Nodal lines of Laplace eigenfunctions

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Nodal hypersurfaces of eigenfunctions

Let (M, g) be a real analytic Riemannian manifold, possibly with boundary.

Consider: real eigenfunctions of the Laplacian

$$\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

on a *real analytic* Riemannian manifolds (M, g).

The nodal hypersurface is the zero set

$$Z_{\varphi_j} = \{x : \varphi_j(x) = 0\}.$$

Why study nodal hypersurfaces?

- To ‘visualize sound’, i.e. modes of vibration of a drum (Chladni, 1800; Rayleigh, 1880). Nodal lines are rest points where the drum is not vibrating.

- To visualize stationary states of atoms. E. Schrödinger: Quantization as an eigenvalue problem - Annalen der Physik, 1926.

What we would like to understand

High energy $\lambda_j \to \infty$ asymptotics of nodal lines: how they snake around (i.e. how they are distributed on M):

- Hypersurface volumes $\mathcal{H}^{m-1}(Z_{\varphi_\lambda}) \sim ??$ of nodal sets.

- Distribution of nodal hypersurfaces:

$$\int_{Z_{\varphi_\lambda}} f d\mathcal{H}^{m-1} \sim ??$$

- Similar problems for critical points of eigenfunctions: $\nabla \varphi_\lambda(x) = 0$:

$$\#\{x_j : \nabla \varphi_\lambda(x_j) = 0\} \sim ??$$

Distribution of nodal hypersurfaces

If $U \subset M$ is a nice open set, we want to determine the total hypersurface volume

$$Vol(Z_{\varphi_j} \cap U)$$

as $\lambda_j \to \infty$ E.g. length in dimension one.

We put arc-length measure on the nodal line, or in higher dimensions the natural Riemannian hyper-surface measure $d\mathcal{H}^{n-1}$. Let f be a function and integrate it over the nodal hypersurface to get

$$\langle [\tilde{Z}_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f(x) d\mathcal{H}^{n-1}. \qquad (1)$$

Problem: Asymptotics of $\langle [\tilde{Z}_{\varphi_j}], f \rangle$ as $\lambda_j \to \infty$ (if exists).

Volumes of nodal hypersurfaces

Theorem **1** *(Donnelly-Fefferman, Inv. Math. 1988) Suppose that* (M, g) *is real analytic. Then*

$$c_1 \lambda \leq \mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \leq C_2 \lambda.$$

(Note that our λ *is the square root of the* Δ*-eigenvalue.)*

Result is sharp: there is no asymptotic formula in general.

Distribution of nodal hypersurfaces seems far out of sight...

Distribution of complex nodal hypersurfaces

Our theme is that it is possible to obtain much stronger results on *complex nodal hypersurfaces*—zeros of analytic continuations of eigenfunctions to the complexification $M_{\mathbb{C}}$ of M, and study their complex zeros and critical points. This can give information on growth of real zeros and critical points.

The origin of complex analysis was the realization that complex zeros of polynomials have a richer and simpler theory than real zeros. We follow that same path on analytic Riemannian manifolds

Distribution of complex nodal hypersurfaces when $\partial M = \emptyset$ and the geodesic flow is ergodic.

Our first result determines the limit distribution of complex nodal hypersurfaces in the ergodic case–definitions later.

Theorem **2** *(Z: Invent. Math. 167 (2007)) Assume (M, g) is real analytic and that the geodesic flow of (M, g) is ergodic. Then there exists a subsequence of full density in the spectrum such that*

$$\frac{1}{\lambda_j} Z_{\varphi^{\mathbb{C}}_{\lambda_j}} \to \overline{\partial}\partial |\xi|_g, \text{ weakly in } B^*_g M.$$

Both sides are viewed as $(1,1)$ currents: dual to $(m-1, m-1)$ forms. Here, $\overline{\partial}$ is the Cauchy-Riemann operator for the complex structure on the unit ball bundle with respect to the complex structure adapted to g.

Example: $\partial M = \emptyset$: the unit circle S^1

The complexification of S^1 is the cylinder $S^1_{\mathbb{C}} = S^1 \times \mathbb{R} = \mathbb{C}/\mathbb{Z}$. The complexified configuration space is similar to the phase space T^*S^1. This is always true.

The (real) eigenfunctions are $\cos k\theta, \sin k\theta$ on a circle. They have the quantum ergodic property: $2|\cos k\theta|^2 \to 1$ in weak sense (rapid oscillations smear out to constant).

The holomorphically extended eigenfunctions are $\cos kz, \sin kz$. They have exponential growth $e^{k|\Im z|}$ as $k \to \infty$.

Complex zeros = real zeros. Become uniformly distributed on $\Im\zeta = 0$.

Distribution of complex zeros for S^1

The zeros of $\sin 2\pi k z$ in the cylinder $\mathbb{C}/\mathbb{Z}$ all lie on the real axis at the points $z = \frac{n}{2k}$. Thus, there are $2k$ real zeros. The limit zero distribution is:

$$\begin{aligned} \lim_{k\to\infty} \frac{i}{2\pi k}\partial\bar{\partial} \log|\sin 2\pi k|^2 &= \lim_{k\to\infty} \frac{1}{k}\sum_{n=1}^{2k} \delta_{\frac{n}{2k}} \\ &= \frac{1}{\pi}\delta_0(\xi) dx \wedge d\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{i}{\pi}\partial\bar{\partial}|\xi| &= \frac{i}{\pi}\frac{d^2}{4d\xi^2}|\xi| \;\; \frac{2}{i} dx \wedge d\xi \\ &= \frac{i}{\pi}\,\frac{1}{2}\,\delta_0(\xi) \;\; \frac{2}{i} dx \wedge d\xi. \end{aligned}$$

Boundary problems: Cauchy data of eigenfunctions on plane domains

Our methods also give results when $\partial M \neq \emptyset$.

New phenomena occur in boundary problems: Let $\Omega \subset \mathbb{R}^2$ be a piecewise analytic plane domain and consider the Neumann problem

$$\begin{cases} -\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda \text{ in } \Omega, \\ \partial_\nu \varphi_\lambda = 0 \text{ on } \partial\Omega, \end{cases}$$

Here, ∂_ν is the interior unit normal.

Cauchy data $= \varphi_\lambda|_{\partial\Omega}$. In the Dirichlet case, the Cauchy data is $\partial_\nu \varphi_\lambda|_{\partial\Omega}$.

Nodal loops versus open nodal lines

For generic piecewise analytic plane domains, zero is a regular value of all the eigenfunctions φ_j i.e. $\nabla \varphi_j \neq 0$ on Z_{φ_j}. The connected components are either open segments touching the boundary at two endpoints or loops contained in Ω^o.

Problem: As $\lambda \to \infty$, count the number of closed nodal loops and the number of open nodal lines.

Counting open nodal lines

Theorem **3** *(with John Toth, in progress) Suppose that $\Omega \subset \mathbb{R}^2$ is a piecewise real analytic plane domain. Then the number $n(\lambda_j) = \# Z_{\varphi_j} \cap \partial\Omega$ of boundary nodal points satisfies $n(\lambda_j) \leq C\lambda_j$, where C is a constant depending only on Ω.*

Corollary **4** *Let Ω be a piecewise analytic domain for which all Z_{φ_j} are regular. Let $n_{\partial\Omega_c}(\lambda_j)$ be the number of open nodal lines, i.e. connected components of $\{\varphi_{\lambda_j} = 0\} \subset \Omega$ which intersect $\partial\Omega_c$. Then $n_{\partial\Omega_c}(\lambda_j) = O(\lambda_j)$.*

It is expected that the total number of nodal components is on the order of λ^2. So *few* nodal lines touch the boundary.

Bruhat-Whitney complexification

Theorem **5 (Bruhat-Whitney, 1959)** *Let M be a real analytic manifold of real dimension n. Then there exists a complex manifold $M_{\mathbb{C}}$ of complex dimension n and a real analytic embedding $M \to M_{\mathbb{C}}$ such that M is a totally real submanifold of $M_{\mathbb{C}}$. The germ of $M_{\mathbb{C}}$ is unique.*

Application: Let $\Omega \subset \mathbb{R}^2$ be a domain with real analytic boundary $\partial\Omega$. Let $Q : S^1 \to \partial\Omega$ be a real analytic parameterization. Then Q has an analytic continuation $Q_{\mathbb{C}}$ to an annulus. Its image is the complexification $(\partial\Omega)_{\mathbb{C}}$ of the boundary.

Analytic continuation of eigenfunctions

The holomorphic extension of φ_λ is obtained by applying the complexified wave group:

$$(2) \qquad e^{i(i\tau)\sqrt{\Delta}}\varphi_\lambda(\zeta) = e^{-\tau\lambda}\varphi_\lambda^{\mathbb{C}}.$$

This implies connections between the geodesic flow and the growth rate and zeros of $\varphi_\lambda^{\mathbb{C}}$.

- When $\partial M = \emptyset$, all eigenfunctions holomorphically extend to the same maximal 'Graurt tube' $M_{\mathbb{C}}$. I.e. 'radius of convergence' is independent of eigenvalue.

- When $\partial M \neq \emptyset$, Neumann eigenfunctions are real analytic on boundary and extend to a fixed 'tube' $|\Im\zeta| \leq \tau$ in $(\partial\Omega)_{\mathbb{C}}$. (Morrey, Garabedian)

Sketch of Proof of Counting open nodal lines for plane domains

Recall: $n_{\partial\Omega_c}(\lambda_j)$ = number of open nodal lines of Neumann eigenfunctions, i.e. connected components of $\{\varphi_{\lambda_j} = 0\} \subset \Omega$ which intersect $\partial\Omega_c$. Result: $n_{\partial\Omega_c}(\lambda_j) = O(\lambda_j)$.

Proof: First we note that endpoints of open nodal lies are zeros of boundary values of eigenfunctions. So it suffices to prove that boundary values have $O(\lambda_j)$ zeros.

To prove this, it suffices to show that the analytic continuation of the boundary values $\varphi_j|_{\partial\Omega}$ to $(\partial\Omega)_{\mathbb{C}}$ has $O(\lambda_j)$ complex zeros.

Value distribution of complexified eigenfunctions

It is a classical topic in complex analysis to relate growth of zeros in discs $D(0, r)$ of radius $r \to \infty$ of an entire function f to growth of $\max_{D(0,r)} \log |f(z)|$. (Jensen's formula, Nevanlinna theory...)

We want to do something analogous on a fixed domain as the eigenvalue tends to infinity (prior studies by Donnelly-Fefferman, Fang-Hua Lin).

Growth of $\varphi_\lambda^{\mathbb{C}}|_{(\partial\Omega)^{\mathbb{C}}}$

In general, growth of complexified Cauchy data (boundary values) does *NOT* reflect the eigenvalue λ.

Example: unit disc $\Omega = D$. Then,

$\varphi_{m,n}(r,\theta) = J_m(\rho_{mn}r)\cos m\theta, \;\; J_m(\rho_{mn}r)\cos m\theta$

Cauchy data: $\varphi_{m,n}|_{\partial D} = \cos m\theta, \sin m\theta$. Note that growth reflects angular momentum m, not the eigenvalue λ_{mn}.

All zeros are real in the complexification of the boundary. The number of zeros is m, even when the eigenvalue tends to infinity (can be bounded as $\lambda_{mn} \to \infty$.

But we may obtain an UPPER BOUND; if billiards are ergodic, then the number of boundary zeros is $\sim \lambda$ (in progress, with J. Toth).

Analytically continuing $\varphi_j|_{\partial\Omega}$

In the boundaryless case, we use the wave kernel to holomorphically continue Neumann eigenfunctions. This is more difficult on domains with boundary. Here we use Green's formula and layer potentials:

(3)
$$\varphi_j(x) = \int_{\partial\Omega} \frac{\partial}{\partial\nu_{q'}} G(\lambda, x, q') u_j(q') d\sigma(q'), \quad x \in \Omega^o.$$

Here, $u_j = \varphi_j|_{\partial\Omega}$ and G is the free ambient space Green's function on $\mathbb{R}^2$ (a Bessel function).

Analytically continuing $u_j = \varphi_j|_{\partial\Omega}$ II

Take limit as $x \to \partial\Omega$ to get ("jumps formula"):

(4)

$$u_j(q) = 2 \int_{\partial\Omega} \frac{\partial}{\partial \nu_{q'}} G(\lambda, q, q') u_j(q') d\sigma(q') \quad \text{(Neumann).}$$

where

(5)

$$G(\lambda, x, y) = R = J_0(\lambda r(x, y)) \log \frac{1}{r} + B(\lambda, x, y).$$

$R =$ Riemann function; B is also a Bessel function.

Difficulty: due to the logarithm, analytic continuation of kernel is multiple valued; but u_j has a single-valued continuation.

Analytically continuing $\varphi_j|_{\partial\Omega}$, cont.

Let q be a real analytic parameterization of $\partial\Omega$ and let $R = J_0(\lambda r)$ be the Riemann function.

(6) $$\Phi(t; z, z^*) = \int_0^t u_\lambda(s) \frac{\partial}{\partial n} R(s, z, z^*) ds.$$

For $\Im t > 0, < 0)$
(7)
$u_\lambda^{\mathbb{C}}(t) + \pm i\Phi(t, q(t), q^*(t))$

$= \int_0^\ell [\Phi(s; q(t), q^*(t)) + iu_\lambda(s)R(s, q(t), q^*(t))] \frac{q'(s)}{q(s)-q(\ldots}$
$+ \int_0^\ell [\Phi(s; q(t), \bar{q}(t)) - iu_\lambda(s)R(s, q(t), q^*(t))] \frac{\bar{q}'(s)}{\bar{q}(s)-q^*(t\ldots}$
$-2 \int_0^\ell u_\lambda(s) \frac{\partial B}{\partial n}(s; q(t), q^*(t)) ds.$

Key point: RHS only uses u_j on real $\partial\Omega$. LHS is a Volterra operator applied to $u_j^{\mathbb{C}}$.

Growth of $u_j^{\mathbb{C}}$ versus growth of complex zeros

Normalize u_λ so that $||u_\lambda||_{L^2(C)} = 1$. Let $A(\epsilon) = \{1 - \epsilon < |t| < 1 + \epsilon\}$. Let $n(\lambda, q^{\mathbb{C}}(A(\epsilon/2)))$ be the number of zeros of $u_j(q_{\mathbb{C}}(t))$ for $t \in A(\epsilon)$.

Proposition **6** *Then, for any $\epsilon > 0$, there exists a constant, $C(\epsilon) > 0$, such that*

$$n(\lambda, q^{\mathbb{C}}(A(\epsilon/2))) \leq C(\epsilon) \max_{q^{\mathbb{C}}(A(\ \epsilon))} \log |u_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t))|.$$

Growth of $u_j^{\mathbb{C}}$

Inverting the Volterra equation, and using knowledge of analytic continuations of Bessel functions, we get the upper bound

(8)

$$\max_{q^{\mathbb{C}}(t)\in Q(A(\epsilon))} |\varphi_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t))| \leq C_2 e^{\lambda\epsilon} \cdot \|u_\lambda\|_{L^2(\partial\Omega)},$$

and so,

(9)

$$\log \max_{q^{\mathbb{C}}(t)\in Q(A(\epsilon))} |\varphi_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t))| \leq \epsilon\lambda + \log \|u_\lambda\|_{L^2(\partial\Omega)}.$$

Known estimates of $\log \|u_\lambda\|_{L^2(\partial\Omega)}$ show that it is of logarithmic order.

Hence, the number of complex zeros is $O(\lambda)$.