Billiards and vibrations of drums

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Billiards and Vibrations of drums

There are two games one can play on bounded, smooth plane domain $\Omega \subset \mathbb{R}^2$:

- Billiards: hit a billiard ball and watch its trajectory over a long time as it bounces off the boundary $\partial\Omega$.
- Vibrate the drumhead Ω: observe the modes of vibration as the frequency tends to infinity. Where is the mode largest? Where smallest? Spill sand on the vibrating drum and watch the nodal pattern.

It is far from obvious, and hard to prove, but there are strong – and subtle– relations between the pattern of billiard trajectories and the patterns of nodal lines or “largest points” for high frequency vibrations.

The relations between billiards and vibrations is the subject of *Global Harmonic Analysis*. In general, one plays both games on any Riemannian manifold, with or without boundary.

Geometric/wave optics and classical/quantum mechanics

To play billiards on a bounded domain $\Omega \subset \mathbb{R}^2$, hit the ball at a point $p \in \Omega$ in a direction ξ. It follows a straight line until it hits the boundary, then reflects by the law of equal angles.

The equation for the 'mode' of a vibrating drum is the eigenvalue problem,

$$\Delta \varphi = -\lambda^2 \varphi.$$

At time t, the vibrating drumhead has height

$$u_\lambda(x, t) = (\cos t\lambda)\, \varphi_\lambda(x).$$

As the frequency of vibration $\lambda \to \infty$ the structure of the eigenfunctions φ reflects the billiard trajectories. The relation between billiards and modes of vibration is the relation between geometric/wave optics or to classical/quantum mechanics.

Chaotic billiards: one billiard ball

We mainly consider ‘ergodic billards’ where almost all billiard trajectories become uniformly dense both in position and in direction.

Swarm of billiards (Videos due to Semyon Dyatlov)

The 'chaos' is more easily visualized if we consider many billiard trajectories which start off close in position and direction.

Nodal lines to visualize vibrations of drums

To visualize a Δ-eigenfunction (mode of vibration) we look at its nodal line (zero set). The sand accumulates where the drum does not vibrate. How does the number of nodal domains grow with the frequency of vibration?

Billiards versus shapes and sizes of eigenfunctions

Nodal lines are just one feature of the "topography of eigenfunctions", i.e. the shape and size of $|\varphi_\lambda(x)|^2$. The nodal set is the minimum set of $|\varphi_\lambda(x)|^2$. One would equally (or more) like to know its nearly maximum set. We always assume $\int |\varphi_\lambda|^2 = 1$.

It is not at all obvious that the shape and size of φ_λ has any relation to billiards as $\lambda \to \infty$. Relations between the two are the subject of "global harmonic analysis' (global means global in the domain and in time).

The same problems occur on any Riemannian manifold (M, g), with or without boundary. We only illustrate them on billiard tables. In general, billiard trajectories = geodesics.

Propagation of singularities of solutions of the wave equation

The relation between geodesics and waves is illustrated by the propagators $U(t, x, y)$ for simply connected spaces of constant curvature:

- $\mathbb{R}^n$: $$U(t, x, y) = (it)\,(t^2 - |x - y|^2)_+^{-\frac{n+1}{2}}.$$
- $\mathbb{S}^n$: $$U(t, \omega, \omega') = i \sin t(\cos t - \cos r(\omega, \omega'))_+^{-\frac{n+1}{2}}.$$
- $\mathbb{H}^n$: $$U(t, z, w) = i \sin t\,(\cosh t - \cosh r(z, w))_+^{-\frac{n+1}{2}}$$

Harmonic analysis

We introduce Harmonic Analysis with the exponential functions $\mathbf{e}_k(x) = e^{2\pi i \langle k, x \rangle}$ on the flat torus $\mathbb{T} = \mathbb{R}^n / \mathbb{Z}^n$ (with $\mathbf{k} \in \mathbb{Z}^n$). The idea is to express any function (or distribution) as a linear combination of the exponentials,

$$f(x) \sim \sum_{\mathbf{k} \in \mathbb{Z}^n} a_{\mathbf{k}} e^{2\pi i \langle \mathbf{k}, x \rangle}, \tag{1}$$

and to relate properties of f to the dual properties of the Fourier coefficients $a_{\mathbf{k}}$. The key property of the exponentials $e^{2\pi i \langle \mathbf{k}, x \rangle}$ is that they form an orthonormal basis of eigenfunctions of the Laplacian $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ of the flat (Euclidean) metric.

Flat eigenfunctions on a flat torus

The exponentials have special properties that are heavily exploited in studying Fourier series. First, they are joint eigenfunctions of the $\frac{\partial}{\partial x_j}$.

Second, they are uniformly bounded,

$$|e^{2\pi i \langle \mathbf{k}, x \rangle}| \leq 1.$$

Thus, the joint eigenfunctions $e^{2\pi i \langle \mathbf{k}, x \rangle}$ are very *flat*. This flatness reflect the dynamics of the geodesic flow of the flat torus $\mathbb{T}$. Geodesics on a flat torus are projections to $\mathbb{R}^n / \mathbb{Z}^n$ of straight lines on $\mathbb{R}^n$.[1]

[1] Theorem (Toth-Z): if joint eigenfunctions of a completely integrable system are uniformly bounded in the eigenvalue, then (M, g) is a flat torus

Harmonic analysis on a Riemannian manifold

The flat Laplacian generalizes to any complete Riemannian manifold (M, g) as

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} g^{ij} \sqrt{g} \frac{\partial}{\partial x_j},$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$ and $g = \det[g_{ij}]$. When M is compact, there is an orthornormal basis $\{\varphi_j\}$ of eigenfunctions,

$$\Delta_g \varphi_j = -\lambda_j^2 \varphi_j, \quad \int_M \varphi_i \varphi_j dV_g = \delta_{ij} \tag{2}$$

with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$ repeated according to their multiplicities. When M has a non-empty boundary ∂M, one imposes boundary conditions such as Dirichlet $Bu = u|_{\partial M}$ or Neumann $Bu = \partial_\nu u|_{\partial M}$.

Local versus Global harmonic analysis

There are two "schools of thought" on eigenfunctions and nodal sets:

- Local methods study solutions of $\Delta\varphi = -\lambda^2\varphi$ on small balls of radius $\frac{\epsilon}{\lambda}$, i.e. ϵ wavelengths. The local analyst rescales the ball $B_{\frac{\epsilon}{\lambda}}(p) \to B_1(p)$ so that the rescaled φ behaves like a harmonic function. E.g. $\sin nx \to \sin \epsilon x$.

- Global methods rewrite the eigenvalue problem in terms of the solution operator (or *propagator*) of the wave equation,

$$U(t)\varphi := e^{it\sqrt{-\Delta}}\varphi = e^{it\lambda}\varphi.$$

  This equation is only valid if φ is a global eigenfunction. $U(t, x, y)$ is related to geodesics, and as $\lambda \to \infty$ the eigenfunction becomes related to geodesics.

Universal sup norm estimate

How does the geometry of (M, g) affect the SIZE of the eigenfunction, as measured by a norm such as the sup norm $||\varphi||_{L^\infty} := \sup_x |\varphi(x)|$? The exponentials $e^{i\langle \vec{k}, x\rangle}$ on $\mathbb{R}^n/\mathbb{Z}^n$ are flat. But this is very special.

There is a universal estimate for compact M:

$$||\varphi_\lambda||_{L^\infty} \leq C_g \, \lambda^{\frac{n-1}{2}},$$

$$\left(\Delta \varphi_\lambda = -\lambda^2 \varphi_\lambda, \;\; ||\varphi_\lambda||_{L^2} = 1, \; n = \dim M\right),$$

where C_g depends only on g and not on λ

The proof is "semi-local": it uses the wave group

$$U(t) = e^{it\sqrt{-\Delta}}, \quad \text{for small } |t|$$

but only for arbitrarily small times [2]

[2] Can you prove this using elliptic estimates?

Global harmonic analysis of sup norms

When does a Riemannian manifold (M, g) have a sequence φ_{j_k} of eigenfunctions satisfying

$$||\varphi_{j_k}||_{L^\infty} \geq C_g' \, \lambda_{j_k}^{\frac{n-1}{2}},$$

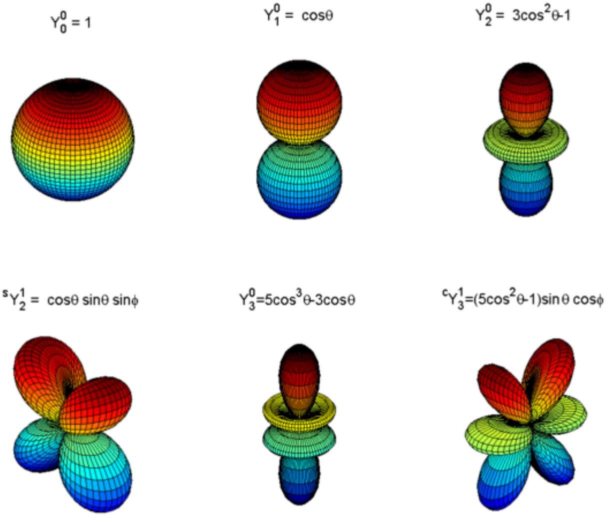
Definition: Say that (M, g) has *maximal sup norm growth* if it possesses a sequence of eigenfunctions $\varphi_{\lambda_{j_k}}$ which saturates the L^∞ bounds.

Example: Any surface of revolution. The eigenfunctions attaining the maximal sup norm bounds are the rotationally invariant ones.
E.g. $\mathbb{S}^2$ The eigenvalues of Δ are $\ell(\ell+1)$. The zonal (rotationally invariant) eigenfunction Y_ℓ^0 has the largest L^∞ norm:

$$\frac{||Y_\ell^0||_{L^\infty}}{||Y_\ell^0||_{L^2}} \simeq \sqrt{\ell},$$

It has huge peaks at the north and south poles.

Graphics of spherical harmonics



Real analytic (M, g) of maximal eigenfunction growth

Zonal spherical harmonics bulge out at the poles. It turns out this is the only way to get maximal sup norm growth[3]:

Theorem

(Sogge-Z, 2014) Suppose (M, g) is a real analytic Riemannian surface without boundary and with maximal eigenfunction growth, i.e. having a sequence $\{\varphi_{\lambda_{j_k}}\}$ of eigenfunctions which achieves (saturates) the bound $||\varphi_\lambda||_{L^\infty} \leq \lambda^{1/2}$.

Then there must exist a pole i.e. a point $x \in M$ such that every geodesic through x is closed. In particular, M must be a topological S^2.

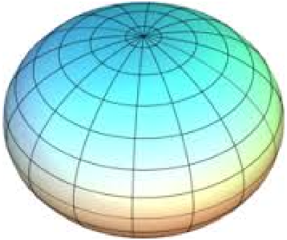
Question: Must (M, g) actually be a surface of revolution? (Need more than just poles!).

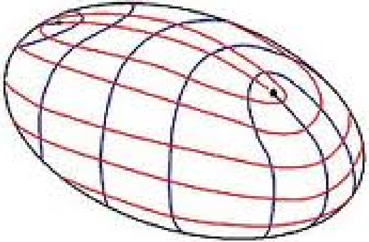
[3]This result is the culmination of a series of works of Y. Safarov, Sogge-Z, Sogge-Toth-Z

Pole of a surface of revolution and umbilic points on an ellipsoid

On the left, the north and south pole of the surface of revolution are "poles": All geodesics through the poles are smoothly closed.

On the right, at the four umbilic points of the ellipsoid, all geodesics starting at an umbilic point P return to P at time 2π, but are not smoothly closed: they are just "loops" at P. The ellipsoid does not have "maximal eigenfunction growth".





Nodal domains

We now turn from size to shape as measured by the nodal set:

The nodal set is $\mathcal{N}_{\varphi_\lambda} = \{x : \varphi_\lambda(x) = 0\}$. A nodal domain is a connected component of $M \backslash \mathcal{N}_{\varphi_\lambda}$. The nodal domains partition M into disjoint open sets:

$$M \backslash \mathcal{N}_{\varphi_\lambda} = \bigcup_{j=1}^{N(\varphi_\lambda)} \Omega_j.$$

When 0 is a regular value of φ_λ the level sets are smooth curves. When 0 is a singular value, the nodal set is a singular (self-intersecting) curve.

$N(\varphi_\lambda)$ = the number of nodal domains of φ_λ.

Counting nodal domains

Question: how large is $N(\varphi_{\lambda_j})$; i.e. how many connected components does the nodal set have? The classical Courant bound is that the number $N(\varphi_{\lambda_j})$ of the *jth* eigenfunction in an orthonormal basis is bounded by $j + 1$;.

There is no non-trivial lower bound for $N(\varphi_\lambda)$: it was shown by H. Lewy that there exist (M, g) and sequences of $\varphi_{\lambda_j}, \lambda_j \to \infty$ with only 2 or 3 nodal domains.

Problem (T. Hoffman-Ostenhof) Let (M, g) be a compact Riemannian manifold. Show that it has *some* sequence of eigenfunctions φ_{λ_j} with $\lambda_j \to \infty$ so that the number of nodal domains of φ_{λ_j} tends to infinity.

$\frac{1}{\lambda}$-density of nodal sets

Nodal sets are useful to visualize the "shape" of an eigenfunction.

The following "fundamental existence theorem' is a local result:

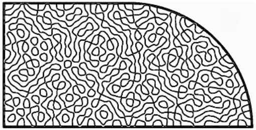
Theorem

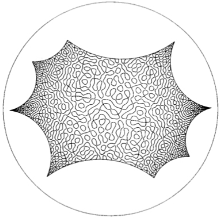
For any (M, g) and $\Delta \varphi_\lambda = -\lambda^2 \varphi_\lambda$, there exists a zero of φ_λ in every ball $B(p, \frac{C_g}{\lambda_j}) \subset M$. I.e. the nodal set $\mathcal{N}_{\varphi_\lambda} = \{x : \varphi_\lambda(x) = 0\}$ is $\frac{1}{\lambda}$-dense.

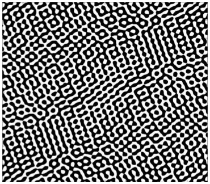
The proof only uses that $\Delta \varphi_\lambda = -\lambda^2 \varphi_\lambda$ in a ball $B(p, r)$ and not globally on (M, g),

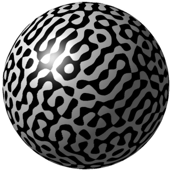
Ergodicity and “chaos” of geodesics and Nodal sets

Global problem: How does “chaotic geodesic flow” impact nodal lines and number of nodal domains. **Intuitive idea**: if the billiard trajectories are ergodic (uniformly distributed), then the nodal lines should also be uniformly distributed, and there should be a lot of nodal domains.









Ergodicity $\implies N(\varphi_j) \to \infty$

The following result with Junehyuk Jung[4] uses ergodicity of the billiard flow to prove that the number of nodal domains of eigenfunctions tends to infinity:

Theorem

[J.Jung-Z, 2014] Let (X, g) be a surface with curvature $k \leq 0$ and with concave boundary. Then for any orthonormal eigenbasis $\{\varphi_j\}$ of Dirichlet (or Neumann) eigenfunctions, one can find a density 1 subset A of $\mathbb{N}$ such that

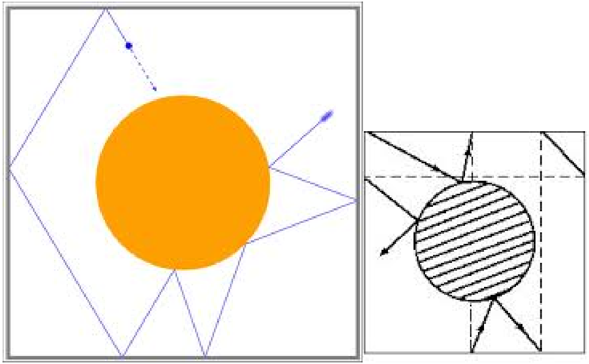
$$\lim_{\substack{j \to \infty \\ j \in A}} N(\varphi_j) = \infty,$$

A density one subset $A \subset \mathbf{N}$ is one for which $\frac{1}{N} \#\{j \in A, j \leq N\} \to 1, \quad N \to \infty$.

[4] Inspired by work of Ghosh-Reznikov-Sarnak on the modular domain, using L-functions

Sinai billiards (chaotic): remove a disc from a square or torus

An example of a non-positively curved surface with concave boundary is a Sinai-Lorentz billiard in which one removes a small disc D from X.



Log lower bound

In a related geometric setting[5], the number of nodal domains grows at least logarithmically for almost all eigenfunctions:

Theorem

(S.Z. 2015) Let (M, J) be a real Riemann surface and let g be a negatively curved invariant metric on M. Then for $j \in A$ (a set of density one),

$$N(\varphi_j) \geq C_g \, (\log \lambda_j)^K, \quad (\forall K < \frac{1}{6}).$$

resp.

$$N(\psi_j) \geq C_g \, (\log \lambda_j)^K, \quad (\forall K < \frac{1}{6}).$$

[5] $N(\varphi_j) \to \infty$ was also proved by J.Jung-S.Z. in this setting

Main ideas of the proof

1. Show that the number $N(\varphi_\lambda)$ of nodal domains is $\geq \frac{1}{2} N(\varphi_\lambda|_{\partial M}, 0, \partial M)$, the number of zeros of φ_λ on the boundary (in the Neumann case). **This is purely topological and is why we need $\partial M \neq \emptyset$.**

2. **Prove that Neumann eigenfunctions have a lot of zeros on ∂M, resp. Dirichlet eigenfunctions have many zeros of $\partial_\nu \varphi_j = 0$ on ∂M. This is where ergodicity is used: Neumann eigenfunctions, restricted to the boundary, are "ergodic".** [6]

3. To prove (2), we show that $\int_\beta \varphi_j ds << \int_\beta |\varphi_j| ds$ on any arc $\beta \subset \partial M$.
4. For log lower bounds, adapt recent log-scale quantum ergodicity results of Hezari-Riviere and X. Han to restrictions of eigenfunctions to curves on surfaces.

[6] the quantum ergodic restriction theorem of Hassel-Z and of Christianson-Toth-Z.

Quantum ergodic restriction theorem

Theorem

[Christianson-Toth-Z, 2013] Let γ be either ∂M for the surface with boundary or $Fix(\sigma)$ for the surface with involution. Then, for a subsequence of Neumann eigenfunctions of density one,

$$\langle f \varphi_j|_\gamma, \varphi_j|_\gamma \rangle_{L^2(\gamma)}$$

$$\to \frac{4}{2\pi Area(M)} \int_\gamma f(s) ds.$$

Similarly for normal derivatives of Dirichlet eigenfunctions. Cauchy data of eigenfunctions to γ are quantum ergodic along γ. This is part of a much more general result.