From Random Polynomials to String Theory

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Our topics

- Random polynomials – How are zeros or critical points distributed?

- Random complex geometry: generalize polynomials to holomorphic sections of line bundles.

- Counting universes in string/M theory–

  'Universes' = 'vacua' of string/M theory = critical points of 'superpotentials' on the moduli space of Calabi-Yau manifolds. How many vacua are there? How are they distributed?

Outline of talk

1. Random polynomials of one complex variable: classical and recent results. Generalization to holomorphic sections of line bundles.

2. String/M vacuum selection problem: Douglas' statistics of vacua program. Rigorous results on counting possible universes = string/M vacua.

3. Geometric problems on critical points of holomorphic sections relative to a hermitian metric or connection.

Random polynomials of one variable

A polynomial of degree N in one complex variable is:

$$f(z) = \sum_{j=1}^{N} c_j z^j, \quad c_j \in \mathbb{C}$$

is specified by its coefficients $\{c_j\}$.

A 'random' polynomial is short for a probability measure P on the coefficients. Let

$$\begin{aligned} \mathcal{P}_N^{(1)} &= \{\textstyle\sum_{j=1}^N c_j z^j, \ (c_1, \ldots, c_N) \in \mathbb{C}^N\} \\ &\simeq \mathbb{C}^N. \end{aligned}$$

Endow $\mathbb{C}^N$ with probability measure dP.

We call $(\mathcal{P}_N^{(1)}, P)$ an 'ensemble' of random polynomials.

Kac polynomials

The simplest complex random polynomial is the 'Kac polynomial'

$$f(z) = \sum_{j=1}^{N} c_j z^j$$

where the coefficients c_j are independent complex Gaussian random variables of mean zero and variance one. Complex Gaussian:

$$\mathbf{E}\,(c_j) = 0 = E(c_j c_k), \quad E(c_j \bar{c}_k) = \delta_{jk}.$$

This defines a Gaussian measure γ_{KAC} on $\mathcal{P}_N^{(1)}$:

$$d\gamma_{KAC}(f) = e^{-|c|^2/2} dc.$$

Expected distribution of zeros

The distribution of zeros of a polynomial of degree N is the probability measure on $\mathbb{C}$ defined by

$$Z_f = \frac{1}{N} \sum_{z:f(z)=0} \delta_z,$$

where δ_z is the Dirac delta-function at z.

Definition: The expected distribution of zeros of random polynomials of degree N with measure P is the probability measure $\mathbf{E}_P Z_f$ on $\mathbb{C}$ defined by

$$\langle \mathbf{E}_P Z_f, \varphi \rangle = \int_{\mathcal{P}_N^{(1)}} \{\frac{1}{N} \sum_{z:f(z)=0} \varphi(z)\} dP(f),$$

for $\varphi \in C_c(\mathbb{C})$.

How are zeros of complex Kac polynomials distributed?

Complex zeros concentrate in small annuli around the unit circle S^1. In the limit as the degree $N \to \infty$, the zeros asymptotically concentrate exactly on S^1:

Theorem **1 (Kac-Hammersley-Shepp-Vanderbei)** *The expected distribution of zeros of polynomials of degree N in the Kac ensemble has the asymptotics:*

$$\mathbf{E}^N_{KAC}(Z^N_f) \to \delta_{S^1} \quad \textit{as} \quad N \to \infty\ ,$$

where $\quad (\delta_{S^1}, \varphi) := \frac{1}{2\pi} \int_{S^1} \varphi(e^{i\theta})\, d\theta.$

Why the unit circle?

Do zeros of polynomials *really* tend to concentrate on S^1?

Answer: yes, for the polynomials which dominate the Kac measure $d\gamma^N_{KAC}$. (Obviously no for general polynomials)

The Kac-Hammersley-Shepp-Vanderbei measure γ^N_{KCA} weights polynomials with zeros near S^1 more than other polynomials.

It did this by an implicit choice of inner product on $\mathcal{P}_N^{(1)}$.

Gaussian measure and inner product

Choice of Gaussian measure on a vector space $\mathcal{H}$ = choice of inner product on $\mathcal{H}$.

The inner product induces an orthonormal basis $\{S_j\}$. The associated Gaussian measure dγ corresponds to random orthogonal sums

$$S = \sum_{j=1}^{d} c_j S_j,$$

where $\{c_j\}$ are independent complex normal random variables.

The inner product underlying the Kac measure on $\mathcal{P}_N^{(1)}$ makes the basis $\{z^j\}$ orthonormal. Namely, they were orthonormalized on S^1. And that is where the zeros concentrated.

Gaussian random polynomials adapted to domains

If we orthonormalize polynomials on the boundary $\partial\Omega$ of any simply connected, bounded domain $\Omega \subset \mathbb{C}$, the zeros of the associated random polynomials concentrate on $\partial\Omega$.

I.e. define the inner product on $\mathcal{P}_N^{(1)}$ by

$$\langle f, \bar{g} \rangle_{\partial\Omega} := \int_{\partial\Omega} f(z)\overline{g(z)}\,|dz| \ .$$

Let $\gamma_{\partial\Omega}^N$ = the Gaussian measure induced by $\langle f, \bar{g} \rangle_{\partial\Omega}$ and say that the Gaussian measure is adapted to Ω.

How do zeros of random polynomials adapted to Ω concentrate?

Equilibrium distribution of zeros

Denote the expectation relative to the ensemble $(\mathcal{P}_N, \gamma^N_{\partial\Omega})$ by $\mathbf{E}^N_{\partial\Omega}$.

Theorem **2**

$$\mathbf{E}^N_{\partial\Omega}(Z^N_f) = \nu_\Omega + O\left(1/N\right) \ ,$$

where ν_Ω *is the equilibrium measure of* $\bar{\Omega}$*.*

The equilibrium measure of a compact set K is the unique probability measure $d\nu_K$ which minimizes the energy

$$E(\mu) = -\int_K \int_K \log|z-w|\, d\mu(z)\, d\mu(w).$$

Thus, in the limit as the degree $N \to \infty$, random polynomials adapted to Ω act like electric charges in Ω.

$SU(2)$ **polynomials**

Is there an inner product in which the expected distribution of zeros is 'uniform' on $\mathbb{C}$, i.e. doesn't concentrate anywhere? Yes, if we take 'uniform' to mean uniform on $\mathbb{CP}^1$ w.r.t. Fubini-Study area form ω_{FS}.

We define an inner product on $\mathcal{P}_N^{(1)}$ which depends on N:

$$\langle z^j, z^k \rangle_N = \frac{1}{\binom{N}{j}} \delta_{jk}.$$

Thus, a random $SU(2)$ polynomial has the form

$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \sqrt{\binom{N}{\alpha}}\, z^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \bar{\lambda}_\beta) = \delta_{\alpha\beta}.$$

Proposition **3** *In the* $SU(2)$ *ensemble,* $\mathbf{E}(Z_f) = \omega_{FS}$*, the Fubini-Study area form on* $\mathbb{CP}^1$*.*

$SU(2)$ **and holomorphic line bundles**

Proof that $\mathbf{E}(Z_f) = \omega_{FS}$ is trivial if we make right identifications:

- $\mathcal{P}_N^{(1)} \simeq H^0(\mathbb{CP}^1, \mathcal{O}(N))$ where $\mathcal{O}(N) = N$th power of the hyperplane section bundle $\mathcal{O}(1) \to \mathbb{CP}^1$.Indeed, $\mathcal{P}_N^{(1)} \iff$ homogeneous polynomials $F(z_0, z_1)$ of degree N: homogenize $f(z) \in \mathcal{P}_N^{(1)}$ to $F(z_0, z_1) = z_0^N f(z_1/z_0)$. Also $H^0(\mathbb{CP}^1, \mathcal{O}(N)) \iff$ homogeneous polynomials $F(z_0, z_1)$ of degree N.

- Fubini-Study inner product on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ $=$ inner product $\int_{S^3} |F(z_0, z_1)|^2 dV$ on the homogeneous polynomials.

- The inner product and Gaussian ensemble are thus $SU(2)$ invariant. Hence, $\mathbf{E}\, Z_f$ is $SU(2)$-invariant.

Gaussian random holomorphic sections of line bundles

The $SU(2)$ ensemble generalizes to all dimensions, and moreover to any positive holomorphic line bundle $L \to M$ over any Kähler manifold.

We endow L with a Hermitian metric h and M with a volume form dV. We define an inner product

$$\langle s_1, s_2 \rangle = \int_M h(s_1(z), s_2(z)) dV(z).$$

We let $\{S_j\}$ denote an orthonormla basis of the space $H^0(M, L)$ of holomorphic sections of L.

Then define Gaussian holomorphic sections $s \in H^0(M, L)$ by

$$s = \sum_j c_j S_j, \quad \langle S_j, S_k \rangle = \delta_{jk}$$

with $\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k)$, $\mathbf{E}(c_j \overline{c_k}) = \delta_{jk}$.

Statistics of critical points

Algebraic geometers are interested in zeros of holomorphic sections. But from now on we focus on critical points

$$\nabla s(z) = 0,$$

where ∇ is a metric connection.

Critical points of Gaussian random functions come up in may areas of physics–

- as peak points of signals (S.O. Rice, 1945);
- as vacua in compactifications of string/M theory on Calabi-Yau manifolds with flux (Giddings-Kachru-Polchinski, Gukov-Vafa-Witten);
- as extremal black holes (Strominger, Ferrara-Gibbons-Kallosh) , peak points of galaxy distributions (Szalay et al, Zeldovich), etc.

Critical points

Definition: *Let* $(L, h) \to M$ *be a Hermitian holomorphic line bundle over a complex manifold* M*, and let* $\nabla = \nabla_h$ *be its Chern connection.*

A critical point of a holomorphic section $s \in H^0(M, L)$ *is defined to be a point* $z \in M$ *where* $\nabla s(z) = 0$*, or equivalently,* $\nabla' s(z) = 0$*.*

In a local frame e critical point equation for $s = fe$ reads:

$$\partial f(w) + f(w) \partial K(w) = 0,$$

where $K = -\log ||e(z)||_h$.

The critical point equation is only C^∞ and not holomorphic since K is not holomorphic.

Statistics of critical points

The distribution of critical points of $s \in H^0(M, L)$ with respect to h (or ∇_h) is the measure on M

$$C_s^h := \sum_{z:\, \nabla_h s(z)=0} \delta_z. \tag{1}$$

Further introduce a measure γ on $H^0(M, L)$.

Definition: The (expected) distribution $\mathbf{E}_\gamma C_s^h$ of critical points of $s \in H^0(M, L)$ w.r.t. ∇_h and γ is the measure on M defined by

$$\langle \mathbf{E}_\gamma C_s^h, \varphi \rangle := \int_{H^0(M,L)} \left[\sum_{z:\nabla_h s(z)=0} \varphi(z) \right] d\gamma(s).$$

The expected number of critical points is defined by

$$\mathcal{N}^{crit}(h, \gamma) = \int_{\mathcal{S}} \#Crit(s, h) d\gamma(s).$$

Problems of interest

1. Calculate $\mathbf{E}_\gamma C^h_s$. How are critical points distributed? (Deeper: how are they correlated?)

2. How large is $\mathcal{N}^{\text{crit}}(h, \gamma)$? How does the expected number of critical points depend on the metric?

3. The 'best' metrics are the ones which minimize this quantity. Which are they?

The vacuum selection problem in string/M theory

These problems have applications to string/M theory.

According to string/M theory, our universe is 10- (or 11-) dimensional. In the simplest model, it has the form $M^{3,1} \times X$ where X is a complex 3-dimensional *Calabi-Yau* manifold.

The **vacuum selection problem**: Which X forms the 'small' or 'extra' dimensions of our universe? How to select the right vacuum?

Complex geometry and effective supergravity

The low energy approximation to string/M theory is effective supergravity theory. It consists of $(\mathcal{M}, \mathcal{L}, W)$ where:

1. $\mathcal{M} = \mathcal{M}_{\mathbb{C}} \times \mathcal{H}$, where $\mathcal{M}_{\mathbb{C}}$ = moduli space of Calabi-Yau metrics on a complex 3-D manifold X, $\mathcal{H}$ = upper half plane;

2. $\mathcal{L} \to \mathcal{M}$ is a holomorphic line bundle with with first Chern class $c_1(\mathcal{L}) = -\omega_{WP}$ (Weil-Petersson Kähler form).

3. the "superpotential" W is a holomorphic section of $\mathcal{L}$.

Hodge bundle and line bundle $\mathcal{L}$

Given a complex structure z on X, let $H^{3,0}(X_z)$ be the space of holomorphic $(3,0)$ forms on X, i.e. type $dw_1 \wedge dw_2 \wedge dw_3$.

On a Calabi-Yau 3-fold, $\dim H^{3,0}(X_z) = 1$. Hence, $H^{3,0}(X_z) \to \mathcal{M}$ is a (holomorphic) line bundle, known as the Hodge bundle. We write a local frame as Ω_z.

$\mathcal{L}$ is the dual line bundle to the Hodge bundle.

(Similarly for $\mathcal{H}$ factor).

Lattice of integral flux superpotentials

Physically relevant sections correspond to integral co-cycles ('fluxes')

$$G = F + iH \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}).$$

Such a G defines a section W_G of $\mathcal{L} \to \mathcal{M}$ by:

$$\langle W_G(z,\tau), \Omega_z \rangle = \int_X [F + \tau H] \wedge \Omega_z.$$

Thus, $G \to W_G$ maps

$$H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}) \to H^0(\mathcal{M}, \mathcal{L}).$$

Let $\mathcal{F}_{\mathbb{Z}} = \{W_G : G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}\}$. Also let $\mathcal{F} = \mathcal{F}_{\mathbb{Z}} \otimes \mathbb{C}$ (flux space).

Possible universes as critical points

A possible universe or vacuum is a Calabi-Yau 3-fold X_z with complex structure z and $\tau \in \mathcal{H}$ s.th.

$$\nabla W_G(z,\tau) = 0,$$

$$\iff \nabla_{\tau,z} \int_X [F + \tau H] \wedge \Omega_z = 0 \quad (z,\tau) \in \mathcal{M}$$

for some $G = F + iH \in H^0(X, \mathbb{Z} \oplus i\mathbb{Z})$: Moreover, the Hessian must be positive definite.

Here, $\nabla = \nabla_{WP}$ is the Weil-Petersson covariant derivative on $H^0(\mathcal{M}, \mathcal{L})$ arising from ω_{WP}.

More on critical points

1. Solutions of $\nabla(G + \tau H) = 0$ are actually supersymmetric vacua. General vacua are critical points of the potential energy $V(\tau) = ||\nabla W(\tau)||^2 - 3||W(\tau)||^2$.

2. The critical point equation for is equivalent to: find (z, τ) s.th.

$$G^{0,3} = G^{2,1} = 0$$

in the Hodge decomposition

$$H^3(X, \mathbb{C}) = H_z^{3,0} \oplus H_z^{2,1} \oplus H_z^{1,2} \oplus H_z^{0,3}.$$

Tadpole constraint

There is one more constraint on the flux superpotentials, called the tadpole constraint. It has the form:

$$\int_X F \wedge H \leq L \iff Q[F + iH] =\leq L \tag{2}$$

where Q is the indefinite quadratic form on $H^3(X, \mathbb{C})$ defined by

$$Q(\varphi_1, \varphi_2) = \int_X \varphi_1 \wedge \overline{\varphi}_2.$$

Since Q is indefinite, the relevant superpotentials are lattice points in the hyperbolic shell (2).

M. R. Douglas' statistical program (studied with Ashok, Denef, Shiffman, Z and others)

1. Count the number of critical points (better: local minima) of all integral flux superpotentials W_G with $Q[G] \leq L$.

2. Find out how they are distributed in $\mathcal{M}$.

3. How many are consistent with the standard model and the known cosmological constant?

Mathematical problem

Given $L > 0$, consider lattice points

$$G = F + iH \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$$

in the hyperbolic shell

$$0 \leq Q[G] \leq L.$$

Let $\mathcal{K} \subset \mathcal{M}$ be a compact subset of moduli space. Count number of critical points in $\mathcal{K}$ for G in shell:

$$\mathcal{N}^{\text{crit}}_{\mathcal{K}}(L) = \sum_{Q[G] \leq L} \#\{(z, \tau) \in \mathcal{K} : \nabla W_G(z, \tau) = 0\}.$$

Problem Determine $\mathcal{N}^{\text{crit}}_{\mathcal{K}}(L)$ where L is the tadpole number of the model. Easier: determine its asymptotics as $L \to \infty$.

Distribution of moduli of universes?

More generally, define

$$N_\psi(L) = \sum_{G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}) : H[G] \leq L} \langle C_G, \psi \rangle,$$

where

$$\langle C_G, \psi \rangle = \sum_{(z,\tau) : \nabla G(z,\tau) = 0} \psi(G, z, \tau).$$

Here, $\psi(G, z, \tau)$ is a smooth function with compact support in $(z, \tau) \in \mathcal{M}$ and polynomial growth in G. D

Example: Cosmological constant $\psi(G, z, \tau) = \{|\nabla W_G(\tau, z)|^2 - 3|W_G(\tau, z)|^2\}\chi(z, \tau)$, with $\chi \in C_0^\infty(\mathcal{M})$.

Problem Find $N_\psi(L)$ as $L \to \infty$.

Radial projection of lattice points

Since $\langle C_G, \psi \rangle$ is homogeneous of degree 0 in G, one can radially project G to the hyperboloid $Q[G] = 1$.

Similar to model problem: take an indefinite quadratic form Q and consider lattice points inside a hyperbolic shell $0 \leq Q \leq L$ which lie inside a proper subcone of the lightcone $Q = 0$. Project onto the hyperboloid $Q = 1$ and measure equidistribution.

[Easier: do it for an ellipsoid].

Critical points of each G give an additional feature.

Discriminant variety/mass matrix

A nasty complication is the *real discriminant hypersurface* $\mathcal{D}$ of $W \in \mathcal{F}$ which have degenerate critical points, i.e the Hessian $D\nabla W(\tau)$ is degenerate. The number of critical points and $\langle C_W, \psi \rangle$ jump across $\mathcal{D}$. So we are not summing a smooth function over lattice points.

But: the Hessian of a superpotential at a critical point is the 'mass matrix' and no massless fermions are observed. So it is reasonable to count vacua away from $\mathcal{D}$.

Rigorous result on lattice sums

Here is a sample result on distribution of vacua:

Theorem **4** *Suppose* Supp $\psi \cap \mathcal{D} = \emptyset$. *Then*

$$\mathcal{N}_\psi(L) = L^{b_3}\left[\int_{\{Q[W]\leq 1\}} \langle C_W, \psi\rangle\, dW + O\left(L^{-\frac{2b_3}{2b_3+1}}\right)\right].$$

Here, $b_3 = \dim H_3(X, \mathbb{C})$, *integral is hyperbolic shell in* $\mathcal{F}$.

Further results:

1. Let $\psi = \chi_{\mathcal{K}}$. Same principal term, but $O(L^{b_3-1})$ remainder.

2. Similarly if we drop assumption Supp $\psi \cap \mathcal{D} = \emptyset$.

Gaussian principal term

The principal coefficient $\int_{\{Q[W]\leq 1\}} \langle C_W, \psi \rangle \, dW$ can be rewritten as:

$$\int_{\mathcal{M}} \int_{\{Q_{z,\tau}[W]\leq 1\}} |\det D\nabla W(z,\tau)| \psi(W,z,\tau) dW \, dV(z,\tau)$$

= density of critical points of Gaussian random superpotentials in the space

$$\mathcal{F}_{z,\tau} = \{W : \nabla W(z,\tau) = 0\}$$

with inner product $Q_{z,\tau} = Q|_{\mathcal{F}_{z,\tau}}$.

It is Gaussian because $Q_{z,\tau} >> 0$ by *special geometry* of $\mathcal{M}$. $dV(z,\tau)$ is a certain volume form on $\mathcal{M}$.

[More precisely, the ensemble is dual to it under the Laplace transform].

Simplification using special geometry

There is a matrix Λ_Z and a space of complex symmetric matrices $\mathcal{H}_Z$ so that

$$\mathcal{K}^{\text{crit}}(Z) = \frac{1}{\sqrt{\det \Lambda_Z}} \int_{\mathcal{H}_Z \oplus \mathbb{C}} \left| \det H^* H - |x|^2 I \right|$$

$$\chi_{\Lambda_Z}(H, x) dH dx,$$

where χ_{Λ_Z} is the characteristic function of the ellipsoid $\{(\Lambda_Z H, H)_{\mathbb{R}} + |x|^2 \leq 1\}$.

Λ_Z may be expressed in terms of the (Strominger) prepotential $\mathcal{F}$.

Upshot: We can estimate the integral for Z in a ball in moduli space using only curvature invariants.

Heuristic estimate

Using "concentration of measure" phenomena for Gromov-Levy families of spaces, and conjectured volume estimates for balls in moduli space, we guess that

$$N_{vac,K_\mu}(L) \sim \frac{(C_1 L)^{b_3}}{b_3!}.$$

The tadpole number usually is in the range $L \sim C b_3$ with $C \in [\frac{1}{3}, 3]$.

So the number of vacua in a ball K_μ satisfying the tadpole constraint would grow at a rate $e^{C b_3}$.

(This might sound reasonable, but in high dimensions, volumes and integrals tend to grow/decay at factorial rates. The exponential estimate is based on a cancelation betweeen factorial growth and decay.)

Gaussian and lattice ensembles

To summarize: we can approximate the discrete ensemble of integral flux superpotentials by a Gaussian random ensemble for large L.

This shows how fundamental Gaussian ensembles are. To understand

$$\int_{\mathcal{M}} \int_{\{Q_{z,\tau}[W] \leq 1\}} |\det D\nabla W(z,\tau)| \psi(W,z,\tau) dW dV_Q(z,\tau$$

we now turn to model Gaussian geometric problems. Even on $\mathbb{CP}^m$ the distribution of critical points is non-obvious.

Geometric study of critical points

Model problem: Given a hermitian holomorphic line bundle $(L, h) \to M$, define the *Hermitian Gaussian measure* γ_h to be the Gaussian measure induced by the inner product $\langle , \rangle_h$, i.e.

$$\langle s_1, s_2 \rangle = \int_M h(s_1(z), s_2(z)) dV(z).$$

We often take $dV = \frac{\omega^m}{m!}$.

Then the distribution $\mathcal{K}^{crit}(h, \gamma)(z)$ and the number $\mathcal{N}^{crit}(h, \gamma)$ of critical points w.r.t. ∇_h are purely metric invariants of (L, h).

How do they depend on h?

Exact formula for $\mathcal{N}^{crit}(h_{FS}, \gamma_{FS})$ on $\mathbb{CP}^1$

Theorem **5** *The expected number of critical points of a random section* $s_N \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$ *(with respect to the Gaussian measure* γ_{FS} *on* $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ *induced from the Fubini-Study metrics on* $\mathcal{O}(N)$ *and* $\mathbb{CP}^1$*) is*

$$\frac{5N^2 - 8N + 4}{3N - 2} = \frac{5}{3}N - \frac{14}{9} + \frac{8}{27}N^{-1} \cdots .$$

Of course, relative to the flat connection d/dz the number is $N-1$. Thus, the positive curvature of the Fubini-Study hermitian metric and connection causes sections to oscillate much more than the flat connection. There are $\frac{N}{3}$ new local maxima and $\frac{N}{3}$ new saddles.

Asymptotic expansion for the expected number of critical points

Theorem **6** *Let (L, h) be a positive hermitian line bundle. Let $\mathcal{N}^{\text{crit}}(h^N)$ denote the expected number of critical points of random $s \in H^0(M, L^N)$ with respect to the Hermitian Gaussian measure. Then,*

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!}\Gamma_m^{\text{crit}}\, c_1(L)^m\, N^m$$

$$+ \int_M \rho dV_\omega N^{m-1}$$

$$+ [C_m \int_M \rho^2 dV_\Omega + \textit{top}] N^{m-2} + O(N^{m-3})\,.$$

Here, ρ is the scalar curvature of ω_h, the curvature of h.

$\Gamma_m^{\text{crit}}\, c_1(L)^m$ is larger than for a flat connection.

To what degree is the expected number of critical points a topological invariant?

The first two terms are topological invariants of a positive line bundle, i.e. independent of the metric! (Both are Chern numbers of L). But the non-topological part of the third term

$$C_m \int_M \rho^2 dV_\Omega N^{m-2}$$

is a non-topological invariant, as long as $C_m \neq 0$. It is a multiple of the Calabi functional.

(These calculations are based on the Tian-Yau-Zelditch (and Catlin) expansion of the Szegö kernel and on Zhiqin Lu's calculation of the coefficients in that expansion.)

Calabi extremal metrics are asymptotic minimizers

As long as $C_m > 0$, we see from the expansion

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!}\Gamma_m^{\text{crit}}\, c_1(L)^m\, N^m + \int_M \rho dV_\omega N^{m-1}$$

$$+C_m \int_M \rho^2 dV_\Omega N^{m-2} + O(N^{m-3})\,.$$

that Calabi extremal metrics asymptotically minimize the metric invariant = average number of critical points. Indeed, they minimize the third term.

We have proved $C_m > 0$ in dimensions $m \leq 5$. We conjecture $C_m > 0$ in all dimensions. Ben Baugher has a new formula for C_m which makes this almost certain.

Summing up

1. Counting candidate universes in string theory amounts to counting critical points of integral superpotentials, which form a lattice in the hyperbolic shell $Q[N] \leq L$.

2. As $L \to \infty$, this ensemble is well-approximated by Lebesgue measure in the shell, which is dual (Laplace transform) to Gaussian measure.

3. As the degree $\deg \mathcal{L} = N \to \infty$, we understand the geometry of the distribution of critical points of Gaussian random sections or the dual Lebesgue one.