Complex zeros of real analytic ergodic Laplace eigenfunctions

Kuranishi Conference, Columbia University
Friday 11: 30, May 6, 2005

Steve Zelditch
Department of Mathematics
Johns Hopkins University

Purpose of talk

Consider the eigenvalue problem

$$\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

for Laplacians on Riemannian manifolds (M, g) satisfying:

- (M, g) is *real analytic*;
- Its geodesic flow $G^t : S_g^* M \to S_g^* M$ is ergodic.

Problem How are nodal hypersurfaces distributed in the limit $\lambda_j \to \infty$.?

Real versus complex nodal hypersurfaces

One would like to know about the real nodal hypersurfaces (zero sets of eigenfunctions):

$$Z_{\varphi_j} = \{x \in M : \varphi_j(x) = 0\}.$$

These are hard. Our results will be about: complex nodal hypersurfaces

$$Z_{\varphi_j^{\mathbb{C}}} = \{\zeta \in B^*M : \varphi_j^{\mathbb{C}}(\zeta) = 0\},$$

where $\varphi_j^{\mathbb{C}}$ is the analytic continuation of φ_j to the ball bundle B^*M for the natural complex structure adapted to g. (Definitions to come).

Distribution of nodal hypersurfaces

Distribution of zeros is determined by the current of integration over the nodal hypersurface

(1) $$\langle [\tilde{Z}_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f(x) d\mathcal{H}^{n-1},$$

where $d\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional (Haussdorf) surface measure on the nodal hypersurface induced by the Riemannian metric of (M, g).

Problem: How does $\langle [\tilde{Z}_{\varphi_j}], f \rangle$ behave as $\lambda_j \to \infty$.

Volumes of nodal hypersurfaces

Even for $f \equiv 1$ this is too hard. The best result to date on volumes of nodal hypersurfaces on analytic Riemannian manifolds is:

Theorem **1** *(Donnelly-Fefferman, Inv. Math. 1988) Suppose that* (M, g) *is real analytic. Then*

$$c_1 \lambda \leq \mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \leq C_2 \lambda.$$

(Note that our λ is the square root of the Δ-eigenvalue.)

Complex zeros

BUT: the distribution of zeros becomes more tractable if we analytically continue φ_j to the complexification $M_{\mathbb{C}}$ of M and study complex zeros.

Definitions later. To state our main result, we identify $M_{\mathbb{C}}$ with the metric (co-)ball bundle $B^*_\epsilon M$ with its adapted complex structure:

Definition: (Lempert-Szoke; Guillemin-Stenzel) The adapted complex structure on B^*M is uniquely characterized by the fact that the maps $(t, \tau) \in \mathbb{C}^+ \to B^*M$,

$$(t, \tau) \to \tau \dot{\gamma}(t), \ \ t \in \mathbb{R}, \ \tau \in \mathbb{R}^+$$

are holomorphic curves for any geodesic γ.

Main result

Theorem **2** *Assume* (M, g) *is real analytic and that the geodesic flow of* (M, g) *is ergodic. Then*

$$\frac{1}{\lambda_j} Z_{\varphi^{\mathbb{C}}_{\lambda_j}} \to \overline{\partial}\partial |\xi|_g, \ \ \textit{weakly in} \ \ B^*_g M.$$

Both sides are viewed as $(1, 1)$ currents: dual to $(m-1, m-1)$ forms. Here, $\overline{\partial}$ is the Cauchy-Riemann operator for the complex structure on the unit ball bundle with respect to the complex structure adapted to g. Also, $|\xi|_g^2 = \sum_{i,j} g^{ij}\xi_i\xi_j$ is the length-squared of a (co-)vector.

Ergodicity

The geodesic flow is the Hamiltonian flow

$$G^t : T^*M - 0 \to T^*M - 0$$

of the metric norm function $|\xi_g|$.

G^t preserves the unit cosphere bundle $S^*_g M = \{|\xi|_g = 1\}$ and all level sets of H.

Ergodic geodesic flow $\iff$ G^t is ergodic on S^*M with respect to Liouville measure $\alpha \wedge \omega^{m-1}$ where $\alpha = \xi \cdot dx$ and $\omega = d\alpha$.

Ergodic means that invariant sets have measure 0 or 1. Only invariant L^2 functions are constants. Time average of a function = space average.

Limit distribution of zeros is singular along zero section

- The Kaehler structure on the cotangent bundle is $\overline{\partial}\partial|\xi|_g^2$. But the limit current is $\overline{\partial}\partial|\xi|_g$. The latter is singular along $M = \{\xi = 0\}$ and the associated volume form is not the symplectic one.

- The reason for the singularity is that the zero set is invariant under the involution $\sigma : T^*M \to T^*M$, $(x, \xi) \to (x, -\xi)$, since the eigenfunction is real valued on M. The fixed point set of σ is M and is also where zeros concentrate. By pushing this further one might be able to prove the conjecture on real zeros.

Simplest case: S^1

The zeros of $\sin 2\pi kz$ in the cylinder $\mathbb{C}/\mathbb{Z}$ all lie on the real axis at the points $z = \frac{n}{2k}$. Thus, there are $2k$ real zeros. The limit zero distribution is:

$$\begin{aligned} \lim_{k\to\infty} \frac{i}{2\pi k} \partial\bar{\partial} \log|\sin 2\pi k|^2 &= \lim_{k\to\infty} \frac{1}{k} \sum_{n=1}^{2k} \delta_{\frac{n}{2k}} \\ &= \frac{1}{\pi} \delta_0(\xi) dx \wedge d\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{i}{\pi} \partial\bar{\partial}|\xi| &= \frac{i}{\pi} \frac{d^2}{4d\xi^2}|\xi| \ \ \frac{2}{i} dx \wedge d\xi \\ &= \frac{i}{\pi} \frac{1}{2} \delta_0(\xi) \ \ \frac{2}{i} dx \wedge d\xi. \end{aligned}$$

Related work

The following prove related results for zeros of holomorphic eigensections of positive line bundles over compact Kähler manifold. Eigensections = eigensection of an ergodic quantum map.

1. B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles. Comm. Math. Phys. 200 (1999)

2. S. Nonnenmacher and A. Voros, Chaotic eigenfunctions in phase space. J. Statist. Phys. 92 (1998), no. 3-4, 431–518.

Outline of proof

- Use holomorphic continuation of wave kernel (or Poisson kernel) at imaginary times to B^*M to analytically continue eigenfunctions $\varphi_\lambda \to \varphi_\lambda^{\mathbb{C}}$.

- Study $U_\lambda(z) \in C^\infty(B^*_\epsilon M)$ defined by:

$$U_\lambda(z) := \frac{\varphi_\lambda^{\mathbb{C}}(z)}{||\varphi_\lambda^{\mathbb{C}}||_{L^2(\partial B^*_\tau M)}}, \quad z \in \partial B^*_\tau M$$

  The normalizing factor means: if $z \in$ the cosphere bundle $\partial B^*_\tau M$, then divide by the L^2-norm of $\varphi_\lambda^{\mathbb{C}}$ restricted to $\partial B^*_\tau M$.

Step I: Ergodicity of complexified eigenfunctions

The first step is to prove quantum ergodicity of the complexified eigenfunctions:

Theorem **3** *Assume the geodesic flow of* (M, g) *is ergodic. Then*

$$|U_\lambda|^2 = \frac{|\varphi_\lambda^\epsilon(z)|^2}{||\varphi_\lambda^\epsilon||^2_{L^2(\partial B^*_\epsilon M)}} \to 1, \text{ weakly in } L^1(B^*_\epsilon M),$$

along a density one subsquence of j.

This is the analogue of what can be proved for the real eigenfunctions (Shnirelman, SZ, Colin de Verdiere).

Weak to strong

Lemma **4** *We have:* $\frac{1}{\lambda_j} \log |U_j|^2 \to 0$ *in* $L^1(B^*_\epsilon M)$.

This is the key input from complex analysis.

It uses that $\log |U_j|^2$ is QPSH = quasi pluri subharmonic.

(A QPSH function ψ is one which may be locally written as the sum of a plurisubharmonic function and a smooth function, or equivalently $i\partial\bar{\partial}\psi$ is locally bounded below by a negative smooth $(1,1)$ form.)

Corollary **5** $\frac{1}{\lambda_j}\partial\bar{\partial} \log |U_j|^2 \to 0,$ *weakly in* $\mathcal{D}'(B^*_\epsilon M)$.

Zeros

Combine

$$\frac{1}{\lambda_j}\partial\bar{\partial}\log|U_j|^2 \to 0, \text{ weakly in } \mathcal{D}'(M_1)$$

with Poincare- Lelong:

$$[\tilde{Z}_j] = \partial\bar{\partial}\log|\tilde{\varphi}_j^{\mathbb{C}}|^2.$$

We get

$$\frac{1}{\lambda_j}\partial\bar{\partial}\log|\tilde{\varphi}_j^{\mathbb{C}}|^2 \sim \frac{1}{\lambda_j}\partial\bar{\partial}\log||\tilde{\varphi}_j^{\mathbb{C}}||^2_{\partial M_\epsilon}.$$

Thus: to find asymptotics of $[\tilde{Z}_j]$ we need asymptotics of $\log|\tilde{\varphi}_j^{\mathbb{C}}|^2$.

Complex FIOs and norm asymptotics

The final step is to prove:

Lemma **6**

$$\frac{1}{\lambda} \log ||\varphi_\lambda^{\mathbb{C}}||_{L^2(\partial B^*_\tau M)} \sim \tau.$$

To prove this we need to study the complexified wave group as a complex Fourier integral operator. We now give details of proofs.

Bruhat-Whitney complexification

Theorem **7 (Bruhat-Whitney, 1959)** *Let M be a real analytic manifold of real dimension n. Then there exists a complex manifold $M_{\mathbb{C}}$ of complex dimension n and a real analytic embedding $M \to M_{\mathbb{C}}$ such that M is a totally real submanifold of $M_{\mathbb{C}}$. The germ of $M_{\mathbb{C}}$ is unique.*

Totally real means: Let $J_p : T_{\mathbb{C}}M_{\mathbb{C}} \to T_{\mathbb{C}}M_{\mathbb{C}}$ denote the complex structure on the (complexified) tangent bundle of $M_{\mathbb{C}}$. Then $J_pT_pM \cap T_pM = \{0\}$. I.e. T_pM contains to complex subspaces.

Examples: Spheres and tori

BH complexifications of spheres and tori can be identified with their full cotangent spaces:

1. S^n It is defined by $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$. Its BH complexification is the complex quadric

$S^2_{\mathbb{C}} = \{(z_1, \ldots, z_n) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 1\}$.

If we write $z_j = x_j + i\xi_j$, the equations become $|x|^2 - |\xi|^1 = 1, \langle x, \xi \rangle = 0$.

2. $T^n = \mathbb{R}^n/\mathbb{Z}^n$ The BH complexification is $\mathbb{C}^n/\mathbb{Z}^n = T^n \times \mathbb{R}^n \equiv T^*M$.

Metric notions and notations

Now equip M with a real analytic Riemannian metric g. Let

- $\pi : T^*M \to M$ =natural projection;
- $\exp : T_x^*M \to M, \quad \exp_x \xi = \pi \circ G^t(x, \xi) =$ exponential map;
- $r(x, y) =$ Riemannian distance between points (well-defined near the diagonal).

Complexified exponential map and complex structure on the cotangent bundle

Consider the complexified exponential map

$$\exp_{\mathbb{C}} : B^*_\epsilon M \to M_{\mathbb{C}}, \ \ (x, \xi) \to \exp_x(\sqrt{-1}\xi).$$

Then:

- $\exists \epsilon_0 : \forall \epsilon < \epsilon_0$, $\exp_x \sqrt{-1}\xi$ is a real analytic diffeo to its image.

- Equip $B^*_{\epsilon_0}$ with the complex structure,

$$J_g = \exp^*_{\mathbb{C}} J_0$$

  where J_0 is the complex structure on $M_{\mathbb{C}}$. J_g is the unique complex structure on $B^*_{\epsilon_0}$ for which $\exp_{\mathbb{C}}$ is a bi-holomorophic map to $M_{\mathbb{C}}$.

Adapted complex structure and radius of Grauert tube

The complex structure J_g on $B^*_{\epsilon_0}M$ is called the *adapted complex structure* to g and the maximal ϵ_0 is called the radius of the Grauert tube. They can be characterized by:

- The metric function $\rho(x,\xi) = |\xi|_g^2$ is plurisubharmonic, i.e. $\overline{\partial}\partial\rho$ is a positive $(1,1)$ form.

- The maps $\iota_{x,\xi} : \mathbb{C}^+_{\epsilon_0} \to B^*_{\epsilon_0}M$,

  $$\iota_{x,\xi}(t,\tau) = \tau G^t(x,\xi)$$

  are holomorphic curves. Here, $\mathbb{C}^+_\epsilon = \{t + itau \in \mathbb{C} : 0 < \tau < \epsilon_0\}$.

Wave kernel

By the wave kernel of (M, g) we mean the kernel

$$E(t, x, y) = \sum_{j=0}^{\infty} e^{it\lambda_j} \varphi_j(x) \varphi_j(y)$$

of $e^{it\sqrt{\Delta}}$.

As observed by Boutet de Monvel, the wave kernel at imaginary times admits a holomorphic extension to $B^*_\epsilon M \times M$ as

$$E(i\epsilon, \zeta, y) = \sum_{j=0}^{\infty} e^{-\epsilon\lambda_j} \varphi_{Cj}(\zeta) \varphi_j(y), \quad (\zeta, y) \in B^*_\epsilon M \times M.$$

Analytic continuation of the wave kernel

Boutet de Monvel observed:

Theorem **8** *$E(i\epsilon, z, y) : L^2(M) \to H^2(\partial M_\epsilon)$ is a complex Fourier integral operator of order $-\frac{m-1}{4}$ associated to the canonical relation*

$$\Gamma = \{(y, \eta, \exp_y(i\epsilon)\eta/|\eta|)\} \subset T^*M \times \Sigma_\epsilon.$$

Moreover,

$$E(i\epsilon) : H^{-\frac{m-1}{4}}(M) \to H^2(\partial B^*_\epsilon M)$$

is an isomorphism.

Euclidean case

On $\mathbb{R}^n$:

$$E(t,x,y) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{i\langle \xi, x-y\rangle} d\xi.$$

Its analytic continuation to $t + i\tau, \zeta = x + ip$ is given by

$$E(t+i\tau, x+ip, y) = \int_{\mathbb{R}^n} e^{i(t+i\tau)|\xi|} e^{i\langle \xi, x+ip-y\rangle} d\xi,$$

which converges absolutely for $|p| < \tau$.

General analytic Riemannian manifold:
(2)

$$E(t,x,y) = \int_{T_y^* M} e^{it|\xi|_{g_y}} e^{i\langle \xi, \mathrm{exp}_y^{-1}(x)\rangle} A(t,x,y,\xi) d\xi$$

where $|\xi|_{g_x}$ is the metric norm function at x, and where $A(t,x,y,\xi)$ is a polyhomogeneous amplitude of order 1.

Analytic continuation of eigenfunctions

The holomorphic extension of φ_λ is obtained by applying a complex Fourier integral operator:

$$E(i\tau)\varphi_\lambda = e^{-\tau\lambda}\varphi_\lambda^{\mathbb{C}}. \tag{3}$$

This implies connections between the geodesic flow and the growth rate and zeros of $\varphi_\lambda^{\mathbb{C}}$.

Corollary **9** *Each eigenfunction φ_λ has a unique holomorphic extensions to M_ϵ satisfying*

$$\sup_{m \in M_\epsilon} |\varphi_\lambda^{\mathbb{C}}(m)| \leq C_\epsilon \lambda^{m+1} e^{\epsilon\lambda}.$$

In particular, eigenfunctions extend holomorphically to the maximal Grauert tube in the adapted complex structure.

Normalized complexified eigenfunctions

Recall:

(4)

$$\begin{cases} \varphi_\lambda^\epsilon = \varphi_\lambda^{\mathbb{C}}|_{\partial B^*_\epsilon(M)} \in H^2(\partial M_\epsilon), \\ U_\lambda(z) \in C^\infty(B^*_\epsilon M) := \frac{\varphi_\lambda^{\mathbb{C}}(z)}{||\varphi_\lambda^\epsilon||_{L^2(\partial M_\epsilon)}}, \quad z \in \partial M_\epsilon. \end{cases}$$

U_λ is CR holomorphic along each sphere bundle in the ball bundle. However, the normalizing factor $||\varphi_\lambda^\epsilon||^{-1}_{L^2(\partial M_\epsilon)}$ depends on ϵ, so $U_\lambda \notin \mathcal{O}(M_\epsilon)$. But $\log |U_\lambda(z)|^2$ is quasi-plurisubharmonic (QPSH) = sum of PSH function and a smooth function.

Step I: Ergodicity of complexified eigenfunctions

Theorem **10** *Assume the geodesic flow of* (M, g) *is ergodic. Then*

$$|U_\lambda|^2 = \frac{|\varphi_\lambda^\epsilon(z)|^2}{||\varphi_\lambda^\epsilon||^2_{L^2(\partial B^*_\epsilon M)}} \to 1, \text{ weakly in } L^1(B^*_\epsilon M)$$

along a density one subsquence of j.

This can be reduced to ergodicity of eigenfunctions on M:

$$\frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} |\langle A\varphi_j, \varphi_j\rangle - \omega(A)|^2 \to 0$$

where $\omega(A) = \int_{S^*M} \sigma_A d\mu_L$.

A picture proof of QE

Let $\rho_j(A) = \langle A\varphi_j, \varphi_j\rangle$. These are invariant states = linear functionals on A which are invariant under conjugation by $e^{it\sqrt{\Delta}}$. The set $\mathcal{E}$ of nvariant states is a compact convex set.

Local Weyl law: $\frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} \rho_j \to \omega(A)$.

Ergodicity: $\omega(A)$ is an extreme point of $\mathcal{E}$. Cannot be a convex function of ρ_j, or even limit of such, unless $\rho_j \to \omega$ (almost all j).

From QE to zeros

Basic fact on PSH and QPSH functions:

Let $\{v_j\}$ be a sequence of subharmonic (or QPSH) functions in an open set $X \subset \mathbb{R}^n$ which have a uniform upper bound on any compact set. Then either $v_j \to -\infty$ uniformly on every compact set, or else there exists a subsequence v_{j_k} which is convergent in $L^1_{loc}(X)$. The hypotheses are satisfied in our example:

i) the functions $\frac{1}{\lambda_j} \log |U_j|^2$ are uniformly bounded above on $B^*_\epsilon M$;

ii) $\limsup_{j \to \infty} \frac{1}{\lambda_j} \log |U_j|^2 \leq 0$.

From QE to zeros

$\{\frac{1}{\lambda_j} \log |U_j|^2\}$ do not tend uniformly to $-\infty$ on compact sets. Inconsistent with QE. So

$$\frac{1}{\lambda_j} \partial\bar{\partial} \log |U_j|^2 \to 0, \text{ weakly in } \mathcal{D}'(M_1).$$

Recall that by Poincare- Lelong:

$$[\tilde{Z}_j] = \partial\bar{\partial} \log |\tilde{\varphi}_j^{\mathbb{C}}|^2.$$

Since

$$\partial\bar{\partial} \log |U_j|^2 = \partial\bar{\partial} \log |\tilde{\varphi}_j^{\mathbb{C}}|^2 - \partial\bar{\partial} \log ||\tilde{\varphi}_j^{\mathbb{C}}||^2_{\partial M_\epsilon},$$

we find asymptotics of $[\tilde{Z}_j]$ from asymptotics of $\log |\tilde{\varphi}_j^{\mathbb{C}}|^2$.

Step III: norm asymptotics

The final step is to prove:

Lemma **11**

$$\frac{1}{\lambda}\log ||\varphi_\lambda^{\mathbb{C}\sqrt{\rho}}||_{L^2(\partial M_{\sqrt{\rho}})} \sim |\xi|_g.$$

This is a new feature. The analogue for positive line bundles is the hermitian metric, but now the norm depends on the eigenfunction.

Idea of norm estimate

$||\varphi_\lambda^{\mathbb{C}}||^2_{L^2(\partial B^*_\epsilon M)}$ equals $e^{2\epsilon\lambda_j}$ times

$$\langle E(i\epsilon)\varphi_\lambda, E(i\epsilon)\varphi_\lambda\rangle = \langle E(i\epsilon)^* \Pi_\epsilon E(i\epsilon)\varphi_\lambda, \varphi_\lambda\rangle.$$

Here, $\Pi_\epsilon : L^2(\partial B^*_\epsilon M) \to H^2(\partial B^*_\epsilon M)$ is the Szego kernel.

But $E(i\epsilon)^* \Pi_\epsilon E(i\epsilon)$ is a pseudodifferential operator of order $\frac{n-1}{2}$. Its symbol $|\xi|^{-\frac{n-1}{2}}$ doesn't contribute to the logarithm.

Summing up

Thus, $\frac{1}{\lambda_j} \log |U_j|^2 \to 0$ in $L^1(B^*_\epsilon M)$. Hence,

$$\frac{1}{\lambda_j} \partial\bar{\partial} \log |U_j|^2 \to 0, \text{ weakly in } \mathcal{D}'(B^*_\epsilon M).$$

Since

$$\partial\bar{\partial} \log |U_j|^2 = \partial\bar{\partial} \log |\tilde{\varphi}_j^{\mathbb{C}}|^2 - \partial\bar{\partial} \log ||\tilde{\varphi}_j^{\mathbb{C}}||^2_{\partial M_\epsilon},$$

we have:

$$\frac{1}{\lambda_j} \partial\bar{\partial} \log |\tilde{\varphi}_j^{\mathbb{C}}|^2 \sim \frac{1}{\lambda_j} \partial\bar{\partial} \log ||\tilde{\varphi}_j^{\mathbb{C}}||^2_{\partial M_\epsilon}$$

$$\to \partial\bar{\partial} |\xi|_g.$$

Final Remarks

- We can study ergodicity further by using the holomorphic curves $(t + i\tau) \in \mathbb{C}^+ \to \tau G^t(x, \xi)$ for some (x, ξ). $\varphi_\lambda^{\mathbb{C}}$ pulls back to a holomorphic function on $\mathbb{C}^+$. Its real zeros are discrete, = intersections of the geodesic with the nodal hypersurface and should be uniformly distributed. This is so in the Grauert tube. Since they concentrate on the real axis, it should be so on M.

- Same results should hold on bounded domains with piecewise analytic boundaries and for boundary traces. The zeros of boundary traces should be uniformly distributed on the boundary.

Conjecture on real nodal hypersurface: ergodic case

Conjecture **12** *Let* (M, g) *be a real analytic Riemannian manifold with ergodic geodesic flow, and let* $\{\varphi_j\}$ *be the density one sequence of ergodic eigenfunctions. Then,*

$$\langle [\tilde{Z}_{\varphi_j}], f \rangle \sim \{\frac{1}{Vol(M, g)} \int_M f dVol_g\} \lambda.$$

At this time of writing, even the asymptotics of the area (even in dimension two) has not been proved.

Kaehler metrics and positive $(1,1)$ forms

$\overline{\partial}\partial\rho$ is a positive $(1,1)$ form means:

$$\left[\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}\right]_{j,k} >> 0.$$

Thus, ρ is a Kaehler potential, i.e.

$$\sum_{j,k} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k} dz_j \otimes d\bar{z}_k$$

is a Kaehler metric ω on $B^*_{\epsilon_0} M$. On the zero section, it restricts to g.

Examples: Sphere

The distance function of S^n of constant curvature 1 is given by:

$$r(x,y) = 2\sin^{-1}\frac{|x-y|}{2} = 2\sin^{-1}(\frac{1}{2}\sqrt{(x-y)^2}.$$

The analytic continuation to $S^n_{\mathbb{C}} \times S^n_{\mathbb{C}}$ is the doubly-branched holomorphic function:

$$r_{\mathbb{C}}(z,w) = 2\sin^{-1}\frac{1}{2}\sqrt{(z-w)^2}.$$

The geodesic flow is:

$$G^t(x,\xi) = (\cos tx + \sin t\xi, -\sin tx + \cos t\xi)$$

so the exponential map is

$$exp_x\xi = (\cos|\xi|)x + (\sin|\xi|)\xi,$$

which complexifies to

$$\exp_{\mathbb{C},x}\sqrt{-1}\xi = (\cosh|\xi|)x + \sqrt{-1}(\sinh|\xi|)\frac{\xi}{|\xi|}.$$

Examples: Sphere

The pluri-subharmonic function $\rho = r_{\mathbb{C}}(z, \bar{z})$ equals

$$r_{\mathbb{C}}(z, \bar{z}) = 2\sin^{-1} \pm\sqrt{-1}|\Im z|) = \pm\sqrt{-1}\cosh^{-1}|z|^2.$$

It is define on $S^n_{\mathbb{C}}$. It pulls back under the complexified exponential map to

$$\begin{aligned} \exp^*_{\mathbb{C}} \rho(x, \xi) &= \cosh^{-1}|(\cosh|\xi|)x + \sqrt{-1}(\sinh|\xi|)\frac{\xi}{|\xi} \\ &= \cosh^{-1}\{(\cosh|\xi|)^2 - (\sinh|\xi|)^2\} \\ &= \cosh^{-1}\cosh 2|\xi| = 2|\xi|. \end{aligned}$$

Examples: Torus

Here, $r(x,y) = |x-y|$ and $r_{\mathbb{C}}(z,w) = \sqrt{(z-w)^2}$. Then $\rho(z,\bar{z}) = \sqrt{(z-\bar{z})^2} = |\Im z| = |\xi|$.

The complexified exponential map is:

$$\exp_{\mathbb{C}x}(i\xi) = x + i\xi.$$

Examples: Hyperbolic space

In this case, the BH complexification only fills out a ball bundle of T^*M:

3. H^n The hyperboloid model of hyperbolic space is the hypersurface in $\mathbb{R}^{n+1}$ defined by

$$H^n = \{x_1^2 + \cdots x_n^2 - x_{n+1}^2 = -1, \;\; x_n > 0\}.$$

Then,

$$H^n_{\mathbb{C}} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \cdots z_n^2 - z_{n+1}^2 = -1\}$$

In real coordinates $z_j = x_j + i\xi_j$, this is:

$$\langle x, x \rangle_L - \langle \xi, \xi \rangle_L = -1, \;\; \langle x, \xi \rangle_L = 0$$

where $\langle , \rangle_L$ is the Lorentz inner product of signature $(n, 1)$.