Asymptotic Geometry

of polynomials

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Statistical algebraic geometry

We are interested in asymptotic geometry as the degree $N \to \infty$ of zeros of polynomial systems

$$\begin{cases} p_1(z_1, \ldots, z_m) = 0 \\ \\ p_2(z_1, \ldots, z_m) = 0 \\ \vdots \\ p_k(z_1, \ldots, z_m) = 0. \end{cases}$$

We are interested both in complex (holomorphic) polynomials with $c_\alpha \in \mathbb{C}, z \in \mathbb{C}^m$ and real polynomials with $c_\alpha \in \mathbb{R}, x \in \mathbb{R}^m$.

More precisely, we are interested in the asymptotics as $N \to \infty$ of statistical properties of random polynomial systems.

Statistical Algebraic Geometry (2)

- Statistical algebraic geometry: zeros of individual polynomials define algebraic varieties. Instead of studying complexities of all possible individual varieties, study the expected (average) behaviour, the almost-sure behaviour.

- There are statistical patterns in zeros and critical points that one does not see by studying individual varieties, which are often 'outliers'.

- Our methods/results concern not just polynomials, but holomorphic sections of any positive line bundle over a Kähler manifold.

Plan of talk

- Review the notion of 'Gaussian random polynomial' and more generally 'Gaussian random section'.
- How are zeros of random holomorphic polynomials distributed/correlated?
- What if the Newton polytopes are constrained?
- How do the results work for Gaussian random real polynomials?
- Few proofs, mainly phenomenology of the subject.

Complex polynomials in m variables

Some background on polynomials in m complex variables:

$$z = (z_1, \ldots, z_m) \in \mathbb{C}^m.$$

- Monomials: $\chi_\alpha(z) = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$, $\alpha \in \mathbb{N}^m$.

- Polynomial of degree p (complex, holomorphic, not necessarily homogeneous):

$$f(z_1, \ldots, z_m) = \sum_{\alpha \in \mathbb{N}^m : |\alpha| \leq p} c_\alpha \chi_\alpha(z_1, \ldots, z_m).$$

- Homogenize to degree p: introduce new variable z_0 and put:

$$\widehat{\chi}_\alpha(z_0, z_1 \ldots, z_m) = z_0^{p-|\alpha|} z_1^{\alpha_1} \cdots z_m^{\alpha_m}.$$

  We write $F = \widehat{f}_\alpha(z_0, z_1 \ldots, z_m)$ for the homogenized f.

Random $SU(m+1)$ complex polynomials

Definition: $\mathcal{P}_N^m :=$ complex polynomials

$$f(z_1, \ldots, z_m) = \sum_{\alpha \in \mathbb{N}^m : |\alpha| \leq N} c_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m},$$

of degree N in m complex variables with $c_\alpha \in \mathbb{C}$.

Random polynomial: a probability measure on the coefficients λ_α.

Gaussian random:

$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \sqrt{\binom{N}{\alpha}}\, z^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \bar{\lambda}_\beta) = \delta_{\alpha\beta}.$$

In coordinates λ_α:

$$d\gamma_p(f) = \frac{1}{\pi^{k_N}} e^{-|\lambda|^2} d\lambda \text{ on } \mathcal{P}_N^m.$$

Gaussian measure versus inner product

The Gaussian measure above comes from the Fubini-Study inner product on the space $\mathcal{P}_N^{\mathbb{C}}$ of polynomials of degree N. Indeed,

$$||z^\alpha||_{FS} = \binom{N}{\alpha}^{-1/2}, \ \ \langle z^\alpha, z^\beta \rangle = 0, \ \alpha \neq \beta.$$

Namely, let $F(z_0, \dots, z_m) = z_0^N f(z'/z_0)$ homogenize f. Then
$||f||_{FS}^2 = \int_{S^{2m+1}} |F|^2 \, d\sigma$, (Haar measure).

Thus, the same ensemble could be written:

$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \ \frac{z^\alpha}{||z^\alpha||_{FS}},$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \overline{\lambda}_\beta) = \delta_{\alpha\beta}.$$

Why the Fubini-Study $SU(m+1)$-ensemble?

One could use any inner product in defining a Gaussian measure: Write

$$s = \sum_j c_j S_j, \quad \langle S_j, S_k \rangle = \delta_{jk}$$

with $\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k)$, $\mathbf{E}(c_j \overline{c_k}) = \delta_{jk}$.

We use the Fubini-Study because the expected distribution of zeros (or critical points etc.) of ‘typical’ polynomials become uniform over $\mathbb{CP}^m$. Thus, the ensemble is natural for projective geometry. Taking $\sum c_\alpha z^\alpha$ with c_α normal biases the zeros towards the torus $|z_j| = 1$.

Random real $O(m+1)$ polynomials

We now consider the same problems for real polynomials.

Let $Poly(N\Sigma)_{\mathbb{R}}$ be the space of real polynomials

$$p(x) = \sum_{|\alpha| \leq N} c_\alpha \chi_\alpha(x), \quad \chi_\alpha(x) = x^\alpha, \quad x \in \mathbb{R}^m, \alpha \in N\Sigma$$

of degree N in m real variables with real coefficients. Define the inner product

$$\langle \chi_\alpha, \chi_\beta \rangle = \delta_{\alpha,\beta} \frac{1}{\binom{N}{\alpha}}.$$

Define a random polynomial in the $O(m+1)$ ensemble as

$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \sqrt{\binom{N}{\alpha}}\, x^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \lambda_\beta) = \delta_{\alpha\beta}.$$

Why $O(m+1)$?

If we homogenize the polynomials $Poly(N\Sigma)$, we obtain a representation of $O(m+1)$. The invariant inner product is

$$\langle P, Q \rangle := P(D)\bar{Q}(0) = \int_{\mathbb{R}^n} P(2\pi i\xi)\bar{\hat{Q}}(\xi)d\xi,$$

where $P(D)$ is the constant coefficient differential operator defined by the Fourier multiplier $P(2\pi i\xi)$.

We may regard the zeros as points of $\mathbb{R}P^m$. The expected distribution of zeros will be uniform there w.r.t. the natural volume form.

Random holomorphic sections

For geometers: The complex polynomial ensemble can be defined for any positive holomorphic line bundle $L \to M$ over any Kähler manifold.

Recall: $\mathcal{P}_p^m \simeq H^0(\mathbb{CP}^m, \mathcal{O}(p))$ (holomorphic sections on the Nth power of the hyperplane section bundle).

More generally: define Gaussian holomorphic sections $s \in H^0(M, L^N)$ of powers of *any* positive line bundle over any Kähler manifold (M, ω):

$$s = \sum_j c_j S_j, \quad \langle S_j, S_k \rangle = \delta_{jk}$$

with $\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k)$, $\mathbf{E}(c_j \overline{c_k}) = \delta_{jk}$. [$\langle , \rangle$ = inner product on $H^0(M, L^N)$ using unique hermitian metric h of curvature form ω.]

How are zeros distributed? Correlated?

- **Problem 1**: How are the simultaneous zeros $Z_s = \{z : s_1(z) = \cdots = s_m(z) = 0\}$ of a m-tuple $s = (s_1, \ldots, s_m)$ of independent random polynomials (holomorphic sections) distributed?

- **Problem 2** How are the zeros correlated? For a full system of m equations in m unkowns, the simultaneous zeros form a discrete set. Do zeros repel each other like charged particles? Or behave independently like particles of an ideal gas? Or attract like gravitating particles?

Definition of 'distribution of zeros'

Since the zeros of a full system of m polynomials in m variables form a discrete set, we define distribution of zeros of the system $(f_1, \ldots, f_m)$ by

$$Z_{f_1,\ldots,f_m} = \sum_{\{z_j : f_1(z_j) = \cdots = f_m(z_j) = 0} \delta_{z_j}.$$

Here, $\delta(z)$ is the Dirac point mass at z. I.e. $\int \psi \delta(z) = \psi(z)$.

Note that $Z_{f_1,\ldots,f_m}$ is not normalized, i.e. its mass is the number of zeros.

Expected zero distributions: Definition

We denote the expected distribution of the simultaneous zeros of a random system of m polynomials by $\mathbf{E}_N(Z_{f_1,\dots,f_m},\varphi)$. It is the average value of the measure $(Z_{f_1,\dots,f_m},\varphi)$ w.r.t. f.

We have:

$$\mathbf{E}_N(Z_{f_1,\dots,f_m})(U) = \int d\gamma_p(f_1)\cdots\int d\gamma_p(f_m)$$

$$\times\Big[\#\{z\in U : f_1(z)=\cdots=f_m(z)=0\}\Big]\,,$$

for $U\subset\mathbb{C}^{*m}$, where the integrals
are over $\mathcal{P}_N^{\mathbb{C}}$.

Similarly for any other complex phase space, or for real polynomials.

Expected distribution of zeros in the $SU(m+1)$ and $O(m+1)$ ensembles

- $SU(m+1)$ ensemble: $\mathbf{E}(Z_f) = \frac{N^m}{Vol(\mathbb{CP}^m)} dVol_{\mathbb{CP}^m}$. It is a volume form on $\mathbb{CP}^m$ invariant under $SU(m+1)$, so is a constant multiple of the invariant volume form. The constant is determined by integrating, which gives the expected number of zeros. This must equal the Bezout number N^m, the product of the degrees of the f_j's.

- $O(m+1)$ ensemble: $\mathbf{E}(Z_f) = \frac{N^{m/2}}{Vol(\mathbb{R}P^m)} dVol_{\mathbb{R}P^m}$. For the same reason, it must be a constant multiple of the invariant volume form. But this time the number of zeros is a random variable. Shub-Smale (1995) showed that the expected number of zeros is the square root of the Bezout number for complex roots.

Expected distribution of zeros on a general Kähler manifold

Less obvious: same result is true asymptotically on any Kähler manifold.

Theorem 1 (Shiffman-Z) We have:

$$\frac{1}{(N)^m}\mathbf{E}_N(Z_{f_1,\dots,f_m}) \to \omega^m$$

in the sense of weak convergence; i.e., for any open $U \subset M$, we have

$$\tfrac{1}{(N)^m}\mathbf{E}_N\Big(\#\{z \in U : f_1(z) = \cdots = f_m(z) = 0\}\Big)$$

$$\to m!\mathrm{Vol}_\omega(U) \ .$$

Zeros concentrate in curved regions. Curvature causes sections to oscillate and hence zeros to occur.

Correlations between zeros

Expected distribution of zeros is uniform, but zeros are not thrown down independently; they are "correlated":

Definition: The 2 *point correlation function of k sections of degree N in m variables* is:

$$K_{2m}^N(z^1, z^2) = E(|Z_{(s_1,\ldots,s_m)}|^2),$$

= the probability density of finding a pair of simultaneous zeros at z^1, z^2
= conditional probability of finding a second zero at z^2 if there is a zero at z^1. Here,

$$|Z_{(s_1,\ldots,s_m)}|^2 = |Z_{(s_1,\ldots,s_m)}| \times |Z_{(s_1,\ldots,s_m)}|$$

is product measure on

$$M_2 = \{(z^1, z^2) \in M^2 : z^1 \neq z^2\}.$$

Scaling limit of correlation functions

As the degree N increases, the density of zeros increases. If we scale by a factor $\sqrt{N}$, the expected density of zeros stays constant. We now scale to keep the density constant.

Fix $z_0 \in M$ and consider the pattern of zeros in a small ball $B(z_0, \frac{1}{\sqrt{N}})$. We fix local coordinates z for which $z^0 = 0$ and rescale the correlation function by $\sqrt{N}$. In the limit we obtain the 2-*point scaling limit zero correlation function*

$$K^{\infty}_{2km}(z^1, z^2) \tag{1}$$
$$= \lim_{N\to\infty} \left(c_m N^k\right)^{-2} K^N_{2m}\left(\frac{z^1}{\sqrt{N}}, \frac{z^2}{\sqrt{N}}\right).$$

Universality of the scaling limit

Theorem **1** *(Bleher-Shiffman-Z) The scaling limit pair correlation functions $K^{\infty}_{2m}(z^1, z^2)$ are universal, i.e. independent of M, L, ω.*

The universal limit correlation function is the two-point correlation function for the Gaussian 'Heisenberg ensemble', namely $H^2(\mathbb{C}^m, e^{-|z|^2})$. The trivial bundle $\mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^m$ with curvature $dz \wedge d\bar{z}$ is local model for all positive line bundles.

Fast decay of correlations

Universal scaling limit pair correlation function = function

$$K^{\infty}_{2km}(z^1, z^2) = \kappa_{km}(|z_1 - z_2|)$$

of distance between points. Very short range even on length scale $\frac{r}{\sqrt{N}}$.

Theorem **2** *(BSZ)* $\kappa_{km}(r) = 1 + O(r^4 e^{-r^2}), \ r \to +\infty$.

$\kappa \equiv 1$ for independent random points ('ideal gas').

Small distance behavior

(In the real case, replace r by $\sqrt{r}$)

Theorem **3** *(Bleher-Shiffman-Z, 2001):*

$$\kappa_{mm}(r) = \begin{cases} \frac{m+1}{4} r^{4-2m} + O(r^{8-2m})\,, & \textit{as} \quad r \to 0, \\ \\ 1 + O(r^4 e^{-r^2}), & r \to +\infty. \end{cases}$$

- When $m = 1, \kappa_{mm}(r) \to 0$ as $r \to 0$ and one has "zero repulsion."
- When $m = 2$, $\kappa_{mm}(r) \to 3/4$ as $r \to 0$ and one has a kind of neutrality.
- With $m \geq 3$, $\kappa_{mm}(r) \nearrow \infty$ as $r \to 0$ and zeros attract (or 'clump together'): One is more likely to find a zero at a small distance r from another zero than at a small distance r from a given point.

Discriminant variety

One can understand this dimensional dependence heuristically in terms of the geometry of the discriminant varieties $\mathcal{D}_N^m \subset H^0(M, L^N)^m$ of systems $S = (s_1, \dots, s_m)$ of m sections with a 'double zero'. The 'separation number' $sep(F)$ of a system is the minimal distance between a pair of its zeros. Since the nearest element of $\mathcal{D}_N^m$ to F is likely to have a simple double zero, one expects: $sep(F) \sim \sqrt{dist(F, \mathcal{D}_N^m)}$. Now,the degree of $\mathcal{D}_N^m$ is approximately N^m. Hence, the tube $(\mathcal{D}_N^m)_\epsilon$ of radius ϵ contains a volume $\sim \epsilon^2 N^m$. When $\epsilon \sim N^{-m/2}$, the tube should cover $PH^0(M, L^N)$. Hence, any section should have a pair of zeros whose separation is $\sim N^{-m/4}$ apart. It is clear that this separation is larger than, equal to or less than $N^{-1/2}$ accordingly as $m = 1, m = 2, m \geq 3$.

Polynomials with fixed Newton polytope

We now ask: how is the distribution of zeros affected by the Newton polytope of a polynomial? How about the mass density? Critical points?

The Newton polytope P_f of a polynomial

$$f(z) = \sum_{|\alpha| \leq N} c_\alpha \, z^\alpha \text{ on } \mathbb{C}^m$$

is the convex hull of its support $S_f = \{\alpha \in \mathbb{Z}^m : c_\alpha \neq 0\}$.

Similarly for $\mathbb{R}^m$.

Counting zeros of complex polynomials: Bezout and Bernstein-Kouchnirenko theorems

- **Bezout's theorem**: m generic homogeneous polynomials $F_1, \ldots, F_m$ of degree p have exactly p^m simultaneous zeros; these zeros all lie in $\mathbb{C}^{*m}$, for generic F_j.

- **Bernstein-Kouchnirenko Theorem** The number of joint zeros in $\mathbb{C}^{*m}$ of m generic polynomials $\{f_1, \ldots, f_m\}$ with given Newton polytope P equals $m!\text{Vol}(P)$.

- More generally, the f_j may have different Newton polytopes P_j; then, the number of zeros equals the 'mixed volume' of the P_j.

Consistency: If $P = p\Sigma$, where Σ is the standard unit simplex in $\mathbb{R}^m$, then $\text{Vol}(p\Sigma) = p^m\text{Vol}(\Sigma) = \frac{p^m}{m!}$, and we get Bézout's theorem.

Themes

The Newton polytopes of a polynomial system $f_1, \ldots, f_m$ also have a crucial influence on the *spatial distribution* of *zeros* $\{f_1 = \cdots = f_m = 0\}$ and critical points $\{df = 0\}$.

- (i) There is a *classically allowed region*

  $$\mathcal{A}_P = \mu_{\Sigma}^{-1}(\frac{1}{p}P)$$

  region where the zeros or critical points concentrate with high probability and its complement, the *classically forbidden region* where they are usually sparse.

  Here,

  $$\mu_{\Sigma}(z) = \left(\frac{|z_1|^2}{1 + \|z\|^2}, \ldots, \frac{|z_m|^2}{1 + \|z\|^2}\right)$$

  is the moment map of $\mathbb{CP}^m$.

Asymptotic and Statistical

These results are statisctical and asymptotic:

- Not *all* polynomials $f \in \mathcal{P}_P^m$ have this behaviour; but typical ones. We will endow $\mathcal{P}_P^m$ with a Gaussian probability measure, and show that the above patterns form the expected behaviour of random polynomials.

- The variance is small compared to the expected value: i.e. the statistics are 'self-averaging' in the limit $N \to \infty$. Here, as $N \to \infty$, we dilate $P \to NP$.

Random polynomials with $P_f \subset P$

Definition of the ensemble: Let $Poly(P)$ denote the space of polynomials with $P_f \subset P$.

Endow $Poly(P)$ with the *conditional probability measure* $\gamma_p|_P$:

(2)

$$d\gamma_{p|P}(s) = \frac{1}{\pi^{\#P}} e^{-|\lambda|^2} d\lambda, \quad s = \sum_{\alpha \in P} \lambda_\alpha \frac{z^\alpha}{\|z^\alpha\|},$$

where the coefficients $\lambda_\alpha =$ independent complex Gaussian random variables with mean zero and variance one. Denote conditional expectation by $\mathbf{E}_{|P}$.

Asymptotics of expected distribution of zeros

Let $\mathbf{E}_{|P}(Z_{f_1,\dots,f_m})$ = expected distribution of simultaneous zeros of $(f_1,\dots,f_m)$, chosen independently from $Poly(P)$. We will determine the asymptotics of the expected density as the polytope is dilated $P \to NP, N \in \mathbb{N}$.

Theorem **4** *(Shiffman-Z) Suppose that P is a simple polytope in $\mathbb{R}^m$. Then, as P is dilated to NP,*

$$\frac{1}{(Np)^m}\mathbf{E}_{|NP}(Z_{f_1,\dots,f_m}) \to \begin{cases} \omega_{\mathrm{FS}}^m & \text{on } \mathcal{A}_P \\ 0 & \text{on } \mathbb{C}^{*m}\backslash\mathcal{A}_P \end{cases}.$$

Thus, the simultaneous zeros of m polynomials with Newton polytope P concentrate in the allowed region and are uniform there, giving a quantitative BK result.

Asymptotics of expected distribution of critical points (from jt. work w/ M. Douglas

Take one random polynomial $f \in poly(NP)$ and consider its distribution of critical points:

$$\frac{\partial f(z_0)}{\partial z_1} = \cdots = \frac{\partial f(z_0)}{\partial z_1} = 0.$$

The polynomials $\frac{\partial f(z_0)}{\partial z_j}$ are of course far from independent! Let $C_f = \sum_{z_j : \nabla f(z_j)=0} \delta(z_j)$.

Let $\mathbf{E}_{|P}(C_f) =$ expected distribution of critical points of $f \in poly(NP)$.

Theorem **5** *(B. Shiffman-Z) Suppose that P is a simple polytope in $\mathbb{R}^m$. Then, as P is dilated to NP,*

$$\frac{1}{(Np)^m}\mathbf{E}_{|NP}(C_f) \to \begin{cases} \omega_{\mathrm{FS}}^m & \text{on } \mathcal{A}_P \\ 0 & \text{on } \mathbb{C}^{*m} \backslash \mathcal{A}_P \end{cases}.$$

Mass asymptotics

A key ingredient is the mass asymtotics of random sections:

Theorem **6**

$$\mathbf{E}_{\nu_{NP}}\left(|f(z)|^2_{\mathrm{FS}}\right) \sim \begin{cases} \frac{\omega^m}{\mathrm{Vol}(P)} + O(N^{-1}), \\ \textit{for} \quad z \in \mathcal{A}_P = \mu^{-1}(\frac{1}{p}P^\circ) \\ \\ N^{-s/2} e^{-Nb(z)} \left[c_0^F(z) + O(N^{-1})\right], \\ \textit{for } z \in (\mathbb{C}^*)^m \backslash \mathcal{A}_P \end{cases}$$

where c_0^F *and* $b|_{\mathcal{R}_F^\circ}$ *are positive.*

b is a kind of Agmon distance, giving decay of ground states away from the classically allowed region.

Results on random real polynomial systems with fixed Newton polytope

The analogous result for the expected number of real roots and the density of real roots forn the conditional $O(m+1)$ ensemble, where we constrain all polynomials to have Newton polytope P:

Theorem **7** *(Shiffman-Zelditch, May 1, 2003)*

$$\mathbf{E}_{NP}(Z_{f_1,\dots,f_m})(x) = \begin{cases} a_m N^{m/2}, & x \in \mathcal{A}_P \\ \\ O(N^{(m-1)/2}), & x \in \mathbb{R}P^m \backslash \mathcal{A}_P. \end{cases},$$

where $a_m = Vol_{\mathbb{R}P^m}(\mathcal{A}_P)$. The coefficient a_m is NOT the square root of the BKK number of complex roots.

Ideas and Methods of Proofs

- For each ensemble, we define the two-point function

$$\Pi_N(z,w) = \mathbf{E}_N(f(z)\overline{f(w)}).$$

  It is the Bergman-Szegö (reproducing) kernel for the inner product space of polynomials or sections of degree N.

- All densities and correlation functions for zeros may be expressed in terms of the joint probability density (JPD) of the random variables $X(f) = f(z_0), \Xi(f) = df(z_0)$. For critical points, we also need $Hf(z_0) = Hessian f(z_0)$.

- For Gaussian ensembles, the JPD is a Gaussian with covariance matrix depending only on $\Pi_N(z,w)$ and its derivatives.

Bergman-Szegö kernels

More precisely:

- Expected distribution of zeros: $\mathbf{E}_N(Z_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \log \Pi_N(z,z) + \omega$.

- Joint probability distribution (JPD) $D_N(x^1,\dots,x^n;\xi^1,\dots,\xi^n;z^1,\dots,z^n)$ of random variables $x^j(s) = s(z^j)$, $\xi^j(s) = \nabla s(z^j)$, = function of Π_N and derivatives.

- Correlation functions in terms of JPD

$$K^N(z^1,\dots,z^n) = \int D_N(0,\xi,z) \prod_{j=1}^n \Big(\|\xi^j\|^2 d\xi^j\Big) d\xi.$$

Scaling asymptotics

Scaling asymptotics of correlation functions reduces to scaling asymptotics of Π_N. Here is the result for $H^0(M, L^N)$:

Theorem **8** *(BShZ) In 'normal coordinates' $\{z_j\}$ at $P_0 \in M$ and in a 'preferred' local frame for L:*

$$\frac{\pi^m}{N^m}\Pi_N(P_0 + \frac{u}{\sqrt{N}}, \frac{\theta}{N}; P_0 + \frac{v}{\sqrt{N}}, \frac{\varphi}{N})$$

$$\sim e^{i(\theta-\varphi)+u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)}\left[1 + b_1(u,v)N^{-\frac{1}{2}} + \cdots\right].$$

Note: $e^{i(\theta-\varphi)+u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)}$ = Bergman-Szegö kernel of Heisenberg group.

Proof based on Boutet de Monvel -Sjostrand parametrix for the Π_N.

Fixed Newton polytope

This requires (exponentially decaying) asymptotics of the conditional Bergman- Szegö kernel

$$\Pi_{|NP}(z,w) = \sum_{\alpha \in NP} \frac{z^\alpha \overline{w}^\alpha}{||z^\alpha||_{FS}||w^\alpha||_{FS}}.$$

This projection sifts out terms with $\alpha \in P$ from the simple Szegö projector of $\mathbb{CP}^m$.

We need asymptotics of $\Pi_{|NP}(z,w)$. For this we use the Khovanskii-Pukhlikov (Brion-Vergne, Guillemin) Euler MacLaurin sum formula.

Final remarks and open problems

- What happens in other Gaussian ensembles? Or more general ensembles?

- Distribution of zeros of random fewnomial systems (real or complex)? Expected number of zeros of random real fewnomial systems and comparison to Khovanskii's bound.

- Geometric quantities of random real polynomials: average Betti numbers, number of components, etc.