Large N limit of YM_2 with gauge group $U(N)$: Problems and Results

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Outline of talk

Two dimensional $U(N)$ gauge theory over a Riemann surface is one of the few mathematically well-defined models in quantum field theory. 't Hooft proposed that the large N limit of gauge theory is or resembles a string theory. Specific asymptotics were conjectured by D.J. Gross, A. Matytsin, W. Taylor, M. R. Douglas, V. Kazakov, V. Kostov, T. Wynter and others. The purpose of my talk is to:

- Review some of the physics conjectures on the large N limit of YM_2;

- Give two rigorous calculation of large N limit asymptotics for $U(N)$ gauge theory which contradict the physics conjectures.

- Propose one way to make them consistent with the physics conjectures.

What is 2D gauge theory?

YM_2 with gauge group $U(N)$ concerns connections on (all) principal $U(N)$-bundles $P \to M$ over a Riemann surface M. The Riemann surface could be a closed surface of genus g, or a surface with boundary such as a cylinder or a three (or more) legged surface.

Denote the space of connections by $\mathcal{A}$. A connection is an equivariant $u(N)$-valued 1-form on P. The moduli space of connections is $\mathcal{A}/\mathcal{G}$ where $\mathcal{G}$ is the gauge group. Define the Yang-Mills action $S(A) = \frac{1}{4}||F_A||^2$ where F_A is the curvature of A and where $||F_A||^2 = \int_M TrF_A^* F_A dV$, where dV is the area form on M. We formally define the YM measure $e^{-S(A)}\mathcal{D}A$ on $\mathcal{A}/\mathcal{G}$.

Migdal's formulae

According to Migdal (Soviet Phys. JETP 42 (1975), no. 3, 413–418), the partition function of YM_2 over a closed Riemann surface M_G of genus G and area A is:

$$\mathcal{Z}_N(A,G) = \sum_{R\in\widehat{G}} d_R^{2-2G} e^{-\frac{A}{2N}C_2(R)}.$$

Here, $R \in U(\widehat{N})$ runs over the irreps of $U(N)$,

- χ_R = character of R, d_R = dimension of R;

- $C_2(R)$ =eigenvalue of Casimir Δ in the irrep R.

Large N limit problems

Define the free energy by

$$F_N(A, G) = \frac{1}{N^2} \log \mathcal{Z}_N(A, G).$$

Problem Find the limiting free energy

$$\lim_{N \to \infty} F_N(A, G).$$

How does it depend on the area A? Does $F_N(A, G)$ admit a complete asymptotic expansion?

Theme (Gross, W. Taylor, 't Hooft): It should admit a limit which is may be described in terms of maps between Riemann surfaces, i.e. as a 'string theory.'

Partition function of a cylinder

In the case of a cylinder, the partition function involves the holonomies of the connection over the two boundary circles. Migdal's formula is:

$$\mathcal{Z}_N(U_1, U_2; A) = \sum_{R \in \widehat{U(N)}} \chi_R(U_1)\chi_R(U_2^*) e^{-\frac{A}{2N} C_2(R)}$$

of YM_2 for a cylinder of area A and gauge group $U(N)$: U_1, U_2 are the boundary holonomies. $\mathcal{Z}(U_1, U_2; A)$ is the *central* heat kernel at time $t = A/2N$, i.e. the kernel of $e^{-t\Delta}$ on central functions. It is a class function in U_1, U_2.

To state the corresponding large N limit problem, we need more notation.

Density of eigenvalues

Since $\mathcal{Z}(U_1, U_2; A)$ is conjugacy invariant in U_1, U_2 we may assume they are diagonal. The eigenvalues of $U \in U(N)$ are denoted $\{e^{i\theta_k},\ k = 1, \ldots, N\}$.

The eigenvalue distribution of U is $\frac{1}{N}\sum_{k=1}^{N}\delta(e^{i\theta_k})$. Given a sequence $U_N \in U(N)$, we write $U_N \to \sigma$ (as $N \to \infty$) if

$$\frac{1}{N}\sum_{k=1}^{N}\delta(e^{i\theta_k}) \to \sigma$$

in the sense of measures, i.e. $\frac{1}{N}\sum_{k=1}^{N} f(e^{i\theta_k}) \to \int_{S^1} f d\sigma$

Density of Young Tableaux

Irreps of $U(N)$ are parametrized by (shifted) highest weights or Young diagrams: $R \iff -\infty < \ell_1 < \ell_2 < \cdots < \ell_N < \infty$. To R we associated the probability measure

$$d\rho_R = \frac{1}{N} \sum_{j=1}^{N} \delta(\frac{\ell_j}{N}).$$

Here, $\delta(t)$ is the point mass at t. Thus,

$$\int_{\mathbb{R}} f(y) d\rho_R(y) = \frac{1}{N} \sum_{j=1}^{N} f(\frac{\ell_j}{N}).$$

Given a sequence R_N of Young tableaux or highest weights for $U(N)$, we write $R_N \to d\rho$ if $d\rho_R \to d\rho$ in the sense of measures. Any weak limit is a probability measure satisfying $\rho_Y([0,T]) \leq T$, since $\ell_{j+1} - \ell_j \geq 1$. If the limit has a density, which is written $d\rho_Y = \rho'_Y(y)dy$, then $\rho'_Y(y) \leq 1$. A limit measure is called a "distribution on Young tableaux".

Large N limit of cylinder partition function

The free energy for YM_2 on the cylinder is again defined by

$$F_N(U_{C_1}, U_{C_2}|A) = \frac{1}{N^2} \log \mathcal{Z}_N(U_{C_1}, U_{C_2}|A).$$

Problem : Find $\lim_{N\to\infty} F_N(U_{C_1}, U_{C_2}|A)$ assuming $U_j^N \to \sigma_j$.
More generally: $F_N(U_{C_1}, \ldots, U_{C_r}|A)$.

Special cases:

- $U_1 = U_2 = I$. The sphere; $U_j \to \delta(1)$.

- $U_1 = 1, U_2$ arbitrary: the disc.

Gross-Taylor conjecture in genus 1

The partition function in genus one equals

$$Z_0(A) = \sum_{R \in \widehat{G}} e^{-\frac{A}{2N} C_2(R)}.$$

Conjecture **1** *(Gross-Taylor)*

$$F(A,N) = \frac{1}{N^2} \log \mathcal{Z}(A,N)$$

$$\to -\frac{A}{24} - 2 \sum_{n=1}^{\infty} \log(1 - e^{-nA/2})$$

$$= -2 \log \eta(A/2).$$

This is the most elementary of the Gross-Taylor conjectures and probably can be (or has been) proved rigorously.

Gross-Taylor conjecture for $B \geq 2$

For YM_2 on a closed surface of genus $G \geq 2$, the conjecture is:

Conjecture **2** *(Gross-Taylor) The free energy has an expansion*

$$F_N(A, G) \sim \sum_{g=0}^{\infty} N^{2-2g} f_g^G(A),$$

where

$$f_g^G(A) \sim \sum_n \sum_i \omega_{g,G}^{n,i} e^{-nA} A^i.$$

The coefficients involve ‘statistics of branched covers of M_G. (Refer to the original papers for the specifics).

Gross-Matytsin and Kazakov-Wynter Character asymptotics

Consider a $U(N)$ character, $\chi_R(U)$, with $R = -\infty < \ell_1 < \ell_2 < \cdots < \ell_N < -\infty$. Then

$$\chi_R(U) = \frac{\det(e^{i\ell_j\theta_k})}{\Delta(e^{i\theta})}, \;\; \Delta = \Pi_{j<k}(e^{i\theta_j} - e^{i\theta_k}).$$

As before, assume that $U \to \sigma, R \to \rho$.

Conjecture **3** *(Gross-Matytsin)*

$$\chi_R(U) \sim e^{N^2 F_0[\rho,\sigma)]}, \;\; \textit{where}$$

$$F_0(\rho,\sigma) = S(\rho,\sigma) + \tfrac{1}{2}\{\textstyle\int_{\mathbb{R}} \rho(x)x^2 dx + \int \sigma(y)y^2 dy\}$$

$$-\tfrac{1}{2}\{\textstyle\int_{\mathbb{R}\times\mathbb{R}} \rho(x)\rho(y) \ln|x-y| dxdy$$

$$+\textstyle\int_{\mathbb{R}\times\mathbb{R}} \sigma(x)\sigma(y) \ln|x-y| dxdy,$$

where S is the classical action corresponding to the Hopf equation

$$\begin{cases} \frac{\partial}{\partial t} + f\frac{\partial f}{\partial x} = 0 \\ \\ \Im f(x,0) = \pi\rho(x), \;\; \Im f(x,1) = \pi\sigma(x). \end{cases}$$

Discussion

The boundary problem for the Hopf (Burgers, 1D Euler) equation

$$\begin{cases} \frac{\partial}{\partial t} + f\frac{\partial f}{\partial x} = 0 \\ \Im f(x,0) = \pi\rho(x), \;\; \Im f(x,1) = \pi\sigma(x). \end{cases}$$

gives 2 boundary conditions for a first order equation. One real boundary condition (initial value problem) already gives a unique solution. the 2 boundary condition problem is well posed because the boundary conditions are complex, and only the imaginary parts are specified.

The cylinder

Conjecture **4** *(Kazakov-Wynter, Gross-Matytsin) Assume that* $U_1 \to \sigma_1, U_2 \to \sigma_2$. *Then*

$$F_N(U_1, U_2|A) \to F(\sigma_1(\theta), \sigma_2(\theta)|A)$$

$$:= S(\sigma_1(\theta), \sigma_2(\theta)|A)$$

$$-\frac{1}{2}\int_{S^1}\int_{S^1} \sigma_1(\theta)\sigma_1(\phi) \log|\sin\frac{\theta-\phi}{2}|d\theta d\phi$$

$$-\frac{1}{2}\int_{S^1}\int_{S^1} \sigma_2(\theta)\sigma_2(\phi) \log|\sin\frac{\theta-\phi}{2}|d\theta d\phi,$$

where

where S is the classical action corresponding to the Hopf equation

$$\begin{cases} \frac{\partial}{\partial t} + f\frac{\partial f}{\partial x} = 0 \\ \Im f(x,0) = \pi\sigma_1(x), \;\; \Im f(x,1) = \pi\sigma_2(x). \end{cases}$$

Rigorous results

- (i) (Tate-Zelditch) The conjectured character asymptotics are not always correct: they fail for any sequence R_N of irreps of $U(N)$ if one evalutes characters on Kostant's elements of type ρ;

- (ii) (Guionnet-Zeitouni) Matytsin's character asymptotics are correct if one evaluates (analytic continuation of) characters on positive Hermitian matrices;

- (iii) (Zelditch) The large N asymptotics of the partition function for YM_2 on the cylinder also fail for certain sequences.

Discussion

- The large N limit is not correct as a pointwise limit for all sequences.

- The counterexamples are to date the only rigorous calculations. It is not clear how generally the conjectures fail.

- It is possible that all of the conjectures can be cured by analytically continuing (Wick rotating) from $U(N)$ to positive Hermitian matrices, i.e. that the Guionnet-Zeitouni methods also work for partition functions. Physical interpretation?

- Douglas has suggested an alternative viewpoint towards large N limites of YM_2 which is not pointwise.

Counterexamples

We now give the counterexamples. They are based on the special sequences

$$U_N = a_N = \quad \text{principal elements of type } \rho.$$

Definition (Kostant) A principal element $a_N \in U(N)$ of type ρ is a regular element of finite order equal to the Coxeter number N of $U(N)$. Thus, it is an element whose eigenvalues are the distinct Nth roots of unity. There is one conjugacy class of such elements. For $SU(2)$ they form the equatorial sphere (with north pole at I).

Theorem (Kostant): the only character values at this element are $\chi_R(a) = -1, 0, 1$.

Rigorous results on cylinder partition function

The partition function of YM_2 on a cylinder, with gauge group equal to G, is given by
(1)

$$\mathcal{Z}_G(U_1, U_2|A) = \sum_{R \in \widehat{G}} \chi_R(U_1)\chi_R(U_2^*)e^{-\frac{A}{2N}C_2(R)}$$

Here, $A \geq 0$ is the area of the cylinder. It is the value at time $t = \frac{A}{2N}$ of the *central* heat kernel of G:
(2)

$$H_G(t, U_1, U_2) = \sum_{R \in \widehat{G}} \chi_R(U_1)\chi_R(U_2^*)e^{-tC_2(R)},$$

i.e. the kernel of the heat operator acting on the space of central functions on $SU(N)$. It is obtained from the usual heat kernel by averaging both variables over conjugacy classes.

Central heat kernel of $SU(2)$

For example, the central heat kernel of $SU(2)$ at these special values is given by

(3) $$H_{SU(2)}(t, e^{ix}, e^{\frac{i\pi}{2}}) = \frac{\theta_1(e^{i\pi x}, it)}{2e^{-\pi t/4} \sin \pi x}$$

where

(4)

$$\theta_1(z,t) = 2\sum_{n=0}^{\infty} (-1)^n \sin\{(2n+1)z\} e^{-\pi(n+1/2)^2 t}$$

is Jacobi's theta function.

Warmup: Unscaled large N limit

We first show that if we do NOT scale $t = A/N$, then the partition function has a large N limit at the Kostant elements.

Theorem **5** *Let $k_N \in \mathbf{su}(N)$ be diagonal matrices with entries θ_j^N, and assume $d\sigma_N := \frac{1}{\ell_N}\sum_{j=1}^{\ell_N} \delta(e^{i\theta_j^N}) \to \sigma$. Then, as $N \to \infty$,*

$$\frac{1}{N^2} \log H_{SU(N)}(t, a_N, e^{k_N}) \to -\frac{1}{2} \log \eta(it)$$

$$+\frac{1}{2} \int_{\mathbb{R}} \log H_{SU(2)}(t, e^{ix}, e^{\frac{i\pi}{2}}) d\sigma * \bar{d\sigma}(x).$$

Here, e^{ix} is short for the diagonal matrix $D(e^{ix}, e^{-ix})$. Also, $d\sigma * \bar{d\sigma}(x)$ is the measure

(5)

$$\int_{S^1} f(e^{ix}) d\sigma * \bar{d\sigma}(x) := \int_{S^1}\int_{S^1} f(e^{i(x-y)}) d\sigma(x) d\sigma(y).$$

True large N asymptotics

At first sight, this result seems to explain the normalization $\frac{1}{N^2}\log \mathcal{Z}_N$. However, the large N limit conjectures concern the scaling limit of the central heat kernel under $t \to \frac{A}{2N}$. This puts into play a simultaneous limit process in $d\sigma_N * \bar{d}\sigma_N \to d\sigma * \bar{d}\sigma$ and in the asymptotics of theta functions. We find that for our cases of the problem, this rescaling changes the growth rate of the free energy.

Theorem **6** *Suppose that* e^{k_N} *is a sequence such that* $d\sigma_N \to d\sigma$*. Then,*

$$F_N(a_N, e^{k_N}|\frac{A}{2N}) = \frac{1}{2}\int_{S^1} \log H_{SU(2)}(A/2N, e^{ix}, e^{\frac{i\pi}{2}})$$

$$d\sigma_N * \overline{d\sigma_N}(e^{ix}) - \frac{1}{2}\log \eta(\frac{iA}{2N}) + O(1/N)$$

$$\sim -\frac{N}{A}\left\{\int_{S^1} \frac{1}{\pi} \min\{d(e^{ix}, e^{i\pi/2}), d(e^{ix}, e^{-i\pi/2})\}^2\right.$$

$$\left. d\sigma * \overline{d\sigma}(e^{ix}) - \frac{\pi}{12}\right\},$$

where $d(e^{ix}, e^{iy})$ *is the distance along* S^1*.*

Large N limit of partition function

There are two terms of opposite sign in the leading order term, but they usually do not cancel. Consider the simplest case, where $U_2 = Id$. MacDonald's identity reads:
(6)
$$H_{SU(N)}(t, a_N, I) = e^{-\dim SU(N)\, t/24} \eta(it)^{\dim SU(N)},$$
where $\eta(t)$ is Dedekind's η-function. The eigenvalue distribution of a_N tends to $\frac{d\theta}{2\pi}$, while that of I is obviously δ_1.

Proposition **7** *When $U_1 = a_N$ and $U_2 = I$, so that $\sigma_1 = d\theta, \sigma_2 = \delta_1$, then*
$$\frac{1}{N^2} \log \mathcal{Z}_{SU(N)}(e^{4\pi i\rho}, 1|A) = -\frac{2N}{A}\{\frac{\pi}{12}\}$$
$$-\frac{1}{2} \log(\frac{A}{2N}) - A/48N + O(e^{-cN}).$$

Anomalous factor of N

The key point is that there is an extra factor of N in the asymptotics which is not consistent with the original idea that the free energy have a stringy expansion.

Why it occurs: The sign oscillation $\chi_R(a_N) = \pm 1, 0$ causes much more cancellation than expected, so the asymptotics have the form $e^{-N^3 \Xi}$ rather than $e^{-N^2 \Xi}$.

Character values might always oscillate this wildly. The oscillation is cured by analytic continuation to positive matrices.

Idea of proof

The asymptotics are based on the use of MacDonald's identities.

Theorem **8** *Let G be compact, connected, semisimple and simply connected, and let a be an element of type ρ. Then*

$$H_G(t,a,e^{-k}) = (e^{-\pi t/12}\eta(it))^{-|R_+|+\ell}$$

$$\Pi_{\alpha\in R_+} H_{SU(2)}(t, e^{i\langle\alpha,k\rangle}, e^{i\pi/2}).$$

[History: The heat kernel proof for $k = 0$ is due to Kostant. The heat kernel proof for general k is due to H.D. Fegan, I. B. Frenkel, D. Bernard. Fegan later published erroneous generalizations.)

Dedekind eta function

The Dedekind eta-function is:
(7)
$$\eta(z) = e^{2\pi i z/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}), \quad \Im z > 0.$$

It is a modular cusp form of weight $1/2$, i.e. it satisfies

$$\eta(\gamma z) = \theta(\gamma) j_\gamma(z)^{1/2} \eta(z), \quad \text{if } \gamma \in SL(2, \mathbb{Z}),$$

where $\theta(\gamma)$ is a certain multiplier. When $\gamma z = -1/z$, we have

$$\eta(-\frac{1}{z}) = (iz)^{1/2} \eta(z).$$

We are particularly interested in $\log \eta(iy)$ as $y \to 0+$. For real numbers $y > 0$ we have:
(8)
$$\log \eta(iy) \;=\; -\frac{\pi}{12y} - \frac{1}{2} \log y + \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{2\pi m/y}}.$$

Dedekind eta function and $SU(2)$

The central heat kernel $H(t, a, y)$ at the special point a may be expressed in terms of Jacobi's theta function:

$$(9) \qquad H_{SU(2)}(t, a, 1) = (e^{-\pi t/12}\eta(it))^3.$$

Partition function: simplest case

When $U_1 = a$ and $U_2 = 1$, Macdonald's identity gives:

$$\mathcal{Z}_{SU(N)}(a, 1; A)$$

$$= e^{-\pi \dim SU(N) A/24N} \eta(iA/2N)^{\dim SU(N)}, \quad .$$

The free energy is then

$$\frac{1}{N^2} \log H_{SU(N)}(A/2N, a, I)$$

$$= \frac{\dim SU(N)}{N^2} \{-\frac{A\pi}{48N} + \log \eta(iA/2N)\}.$$

We note that $\frac{\dim SU(N)}{N^2} = \frac{1}{2} + O(\frac{1}{N})$. We substitute $y = A/2N$ in the right side to get:

$$\log \eta(iA/2N) = -\frac{2N}{A}\frac{\pi}{24} - \frac{1}{2}\log(\frac{A}{2N})$$

$$+ \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{4N\pi m/A}}$$

The sum is an exponentially small correction.

Unscaled limit

We now show that the unscaled large N limit exists.

We take e^k to be a diagonal matrix with entries $e^{2\pi i \lambda_j(N)}$. Then

$$\frac{1}{N^2} \log H_{SU(N)}(t, a, e^k) = \frac{1}{N^2}\{(-|R_+| + N)[-\frac{\pi t}{12}+$$

$$\log \eta(-it/4\pi)\} + \frac{1}{N^2} \sum_{\alpha>0} \log H_{SU(2)}(t, e^{i\langle\alpha,k\rangle}, e^{i\pi/2})$$

The roots of $SU(N)$ are $e_i - e_j$ and its positive roots satisfy $i < j$. Hence, $\langle\alpha, k\rangle = \lambda_i - \lambda_j \implies$

$$\sum_{\alpha\in R_+} \log H_{SU(2)}(t, e^{i\langle\alpha,k\rangle}, e^{i\pi/2})$$

$$= \frac{1}{2} \sum_{i\neq j} \log H_{SU(2)}(t, e^{i(\lambda_i-\lambda_j)}, e^{i\pi/2}).$$

Unscaled limit

Since $\ell_N \sim N$ for $SU(N)$, we have

$\frac{1}{N^2}\sum_{\alpha \in R_+} \log H_{SU(2)}(t, e^{i\langle \alpha, k\rangle}, e^{i\pi/2})$

$= \int_{S^1} \log H_{SU(2)}(t, e^{ix}, e^{i\pi/2}) d\sigma_N * \overline{d\sigma_N}(e^{ix})$

$-\frac{1}{N} \log H_{SU(2)}(t, 1, e^{i\pi/2})$

where

$$d\sigma_N * \overline{d\sigma_N}(e^{ix}) = \int_{S^1} d\sigma_N(e^{i(x-x')})\, d\sigma_N(e^{ix'}).$$

If

$$\sigma_N := \frac{1}{\ell_N}\sum_{j=1}^{\ell_N} \delta(e^{2\pi i \lambda_j(N)}) \to \sigma \in \mathcal{M}(S^1),$$

then

$$d\sigma_N * \overline{d\sigma_N} \to d\sigma * \overline{d\sigma},$$

since the Fourier coefficients of the left side tend to those of the right side. Thus, we obtain the stated limit.

Scaled limit

We now re-do the calculation but make the scaling $t = \frac{A}{2N}$. We additionally use the uniform off-diagonal asymptotics of the central heat kernel of $SU(2)$.

The central heat kernel is related to the actual heat kernel by

$$\log\ H_{SU(2)}(t, e^{ix}, a) = \log \int_{SU(2)}$$

$$k_{SU(2)}(t, e^{ix}, g^{-1}ag)dg$$

where $k_{SU(2)}(t, x, y)$ is the heat kernel. Therefore, we are interested in the uniform asymptotics of

$$\log\ H_{SU(2)}(\tfrac{A}{2N}, e^{ix}, a) = \log \int_{SU(2)}$$

$$k_{SU(2)}(\tfrac{A}{2N}, e^{ix}, g^{-1}ag)dg$$

in x for each A.

Scaled limit (cont)

There exists a uniform heat kernel parametrix for $k_{SU(2)}$ given by:

(10) $$k_{SU(2)}(t,u,v) \sim t^{-3/2} e^{-\frac{d(u,v)^2}{\pi t}} V(t,u,v)$$

where $V(t,u,v) \sim \sum_{j=0}^{\infty} V_j(u,v) t^j$. Thus,

$$\log\ H_{SU(2)}(\frac{A}{2N}, e^{ix}, a) \sim \log\{(\frac{N}{A})^{3/2} \int_{SU(2)}$$

$$e^{-\frac{2N}{\pi A} d(e^{ix}, g^{-1}ag)^2} V(A/2N, e^{ix}, g^{-1}ag) dg\},$$

where $V(A/2N, e^{ix}, g^{-1}ag)$ is a semiclassical amplitude in N.

Scaled limit (cont.)

The asymptotics are determined by the minimum point of the phase $d(e^{ix}, g^{-1}ag)^2$, namely by the distance $d(e^{ix}, C_a)$ from e^{ix} to the conjugacy class of a. We note that the conjugacy class $C(a) = \{g^{-1}ag : g \in SU(2)\}$ is a great (equatorial) 2-sphere of radius $\pi/2$ from (the north pole) I.

Thus, as $e^{ix} \to 1$,
(11)

$$H_{SU(2)}(\frac{A}{2N}, e^{ix}, a) \sim \begin{cases} |x|^{-3/2} e^{-\frac{2N}{\pi A} d(e^{ix}, C_a)^2}, & x \neq 0 \\ \\ (\frac{N}{A})^{3/2} e^{-2N\pi^2/4\pi A}, & x = 0. \end{cases}$$

Scaled limit (cont.)

We eventually obtain the leading order term:

$$(12) \qquad -\frac{2N}{\pi A}\int_{S^1} d(C(a), e^{ix})^2 d\sigma_N * \overline{d\sigma}_N(e^{ix})$$

plus the canonical terms $\frac{1}{N}\log H_{SU(2)}(A/2N, 1, a)$ and $\log \eta(\frac{iA}{2N})$ which are independent of $d\sigma$. We then recognize that $d(e^{ix}, C_a) = \min\{d(e^{ix}, e^{\pm i\pi/2})\}$, completing the proof.

Once again, we see the extra factor of N which came from the time-rescaled heat kernel.

Character asymptotics

The following result shows that the oscillation of values of $\chi_R(a_N)$ is much too regular for any such results. There is simply no separation of the possible limiting shapes of Young diagrams into the three discrete classes of possible limits $0, \pm 1$; all possible limit shapes are consistent with the limit 0.

Theorem **9** *Given any sequence of irreducibles $R_N \in \widehat{SU(N)}$, with $R_N \to \rho$, there exists a sequence $R'_N \in \widehat{SU(N)}$ with $R'_N \to \rho$ with the property that $\chi_{R'_N}(a_N) = 0$. Hence, there cannot exist a limit functional $F_0(d\theta, d\rho)$ depending only on the limit densities $d\sigma, d\rho$.*

Character asymptotics: idea of proof

The basic idea of the proof is the following: suppose that the highest weight R is such that $\chi_R(a_N) = \pm 1$. Then, by changing one component of R by one unit, one obtains a highest weight R' such that $\chi_{R'}(a_N) = 0$. Taking a sequence $R_N \to \rho$ and changing $R_N \to R'_N$ one obtains a new sequence with $R'_N \to \rho$ and with $\chi_{R'_N}(a_N) \equiv 0$. The reason why this works is that 0 is by far the likeliest value of $\chi_R(a_N)$. The non-zero values ± 1 are surrounded by a sea of 0's.

Discussion/background

The original conjecture of Matytsin pertained to Itzykson-Zuber integrals

$$I(A,B) \equiv \int_{SU(N)} e^{Ntr[AUBU^\dagger]} dU, \tag{13}$$

where A and B are $N \times N$ Hermitian matrices and dU is (unit mass) Haar measure on $SU(N)$. This was proved by Guionnet-Zeitouni.

As I understand it, Gross-Matytsin and Kazakov-Wynter then stated the analogous conjecture for characters of $U(N)$.

In the words of Gross-Matytsin

Gross-Matytsin: "for large N the $U(N)$ characters behave asymptotically as

$$\chi_R(U) \simeq \mathrm{e}^{N^2 \Xi[\rho_Y(l/N),\sigma(\theta)]} \tag{14}$$

with some finite functional $\Xi[\rho_Y, \sigma]$... we observe that the $U(N)$ characters can be represented as analytic continuations of the Itzykson–Zuber integral (13). Setting $a_k = l_k$, $b_j = \theta_j$ and analytically continuing $a_k \to ia_k$, we see that

$$\frac{\det[e^{Na_i b_j}]}{\Delta(a)\Delta(b)} \to J\Big(\mathrm{e}^{i\theta_s}\Big)\,\chi_R(U). \tag{15}$$

Therefore, we can use the known expressions for the large N limit of the Itzykson–Zuber integral to find the functional Ξ..."

In the words of Kazakov-Wynter

They also write that character values for $U(N)$ are, ‘up to a factor of i...the Itzykson-Zuber determinant...From Matysin’s paper we quote the result (with the minor change of an extra factor of i)...’

Moral: It is precisely the analytic continuation of the large N asymptotics of the Itzykson -Zuber integral from Hermitian to skew-Hermitian matrices (the Lie algebra of $U(N)$), i.e. the extra factor of i as we go from e^A to e^{iA} which leads to incorrect results.

Conjectures/Speculations

Our first conjecture is to Wick rotate the Kazakov-Wynter, Gross-Matytsin partition function asymptotics to positive matrices. We let H_1, H_2 denote sequences of Hermitian matrices and write $H_j \to d\sigma_j(x)$ if the eigenvalue distribution of H_j tends to the probability measure $d\sigma_j$ on $\mathbb{R}$.

Conjecture **10** *(SZ) Assume that* $H_1 \to \sigma_1, H_2 \to \sigma_2$. *Then*

$$F_N(e^{H_1}, e^{H_2}|A) \to F(\sigma_1, \sigma_2|A)$$

$$:= S(\sigma_1, \sigma_2|A)$$

$$-\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \log|x-y| d\sigma_1(x) d\sigma_1(y)$$

$$-\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \log|x-y| d\sigma_1 2(x) d\sigma_2(y)$$

where S *is the classical action corresponding to the Hopf equation.*

Idea

According to a theorem of I. B. Frenkel, the central heat kernel may be expressed as a spherical integral for the loop-group $\mathcal{L}U(N)$:

(16)

$$\mathcal{Z}_N(A, e^{Th}, e^{Tk}) = \int_{C_G(\mathcal{O}_{e^{Th}})} e^{-\frac{1}{t}\langle g^{-1}g', k\rangle} dw^t_{G,\mathcal{O}_{e^{Th}}}(g).$$

This suggests using the large deviations method of Guionnet-Zeitouni for such spherical integrals. If so, one needs to consider Brownian motion on the loop algebra $\mathcal{L}\mathbf{u}(N)$.