of [Fr]: For $t > 0$,

$$\begin{aligned} \theta(k, it) &= e^{-2\pi \|\rho\|^2} \prod_{\alpha \in R_+} (e^{\frac{\langle \alpha, k \rangle}{2}} - e^{-\frac{\langle \alpha, k \rangle}{2}}) \\ &\cdot \prod_{n=1}^{\infty} (1 - e^{-2\pi n t})^\ell \prod_{\alpha \in R} (1 - e^{-2\pi n t + \langle \alpha, k \rangle}) \end{aligned} \quad (17)$$

For our purposes the key formula is the following ([Fr], Proposition (4.4.4)):

$$\theta(k, it) = \eta(it)^{-|R_+| + \ell} \prod_{\alpha \in R_+} \theta_1(\langle \alpha, k \rangle, it). \quad (18)$$

We put $\tilde{\theta}(x, it) = \frac{\theta_1(x, it)}{2e^{-t/4} \sin \pi x}$. The following is the version of MacDonald's identities proved in Theorem 1.5 of [F1] (together with its Proposition 1.3). As above, an element $e^{i\theta}$ is identified with a diagonal element $D(e^{i\theta}, e^{-i\theta})$ of $SU(2)$.

LEMMA 2.1. *Let G be compact, connected, semi-simple and simply connected, and let a be an element of type ρ . Then*

$$H_G(t, a, e^{-k}) = (e^{-\pi t/12} \eta(it))^{-|R_+| + \ell} \prod_{\alpha \in R_+} H_{SU(2)}(t, e^{i\langle \alpha, k \rangle}, e^{i\pi/2}).$$

Proof. Putting together (16), (18) and (13), we get

$$\begin{aligned} H_G(t, a, e^{-k}) &= \sigma(k)^{-1} e^{A/4N \|\rho\|^2} \theta(k, it) \\ &= e^{t \|\rho\|^2} \eta(it)^{-|R_+| + \ell} \prod_{\alpha \in R_+} \tilde{\theta}(\langle \alpha, k \rangle, it). \end{aligned} \quad (19)$$

The stated result now follows from (6). □

As mentioned above, the simplest case (20) comes from combining Lemma 2.1 and (14):

$$H_G(t, a, I) = e^{-\dim G t/24} \eta(it)^{\dim G}, \quad (20)$$

3. LARGE N LIMIT OF $\mathcal{Z}_{SU(N)}(a, U|A)$: PROOF OF THEOREM 1.3

3.1. **Simplest case.** We first consider the simplest case (20). Since $k = 0$ (i.e. $U_2 = I$), we are essentially dealing with YM_2 on the disc. This case is discussed in detail in [GM2], §4.

PROPOSITION 3.1. *When $U_1 = a$ and $U_2 = 1$, so that $\sigma_1 = d\theta, \sigma_2 = \delta_1$, then*

$$\frac{1}{N^2} \log \mathcal{Z}_{SU(N)}(a, 1; A) = -\frac{2N}{A} \frac{\pi}{24} - \frac{1}{2} \log\left(\frac{A}{2N}\right) - A/48N + O(e^{-cN}).$$

Thus, no phase transition occurs.

Proof. Macdonald's identity gives:

$$\mathcal{Z}_{SU(N)}(a, 1; A) = e^{-\pi \dim SU(N) A/24N} \eta(iA/2N)^{\dim SU(N)}, \quad (21)$$

The free energy is then

$$\frac{1}{N^2} \log H_{SU(N)}(A/2N, a, I) = \frac{\dim SU(N)}{N^2} \left\{ -\frac{A\pi}{48N} + \log \eta(iA/2N) \right\}. \quad (22)$$

We note that $\frac{\dim SU(N)}{N^2} = \frac{1}{2} + O(\frac{1}{N})$. We substitute $y = A/2N$ in the right side of (10) to get:

$$\log \eta(iA/2N) = -\frac{2N}{A} \frac{\pi}{24} - \frac{1}{2} \log\left(\frac{A}{2N}\right) + \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{4N\pi m/A}} \quad (23)$$

The summand of the sum $\sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{4N\pi m/A}}$ is smaller than $1/2e^{-4N(\pi/A)m}$ for N sufficiently large, so we may bound the sum by the geometric series and obtain a bound of

$$\frac{1}{1 - e^{-4N(\pi/A)}} - 1 \leq C e^{-4N(\pi/A)}, \quad \text{for } N \text{ sufficiently large.}$$

Thus, the sum is an exponentially small correction, and we have

$$\begin{aligned} \frac{1}{N^2} \log \mathcal{Z}_{SU(N)}(a, 1; A) &= \frac{\dim SU(N)}{N^2} \left\{ -A/48N - \frac{\pi N}{6A} - \frac{1}{2} \log\left(\frac{A}{2N}\right) \right\} + O(e^{-cN/A}). \\ &\sim -\frac{\pi}{24} \frac{2N}{A} - \frac{1}{2} \log\left(\frac{A}{2N}\right) - A/48N. \end{aligned} \quad (24)$$

□

We see clearly the anomalous factor of N . Also, we see that there is no phase transition, despite the fact that the two eigenvalues densities, $\delta(1), d\theta$ have very different support properties. In almost exactly this case, the lack of phase transition is predicted by Gross-Matytsin in [GM2], §4. They explain that when the support of σ_1 is the whole circle and the support of σ_2 is a single point, then the system is always in the strong coupling phase.

They further explain that when $\sigma_1 = \delta(\theta = 0)$, the solution of the Hopf equation can be obtained as the solution of the integral equation

$$\left(1 - \frac{t}{A}\right) f(t, \theta) = \frac{-\theta}{A} + \int \frac{\sigma_1(\theta') d\theta'}{\theta - \theta' - t f(t, \theta)}.$$

In our case, $\sigma_1 = 1$ so the equation is simply

$$\left(1 - \frac{t}{A}\right) f(t, \theta) = \frac{-\theta}{A} - \log(\theta - t f(t, \theta)).$$

It would be interesting to solve this equation and check that the solution is analytic in A .

3.2. Proof of Theorem (1.2). We may take e^k to be a diagonal matrix with entries $e^{2\pi i\lambda_j(N)}$. First, we put $t = A/2N$. By Lemma 2.1 we have:

$$\begin{aligned} \frac{1}{N^2} \log H_{SU(N)}(t, a, e^k) &= \frac{1}{N^2} \{(-|R_+| + N) [-\frac{\pi t}{12} + \log \eta(-it/4\pi)]\} \\ &+ \frac{1}{N^2} \sum_{\alpha > 0} \log H_{SU(2)}(t, e^{i\langle \alpha, k \rangle}, e^{i\pi/2}) \end{aligned} \quad (25)$$

We note that both sides of this equation are real, so that we take the real part \Re without changing the equation.

We now specialize to the case of $SU(N)$. Its roots are $e_i - e_j$ and its positive roots satisfy $i < j$. Hence, $\langle \alpha, k \rangle = \lambda_i - \lambda_j$. Hence

$$\begin{aligned} \sum_{\alpha \in R_+} \log H_{SU(2)}(t, e^{i\langle \alpha, k \rangle}, e^{i\pi/2}) &= \sum_{i < j} \log H_{SU(2)}(t, e^{i(\lambda_i - \lambda_j)}, e^{i\pi/2}) \\ &= \frac{1}{2} \sum_{i \neq j} \log H_{SU(2)}(t, e^{i(\lambda_i - \lambda_j)}, e^{i\pi/2}). \end{aligned} \quad (26)$$

In the last equality we use that $H_{SU(2)}(t, e^{i(\lambda_i - \lambda_j)}, e^{i\pi/2}) = H_{SU(2)}(t, e^{i(\lambda_j - \lambda_i)}, e^{i\pi/2})$.

We further have

$$\sum_{i \neq j} \log H_{SU(2)}(t, e^{i(\lambda_i - \lambda_j)}, e^{i\pi/2}) = \sum_{i, j} \log H_{SU(2)}(t, e^{i(\lambda_i - \lambda_j)}, e^{i\pi/2}) - N \log H_{SU(2)}(t, 1, e^{i\pi/2}).$$

Since $\ell_N \sim N$ for $SU(N)$, we have

$$\begin{aligned} \frac{1}{N^2} \sum_{\alpha \in R_+} \log H_{SU(2)}(t, e^{i\langle \alpha, k \rangle}, e^{i\pi/2}) &= \int_{S^1} \int_{S^1} \log H_{SU(2)}(t, e^{ix}, e^{i\pi/2}) d\sigma_N(e^{i(x-x')}) \overline{d\sigma_N}(e^{ix'}) \\ &- \frac{1}{N} \log H_{SU(2)}(t, 1, e^{i\pi/2}) \\ &= \int_{S^1} \log H_{SU(2)}(t, e^{ix}, e^{i\pi/2}) d\sigma_N * \overline{d\sigma_N}(e^{ix}) \\ &- \frac{1}{N} \log H_{SU(2)}(t, 1, e^{i\pi/2}) \end{aligned}$$

where

$$d\sigma_N * \overline{d\sigma_N}(e^{ix}) = \int_{S^1} d\sigma_N(e^{i(x-x')}) d\sigma_N(e^{ix'}).$$

If

$$\sigma_N := \frac{1}{\ell_N} \sum_{j=1}^{\ell_N} \delta(e^{2\pi i\lambda_j(N)}) \rightarrow \sigma \in \mathcal{M}(S^1),$$

then

$$d\sigma_N * \overline{d\sigma_N} \rightarrow d\sigma * \overline{d\sigma},$$

since the Fourier coefficients of the left side tend to those of the right side. Thus, we obtain the stated limit.

3.3. Proof of Theorem 1.3. We now re-do the calculation but make the scaling $t = \frac{A}{2N}$. We thus have

$$\begin{aligned} F_{SU(N)}(a, e^k; A) &= \frac{1}{2} \int_{S^1} \log H_{SU(2)}\left(\frac{A}{2N}, e^{ix}, e^{i\pi/2}\right) d\sigma_N * \overline{d\sigma_N}(e^{ix}) \\ &- \frac{1}{N} \log H_{SU(2)}\left(\frac{A}{2N}, 1, e^{i\pi/2}\right) \end{aligned} \quad (27)$$

We now obtain the limit by using the uniform off-diagonal asymptotics of the central heat kernel.

The central heat kernel is related to the actual heat kernel by

$$\log H_{SU(2)}(t, e^{ix}, a) = \log \int_{SU(2)} k_{SU(2)}(t, e^{ix}, g^{-1}ag) dg$$

where $k_{SU(2)}(t, x, y)$ is the heat kernel. Therefore, we are interested in the uniform asymptotics of

$$\log H_{SU(2)}\left(\frac{A}{2N}, e^{ix}, a\right) = \log \int_{SU(2)} k_{SU(2)}\left(\frac{A}{2N}, e^{ix}, g^{-1}ag\right) dg \quad (28)$$

in x for each A . There exists a uniform heat kernel parametrix for $k_{SU(2)}$ given by:

$$k_{SU(2)}(t, u, v) \sim t^{-3/2} e^{-\frac{d(u,v)^2}{\pi t}} V(t, u, v) \quad (29)$$

where $V(t, u, v) \sim \sum_{j=0}^{\infty} V_j(u, v) t^j$. The amplitude V is to leading order the volume density in normal coordinates. We will never be evaluating it at a pair of conjugate points, so it is uniformly bounded below by a positive constant and introduces only lower order terms into the logarithm. See for instance [Ka] for general results of this kind. Instead of a parametrix, the reader might prefer to use the (6) in terms of Jacobi's theta function. This formula and also (14) can be used to check the various constants in the formula.

Thus, we have

$$\log H_{SU(2)}\left(\frac{A}{2N}, e^{ix}, a\right) \sim \log \left\{ \left(\frac{N}{A}\right)^{3/2} \int_{SU(2)} e^{-\frac{2N}{\pi A} d(e^{ix}, g^{-1}ag)^2} V(A/2N, e^{ix}, g^{-1}ag) dg \right\}, \quad (30)$$

where $V(A/2N, e^{ix}, g^{-1}ag)$ is a semiclassical amplitude in N . The asymptotics are determined by the minimum point of the phase $d(e^{ix}, g^{-1}ag)^2$, namely by the distance $d(e^{ix}, C_a)$ from e^{ix} to the conjugacy class of a . We note that the conjugacy class $C(a) = \{g^{-1}ag : g \in SU(2)\}$ is a great (equatorial) 2-sphere of radius $\pi/2$ from (the north pole) I .

There is a somewhat different expansion accordingly as $e^{ix} = \pm 1$, $e^{ix} \in C(a)$ or for e^{ix} not of this form, which we will call a general element. If e^{ix} is a general element, then there is a unique closest point $g^{-1}ag$ and the phase is non-degenerate, so we obtain

$$\begin{aligned} & \int_{SU(2)} e^{-\frac{2N}{\pi A} d(e^{ix}, g^{-1}ag)^2} V(A/2N, x, g^{-1}ag) dg \\ & \sim A^{-3/2} e^{-\frac{2N}{\pi A} d(e^{ix}, C_a)^2} \cdot \frac{1}{\sqrt{\det Hess(A^{-1} d(e^{ix}, g^{-1}ag)^2)}}. \end{aligned} \quad (31)$$

We note that the powers of A cancel. If $e^{ix} \in C(a)$, then since e^{ix} also represents a point in the maximal torus, $e^{ix} = a$ or $e^{ix} = a^{-1}$, hence the unique point of minimal distance to e^{ix} in $C(a)$ is of course e^{ix} itself. There is no essential change in the calculation except that the exponent vanishes. We now consider the behavior of $\frac{1}{\sqrt{\det Hess(d(e^{ix}, g^{-1}ag)^2)}}$ as $e^{ix} \rightarrow \pm 1$. In the case $e^{ix} \rightarrow 1$), for instance, $\det Hess(A^{-1} d(e^{ix}, g^{-1}ag)^2) \sim |x|^3$ and hence

$$\log \det Hess(d(e^{ix}, g^{-1}ag)^2) \sim \log |x| \text{ as } x \rightarrow 0.$$

At $e^{ix} = \pm 1$, the entire $C(a)$ becomes a critical manifold for the phase. Exactly at the poles, we have

$$H_{SU(2)}(t, \pm 1, a) = k_{SU(2)}\left(\frac{A}{2N}, \pm 1, a\right) \sim \left(\frac{N}{A}\right)^{3/2} e^{-2N\pi^2/4\pi A}. \quad (32)$$

Thus, as $e^{ix} \rightarrow 1$,

$$H_{SU(2)}\left(\frac{A}{2N}, e^{ix}, a\right) \sim \begin{cases} |x|^{-3/2} e^{-\frac{2N}{\pi A} d(e^{ix}, C_a)^2}, & x \neq 0 \\ \left(\frac{N}{A}\right)^{3/2} e^{-2N\pi^2/4\pi A}, & x = 0. \end{cases} \quad (33)$$

Similarly at $x = \pi$. We note that the exponent is continuous and only the power of N changes at the special points $x = 0, \pi$.

Since we are interested in log asymptotics, the factor $e^{-\frac{2N}{\pi A} d(e^{ix}, C_a)^2}$ is dominant as long as the remainder can be integrated against $d\sigma_N * d\bar{\sigma}_N$. If this measure has a point mass at either 0 or π , there are singularities at $x = 0, \pi$ in the coefficients of the asymptotics which are not integrable $d\sigma * d\bar{\sigma}$. We now discuss this point in detail.

We fix a positive constant $C > 0$ and break up the integral (35) as

$$\begin{aligned} & \int_{S^1 \setminus [-C/N, C/N] \cup [\pi - C/N, \pi + C/N]} \log H_{SU(2)}\left(\frac{A}{2N}, e^{ix}, e^{i\pi/2}\right) d\sigma_N * \overline{d\sigma}_N(e^{ix}) \\ & + \int_{[-C/N, C/N] \cup [\pi - C/N, \pi + C/N]} \log H_{SU(2)}\left(\frac{A}{2N}, e^{ix}, e^{i\pi/2}\right) d\sigma_N * \overline{d\sigma}_N(e^{ix}). \end{aligned} \quad (34)$$

It follows from the pointwise asymptotics in (33) that

$$\begin{aligned} & \int_{S^1 \setminus [-C/N, C/N] \cup [\pi - C/N, \pi + C/N]} \log H_{SU(2)}\left(\frac{A}{2N}, e^{ix}, e^{i\pi/2}\right) d\sigma_N * \overline{d\sigma}_N(e^{ix}) \\ & \sim -\frac{2N}{\pi A} \int_{S^1 \setminus [-C/N, C/N] \cup [\pi - C/N, \pi + C/N]} d(C(a), e^{ix})^2 d\sigma_N * \overline{d\sigma}_N(e^{ix}). \end{aligned} \quad (35)$$

For the second integral of (34) over $[-\frac{C}{N}, \frac{C}{N}]$ and $[\pi - C/N, \pi + C/N]$, we use the scaling asymptotics of the heat kernel rather than its pointwise asymptotics. Since the details are similar for both intervals we only carry them out around the interval $[-\frac{C}{N}, \frac{C}{N}]$. Namely, we write $x = \frac{u}{N}$ with $|u| \leq C$. We have:

$$d\left(e^{i\frac{u}{N}}, g^{-1}ag\right)^2 = \left(\frac{\pi}{2}\right)^2 + \frac{u}{N} Q(u, N, A, g^{-1}ag)$$

for a bounded smooth function Q , hence

$$\begin{aligned} \log H_{SU(2)}\left(\frac{A}{2N}, e^{i\frac{u}{N}}, a\right) & \sim \left\{ \left(\frac{N}{A}\right)^{3/2} e^{-\frac{2N}{\pi A} \frac{\pi^2}{4}} \int_{SU(2)} e^{-Q(u, A, N) g^{-1}ag)^2} V(A/2N, e^{i\frac{u}{N}}, g^{-1}ag) dg \right\} \\ & \sim \left(\frac{N}{A}\right)^{3/2} e^{-\frac{2N}{\pi A} \frac{\pi^2}{4}} [q(u, A) + \frac{1}{N} q_1(u, A) + \dots] \end{aligned} \quad (36)$$

where $q = e^{-Q(u, A, 0)}$ is strictly positive. Substituting (36) into the second term of (34), we get

$$\left[-\frac{2N}{\pi A} \frac{\pi^2}{4} - \frac{3}{2} \log\left(\frac{N}{A}\right)\right] \int_{-C}^C d\sigma_N * d\bar{\sigma}_N\left(\frac{u}{N}\right) - \int_{-C}^C Q(u, A) d\sigma_N * d\bar{\sigma}_N\left(\frac{u}{N}\right). \quad (37)$$

Since Q is bounded and $d\sigma_N(\frac{u}{N}) * d\bar{\sigma}_N(\frac{u}{N})$ has mass at most 1, the last term is $O(1)$ as $N \rightarrow \infty$. Thus, the first term of (37) dominates. Thus, we have

$$\begin{aligned} & \int_{[-C/N, C/N] \cup [\pi - C/N, \pi + C/N]} \log H_{SU(2)}\left(\frac{A}{2N}, e^{ix}, e^{i\pi/2}\right) d\sigma_N * \overline{d\sigma}_N(e^{ix}) \\ & \sim \left[-\frac{2N}{\pi A} \frac{\pi^2}{4} - \frac{3}{2} \log\left(\frac{N}{A}\right)\right] \int_{-C/N}^{C/N} d\sigma_N * d\bar{\sigma}_N. \end{aligned} \quad (38)$$

We note that $d(C_a, 1) = d(C_a, -1) = \frac{\pi}{2}$. Since $d(C_a, e^{ix})$ is continuous, we may rewrite (37)-(38) to leading order as

$$-\frac{2N}{\pi A} \int_{-C/N}^{C/N} d(e^{ix}, C_a)^2 d\sigma_N * \overline{d\sigma}_N. \quad (39)$$

We add this back to (35) to obtain the leading order term

$$-\frac{2N}{\pi A} \int_{S^1} d(C(a), e^{ix})^2 d\sigma_N * \overline{d\sigma}_N(e^{ix}) \quad (40)$$

plus the canonical terms $\frac{1}{N} \log H_{SU(2)}(A/2N, 1, a)$ and $\log \eta(\frac{iA}{2N})$ which are independent of $d\sigma$. We then recognize that $d(e^{ix}, C_a) = \min\{d(e^{ix}, e^{\pm i\pi/2})\}$, completing the proof. \square

Remark:

- It would be interesting to know when the leading order term cancels, but even this cancellation would not cure the anomaly, since there is no control over the rate of the weak convergence $d\sigma_N \rightarrow d\sigma$. Hence, there is no well-defined growth rate of the lower order terms.
- As noted above, the exponent in (33) is continuous. Since we are taking the logarithm, only the exponent is important to leading order, and that is why we obtain a unified formula, even when $d\sigma$ has a point mass at $e^{ix} = \pm 1$.

4. FINAL REMARKS

Some final comments and remarks.

- It would of course be desirable to remove the assumption that one of the matrices U_1 should be a principal element of type ρ . In [F2], H. Fegan states a product formula for the full fundamental solution, but unfortunately this formula is erroneous. It would be interesting to see if there exists some form of MacDonald's identities for the full central heat kernel.
- It would also be desirable to extend the results of this paper to more general groups such as $U(N)$, $SO(N)$, $Sp(N)$. We specialize to $G_N = SU(N)$ because it amply illustrates the main point of this paper. Further, MacDonald's identities pertain to simply connected compact semi-simple groups, and presumably require some modifications for $U(N)$, $SO(N)$ etc.
- If we ignore the unexpected growth in N , the phase transition conjecture suggests that the limit free energy should equal

$$S(d\theta, d\sigma|A) - \frac{1}{2} \int_{S^1} \int_{S^1} \log \left| \sin \frac{\theta-\phi}{2} \right| d\sigma(\theta) d\sigma(\phi) - \frac{1}{2} \int_{S^1} \int_{S^1} \log \left| \sin \frac{\theta-\phi}{2} \right| d\theta d\phi,$$

The last two terms can be obtained from our limit formula by unravelling the denominator in the logarithm. Thus, aside from orders of magnitude, the phase transition conjecture appears to state in these cases that

$$S(d\theta, d\sigma|A) = \frac{1}{2A} \int_{S^1} d(C(a), e^{ix})^2 d\sigma * \overline{d\sigma}(e^{ix}).$$

We have not been able to compare our results directly with the physics predictions by solving the complex Burghers equation.

- In addition to considering ‘pointwise’ asymptotics of the partition function, one could consider weak asymptotics where one averages in various ways over the variables U_1, U_2 , considers the variance of the integrals and so on.
- As mentioned in the introduction, it is possible that an anomaly also occurs in Matytsin’s conjectured asymptotics $[M]$ at the special character values $\chi_R(a)$.

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