Thermodynamics of group representations

Steve Zelditch
Department of Mathematics
Johns Hopkins University

Joint Work with
Tatusya Tate
Department of Mathematics
Keio University
Nagoya University

Topic

This talk is about decomposing a high tensor power $V_\lambda^{\otimes N}$ of an irreducible representation V_λ a compact Lie group such as $SU(k)$ into irreducibles.

There are three parameters: λ, N, k. It is important to understand the decomposition as all three get large. In this talk, λ, k are fixed and $N \to \infty$. In the language of spin chains, this is a thermodynamic limit.

Notation

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. For simplicity, assume G semi-simple

- $T \subset G =$ maximal torus, $\mathfrak{t} =$ Lie algebra of T.

- $\langle \cdot, \cdot \rangle$ = invariant inner product on $\mathfrak{g}$ under adjoint action (= negative Killing form when (g) is semi-simple). We sometimes identify the spaces $\mathfrak{g}$ and $\mathfrak{t}$ with their duals $\mathfrak{g}^*$ and $\mathfrak{t}^*$ by $\langle , \rangle$.

- W = Weyl group acting on (t).

- $C \subset \mathfrak{t}^* =$ (open) dual Weyl chamber.

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Weights and roots

Further let:

- $I \subset \mathfrak{t}$ = integral lattice, *i.e.*, $I = \exp^{-1}(1)$;

- $I^* \subset \mathfrak{t}^*$ = dual lattice := the lattice of weights.

- Φ = roots, Φ_+ = positive roots.

- Let $B \subset \Phi_+$ be the set of the simple roots, so that $f \in C$ if and only if $\langle f, \alpha \rangle > 0$ for all $\alpha \in B$. Let X^* = linear span of the simple roots in $\mathfrak{t}^*$ Since G is semisimple, $X^* = \mathfrak{t}^*$.

Weights in an irrep

Cartan-Weyl: dominant weights $\lambda \in \overline{C} \cap I^* \iff$ irreducible unitary representation (π_λ, V_λ).

Let $m_1(\lambda;\mu)$ = multiplicity of the weight μ in V_λ, i.e. multiplicity of $\chi_\mu(t) = t^\mu$ in $\pi_\lambda|_T$.

Define: $M_\lambda := \{\mu \in I^* \,;\, m_1(\lambda;\mu) \neq 0\}$ = weights occurring in V_λ.

Define: $Q(\lambda)$ = convex hull of the W-orbit of λ.

Basic fact: $Q(\lambda)$ = convex hull of M_λ. I.e. all weights occurring in V_λ lie in Q_λ.

Multiplicities in a tensor proudct

Let V_λ, V_γ be two irreps of a compact Lie group. We are interested in two multiplicity problems:

- Multiplicity of an irrep V_μ in $V_\lambda \otimes V_\gamma$;

- Multiplicity of a weight ν in $V_\lambda \otimes V_\gamma$;

We are interested in large tensor products. First, review known formulae for two.

Multiplicity of irreps in a tensor proudct

Define the multiplicity $m^{\mu}_{\lambda,\gamma}$ by

$$V_\lambda \otimes V_\gamma = \bigoplus_{\mu \in \bar{C} \cap I^*} m^{\mu}_{\lambda,\gamma} V_\mu.$$

Steinberg's formula

$$m^{\mu}_{\lambda,\gamma} = \sum_{v,w \in W} \det(v \cdot w)$$

$$\cdot\, \mathfrak{p}(v(\lambda+\rho) + w(\gamma+\rho) - (\mu + 2\rho)).$$

where $\mathfrak{p} : I^* \to \mathbb{N}$ is Kostant's partition function: $\mathfrak{p}(\zeta)$ = number of decompositions of ζ into positive roots, i.e.

$$\zeta = \sum_{\alpha \in \Phi_+} n_\alpha \alpha, \quad n_\alpha \in \mathbb{N}.$$

Clebsch-Gordon vs. Steinberg

Simplest case is $SU(2)$. $\exists!1 > 0$ root α. I generated by α, I^* by $\alpha/2$. Dominant weights: $n\alpha/2, n \geq 0$.

Steinberg's formula (where $n \geq p$)

$m_{n\cdot\alpha/2,p\cdot\alpha/2}^{\mu\cdot\alpha/2} = \mathfrak{p}((n+p-\mu)\cdot\alpha/2) - \mathfrak{p}((n-p-\mu-2)\alpha/2))$.

Clebsch-Gordon:

$$V_n \otimes V_p = V_{n+p} \oplus V_{n+p-2} \oplus + \cdots + \oplus V_{n-p}.$$

$\mathfrak{p}(k \cdot \alpha/2) = 1$ if $k \in \mathbb{N}$ is even and $\mathfrak{p}(k \cdot \alpha/2) = 0$ otherwise.

Other multiplicity formulae

There are other multiplicity formulae, at least in special cases.

- A recursive formula of Freudenthal;
- Littlewood-Richardson rule: For irreps of $SU(N)$, $m^{\mu}_{\lambda,\gamma} = \#$ "tableaux of shape $\lambda - \gamma$ and weight μ such that the word $w(T)$ determined by T is a lattice permutation."

Fair to say: All formulae are very hard to apply for big groups, big representations or big tensor products.

Multiplicities in tensor powers

Our interest is in calculating multiplicities in high tensor powers $V_\lambda^{\otimes N}$ of irreps of $SU(k)$ or any compact Lie group. We define:

- $m_N(\lambda;\nu)$ = the multiplicity of a weight ν in the N-th tensor power $V_\lambda^{\otimes N}$;

- $a_N(\lambda;\nu)$= the multiplicity of an irreducible summand V_ν in $V_\lambda^{\otimes N}$ with the highest weight ν.

Analogy: multiplicity vs entropy in thermodynamics

View a unitary representation $\mathcal{H}$ as the Hilbert space of a quantum mechanical system. The multiplicity $m(\mu)$ of a weight μ (or irreducible) measures how many states of the system have this weight. Fixing the weight does not determine the state uniquely. The indeterminacy is measured by $\log m(\mu)$. As Boltzmann wrote, the entropy of a weight is defined by $S = \log m$.

Tensor powers $V_\lambda^{\otimes N}$ arise concretely in quantum spin chains with N sites. At each site the Hilbert space is V_λ and the Hilbert space of the chain is the tensor product over sites of V_λ

Okounkov's log-concavity conjectures

In A. Okounkov's paper, "Why should multiplicities be log-concave, " this analogy motivates conjectures that multiplicities should be log-concave in many settings.

Definition: Let $F : \mathbf{A} \to \mathbf{O}$ be a function from an abelian semi-group (e.g. dominant weights) to an ordered abelian semi-group (e.g. representations). Say F is concave if

$$(p+q)F(C) \geq pF(A) + qF(B)$$

for any $A, B, C \in \mathbf{A}$ satisfying

$$(p+q)C = pA + qB, \quad p, q \in \mathbb{N}.$$

Conjecture (Okounkov) Littlewood-Richardson coefficients $m^{\mu}_{\lambda\gamma}$ are log-concave in (λ, γ, μ). More generally, the representation valued function $V : \lambda \to V_\lambda$ is log-concave w.r.t. the natural ordering and tensor multiplication.

Theme of results

Our main results give asymptotic formulae for multiplicities $m_N(\lambda, \mu)$ of *moving weights* $\mu = \mu_N$ or moving irreducibles $a_N(\lambda, \mu)$ in high tensor powers. The overall picture is:

- multiplicities peak at weights near the center of gravity $Q^*(\lambda)$ of $Q(N\lambda)$;

- They have a common exponential rate for weights in a ball of radius $O(\sqrt{N})$ around the center of mass;

- The exponential rate declines as the weight moves from a moderate to a strong deviations region towards the boundary of $Q(N\lambda)$.

- At the boundary point $N\lambda$ of $Q(N\lambda)$, the multiplicity equals one.

Comparison to multi-nomial coefficients

The multiplicity picture resembles Boltzmann's analysis of the asymptotics of multinomial coefficients

$$\begin{cases} m_N : \{k = (k_1, \ldots, k_m) \in \mathbb{N}^m : |k| := k_1 + \cdots + k_m \leq \\ \quad m_N(k) = \binom{N}{k} = \frac{N!}{(N-|k|)!k_1! \cdots k_m!}. \end{cases}$$

, which are multiplicities for states of an ideal gas. As we will see, this is more than an analogy.

Binomial coefficients

The binomial coefficient $b_N(k) = \binom{N}{k}$ peaks at the center $k = \frac{N}{2}$ where $b_N(\frac{N}{2}) \sim N^{-1/2}2^N$. Measuring distance from the center by $d_N(k) = k - \frac{N}{2}$,

$$b_N(k) \sim \begin{cases} (CL) \quad C\ N^{-1/2}2^N\ e^{-\frac{2d_N(k)^2}{N}}, & \\ (MD) \quad C\ N^{-1/2}2^N\ e^{-\frac{2d_N(k)^2}{N} - Nf(\frac{2d_N(k)}{N})} & , \\ \qquad\qquad \text{with } f(x) = \sum_{n=2}^{\infty} \frac{x^{2n}}{(2n)(2n-1)} & \\ (SD) \quad \frac{1}{\sqrt{2\pi Na(1-a)}}\ a^{aN}\ (1-a)^{(1-a)N}, & \\ (RE) \quad C_0\ N^{k_0}, & k \end{cases}$$

Binomial coefficients The first region is the central limit region (CL), where the asymptotics are normal (i.e. have the form

$$N^{-1/2}2^N\varphi(\frac{d_N(k)}{\sqrt{N}}),$$

where φ is the Gaussian). The exponential growth is fixed at $\log 2$ as long as $d_N(k) = O(\sqrt{N})$. In the next region (MD) of moderate deviations, the exponent is decreased by the function f. In the next regime (SD) of strong deviations, the growth exponent decreases from $\log 2$ to

$$\log 2 > a\log\frac{1}{a} + (1-a)\log\frac{1}{1-a}\ (\to 0, a \to 1)$$

as a increases from 1/2 to 1. In the final boundary (RE) region of rare events, the exponent vanishes and the growth rate is algebraic.

Multiplicity regimes

Similarly, for multiplicities of weights and irreducibles in $V_\lambda^{\otimes N}$, there exist:

- A central limit regime of radius $N^{2/3}$ around the center of mass of $Q(N\lambda)$, where multiplicities grow at a rate $\dim V_\lambda^N$;

- A moderate deviations regime, where the exponent $N \log \dim V_\lambda$ decreases of order $o(N)$;

- A large deviations regime of weights of distance $\sim cN$ from the center of mass, where the top order term in N of the exponent decreases;

- A boundary regime, where the exponent is zero.

Center of mass for weight multiplicities

The simplest problem is to determine the asymptotic distribution of multiplicities of weights in $V_\lambda^{\otimes N}$. Let us define a probability measure on $Q(\lambda)$ as follows:

(1)
$$dm_{\lambda,N} := \frac{1}{\dim V_\lambda^{\otimes N}} \sum_{\nu \in Q(N\lambda)} m_N(\lambda,\nu)\delta_{N^{-1}\nu}.$$

This measure charges each possible weight ν of $V_\lambda^{\otimes N}$ with its relative multiplicity $\frac{m_N(\lambda,\nu)}{\dim V_\lambda^{\otimes N}}$ and then dilates the weight back to $Q(\lambda)$. As $N \to \infty$, the dilated weights become denser in $Q(\lambda)$ and we may ask how they become distributed. In particular, which are the most probable weights?

Center of mass for multiplicities of irreps

The analogous measures weighting $\mu \in Q(N\lambda)$ by the multiplicity of the irreducible representation V_μ in $V_\lambda^{\otimes N}$ is defined by

(2)

$$dM_{\lambda,N} := \frac{1}{B_N(\lambda)} \sum_{\nu \in Q(N\lambda)} a_N(\lambda, \nu) \delta_{N^{-1}\nu},$$

where $B_N(\lambda) = \sum_\nu a_N(\lambda; \nu)$.. The measures $dM_{\lambda,N}$ are measures on the closed positive Weyl chamber $\overline{C}$.

Center of mass result

Theorem **1** *Assume that λ is a dominant weight in the open Weyl chamber. Then, we have*

$$m_{\lambda,N}, \;\; M_{\lambda,N} \to \delta_{Q^*(\lambda)}$$

weakly as $N \to \infty$, where $\delta_{Q^*(\lambda)}$ is the Dirac measure at the (Euclidean) center of mass $Q^*(\lambda)$ of the polytope $Q(\lambda)$ given by

$$Q^*(\lambda) = \frac{1}{\dim V_\lambda} \sum_{\nu \in M_\lambda} m_1(\lambda;\nu)\nu. \tag{3}$$

Central limit theorem

Our next result concerns the 'central limit region' of weights which are within a ball of radius $O(\sqrt{N})$ around the center of mass. Fix a dominant weight λ in the open Weyl chamber C. Let

$$d\mu_N^\lambda := \frac{1}{\dim V_\lambda^{\otimes N}} \sum_{\nu \in Q(N\lambda)} m_N(\lambda;\nu)\delta_{\frac{1}{\sqrt{N}}(\nu - NQ^*(\lambda))}.$$

Theorem 1 *As measures on X^*, the linear span of the simple roots, we have*

$$\text{w-}\lim_{N\to\infty} d\mu_N^\lambda = \frac{e^{-\langle A_\lambda^{-1} x\, x\rangle/2}}{(2\pi)^{m/2}\sqrt{\det A_\lambda}}\, dx,$$

where $m = \dim X^*$ and

$$A_\lambda = \frac{1}{\dim V_\lambda} \sum_{\mu \in M_\lambda} m_1(\lambda;\mu)\mu\otimes\mu - Q^*(\lambda)\otimes Q^*(\lambda).$$

Proofs

The center of mass and central limit theorem are elementary results because the character χ_{V_λ} satisfies:

(4)

$$\chi_{V_\lambda^{\otimes N}} = \chi_{V_\lambda}^N \implies dm_{\lambda,N} = D_{\frac{1}{N}} dm_\lambda * \cdots * dm_\lambda,$$

where $dm_\lambda = dm_{\lambda,1}$ and where $D_{\frac{1}{N}}$ is the dilation operator by $\frac{1}{N}$ on the dual Cartan subalgebra $\mathfrak{t}^*$.

A sequence of convolution powers of probability measures satisfies the law of large numbers, the central limit theorem and the (Laplace) large deviations principle.

Recap of large deviations principle

Let us recall the definitions: Let m_N ($N = 1, 2, \ldots$) be a sequence of probability measures on a closed set $E \subset \mathbb{R}^n$. Let $I : E \to [0, \infty]$ be a lower semicontinuous function. Then, the sequence m_N is said to satisfy the *large deviation principle with the rate function* I (and with the speed N) if the following conditions are satisfied:

(a) The level set $I^{-1}[0, c]$ is compact for every $c \in \mathbb{R}$.

(b) For each closed set F in E,

$$\limsup_{N \to \infty} \frac{1}{N} \log m_N(F) \leq - \inf_{x \in F} I(x).$$

(c) For each open set U in E,

$$\liminf_{N \to \infty} \frac{1}{N} \log m_N(U) \geq - \inf_{x \in U} I(x).$$

Cramer's theorem for weights

Theorem **2** *Assume that G is semisimple. Then, the sequence $\{dm_{\lambda,N}\}$ of measures on $Q(\lambda)$ satisfies a large deviations principle with speed N and rate function:*

(5)

$$I_\lambda(x) = \sup_{\tau \in \mathfrak{t}} \left\{ \langle\, \tau, x \,\rangle - \log\left(\frac{\chi_\lambda(\tau/(2\pi i))}{\dim V_\lambda} \right) \right\}, \quad x \in \mathfrak{t}^*,$$

where $\chi_\lambda(\tau/(2\pi i)) = \sum_{\nu \in M_\lambda} m_1(\lambda;\nu) e^{\langle\, \nu, \tau \,\rangle}$ denotes the character of V_λ extended on $\mathfrak{t} \otimes \mathbb{C}$.

Results for irreducibles

We have:

(6)

$$dM_{\lambda,N}(\mu) = \frac{(\dim V_\lambda)^N}{B_N(\lambda)} \sum_{w\in W} \mathrm{sgn}(w) dm_{\lambda,N}(\mu+\rho-w\rho).$$

We can thus deduce the upper-bound half of the large deviation principle for the measure $dM_{\lambda,N}$ from that for $dm_{\lambda,N}$. It follows from Theorem 2 that:

Corollary **3** *Assume that G is semisimple. The sequence $\{dM_{\lambda,N}\}$ of measures on $Q(\lambda)$ satisfies the upper-bound in a large deviations principle with speed N and rate function $I_\lambda(x)$ given by* (5)*.*

(The large deviations principle with the rate function (5) has already been proved by Duffield for $dM_{\lambda,N}$ by a different method.

Main results: pointwise asymptotics

We now turn to the main results, which pertain to pointwise asymptotics rather than bulk weak convergence results. They resemble the asymptotics of multinomial coefficients above. As will be seen, the unifying thread is the combinatorics of lattice paths with steps in a convex polytope.

We first state the multiplicity results for weights, and then for irreps.

Central limit asymptotics for weights

Our first asymptotic result concerns the 'central limit region' of weights which are within a ball of radius $O(\sqrt{N})$ around the center of mass.

Theorem **4** *Fix a dominant weight λ in the open Weyl chamber C. Let ν_N be a sequence of weights such that $|\nu_N| = O(N^{1/2})$. Then,*

(7)

$$m_N(\lambda;\nu_N) = (2\pi N)^{-m/2}|\Pi(G)|(\dim V_\lambda)^N$$

$$\times \quad \left(\frac{e^{-\langle A_\lambda^{-1}\nu_N, \nu_N \rangle/(2N)}}{\sqrt{\det A_\lambda}} + O(N^{-1/2}) \right),$$

where $|\Pi(G)|$ is the order of a certain finite group, where $m = \dim \mathfrak{t}$ is the rank of G and the positive definite linear transform $A_\lambda : \mathfrak{t} \to \mathfrak{t}^*$ is given by

$$A_\lambda = \frac{1}{\dim V_\lambda} \sum_{\mu \in M_\lambda} m_1(\lambda;\mu)\mu \otimes \mu. \tag{8}$$

Exponent of growth

We observe that, in the Central Limit regime, the exponent of growth of multiplicities is the constant $\log \dim V_\lambda$. As noted by Okounkov, by the Weyl dimension formula, $\dim V_\lambda$ is a concave function of λ. Hence, $\log m_N(\lambda, \mu)$ is asymptotically concave in both λ and μ in the CLT region.

The central limit regime actually extends to weights $\nu_N \in NQ(\lambda)$ of the form

(9)

$$\nu_N = NQ^*(\lambda) + d_N(\nu_N), \quad |d_N(\nu_N)| = o(N^s).$$

with $0 \leq s \leq 2/3$. Here, as in the case of binomial coefficients, $d_N(\nu_N)$ represents the distance to the center of gravity of $Q(\lambda)$.

Moving weights in the strong deviations region

We now consider the moderate and strong deviations regions. As suggested by the behavior of multinomial coefficients, the exponent must decrease as we move away from the center of gravity of $Q(N\lambda)$. A key role in the exponent correction will be played by the homeomorphism $\mu_\lambda : X \to Q(\lambda)$

(10)

$$\mu_\lambda(x) := \frac{1}{\sum_{\mu\in M_\lambda} m_1(\lambda;\mu)e^{\langle\, \mu,x\,\rangle}} \sum_{\mu\in M_\lambda} m_1(\lambda;\mu)e^{\langle\, \mu,x\,\rangle}\mu$$

(it resembles the moment map of a toric variety, restricted to the real torus in $(\mathbb{C}^*)^m$.)

The rate function

Define the function δ_λ on the interior $Q(\lambda)^o$ of the polytope $Q(\lambda)$ by

(11)
$$\delta_\lambda(x) = \log\left(\sum_{\mu\in M_\lambda} m_1(\lambda;\mu)e^{\langle\,\mu - x, \tau_\lambda(x)\,\rangle}\right),$$

where $\tau_\lambda = \mu_\lambda^{-1} : Q(\lambda)^o \to X$. It is clear that $\delta_\lambda(\nu) > 0$ for $\nu \in Q(\lambda)^o \cap M_\lambda$. It will turn out that

$$\delta_\lambda(x) = \log\dim V_\lambda - I_\lambda(x), \;\; x \in Q(\lambda)^o$$

where I_λ is the rate function. Hence, $\delta_\lambda(x)$ is concave as a function of (x, λ), bearing out Okounkov's conjectures asymptotically in this problem.

Another ingredient

For $\nu \in Q(\lambda)^o$, we further define the linear map $A^0_\lambda(\nu) : \mathfrak{t} \to \mathfrak{t}^*$ by

(12)

$$A^0_\lambda(\nu) = \sum_{\mu \in M_\lambda} \frac{m_1(\lambda;\mu) e^{\langle\, \mu, \tau_\lambda(\nu)\,\rangle}}{\sum_{\mu' \in M_\lambda} m_1(\lambda;\mu') e^{\langle\, \mu', \tau_\lambda(\nu)\,\rangle}} \mu \otimes \mu - \nu \otimes \nu.$$

Its restriction to the subspace X,

$$A_\lambda(\nu) := A^0_\lambda(\nu)|_X, \tag{13}$$

is positive definite as a linear map from $X \to X^*$.

Weight asymptotics in the strong deviations region

First, we consider the ‘strong deviations’ regime where the weight in question has the form $\nu = N\nu_0 + f$.

Theorem **5** *Let $\lambda \in C \cap I^*$ be a dominant weight, and let $\nu_0 \in M_\lambda$ be a weight of V_λ which lies in the interior $Q(\lambda)^o$ of the polytope $Q(\lambda)$. We fix a weight f in the root lattice Λ^*. Then, we have the following asymptotic formula:*

$$m_N(\lambda; N\nu_0 + f) \sim CN^{-m/2} \frac{|\Pi(G)| e^{N\delta_\lambda(\nu_0) - \langle f, \tau_\lambda(\nu_0) \rangle}}{\sqrt{\det A_\lambda(\nu_0)}},$$

where m is the number of the simple roots, $|\Pi(G)|$ is the order of a certain finite group, and $\tau_\lambda(\nu_0) = \mu_\lambda^{-1}(\nu_0) \in X$.

Weight asymptotics in the moderate deviations regime

Next, we consider a general weight ν. We have just handled the case where $d_N(\nu) \sim N\nu_0$, so now we assume that $|d_N(\nu)| = o(N)$, i.e. the weight lies in the moderate deviations region.

Theorem **6** *Let* $\lambda \in C \cap I^*$ *be a dominant weight, and let* $\nu_N \in NQ(\lambda)$ *be a weight of the form*

$$\nu_N = Nx + d_N(\nu_N), \quad |d_N(\nu_N)| = o(N),$$

where $|d_N(\nu_N)|$ *denotes the norm of the vector* $d_N(\nu_N)$ *with respect to the fixed* W*-invariant inner product on* $\mathfrak{t}^*$*, and where* $x \in Q(\lambda)^o$ *is not necessarily a weight. Then,*

$$m_N(\lambda;\nu_N) \sim (2\pi N)^{-m/2} \frac{|\Pi(G)| e^{N\delta_\lambda(\nu_N/N)}}{\sqrt{\det A_\lambda(\nu_N/N)}}.$$

Furthermore,

$$\lim_{N\to\infty} \frac{1}{N} \log m_N(\lambda;\nu_N) = \delta_\lambda(x).$$

Biane's central limit theorem for irreps

We now state analogous results for multiplicities of irreps, beginning withe the central limit region. Our weight results could be used to find irrep multiplicities, but there is a more direct path due to P. Biane.

Theorem **7** *(Biane) Assume that* G *is semisimple. For every positive integer* N*, let* NM_λ *be the set of weights of the form* $\nu_1 + \cdots + \nu_N$ *with* $\nu_j \in M_\lambda$*. Then, for* $\mu \notin NM_\lambda$*,* $a_N(\lambda;\mu) = 0$*. For,* $\mu \in NM_\lambda$ *with* $|\mu| \leq C\sqrt{N}$*, we have:*

$$a_N(\lambda;\mu) \sim \frac{|\Pi(G)|(\dim V_\lambda)^N (\dim V_\mu) \prod_{\alpha\in\Phi_+}\langle A_\lambda^{-1}\alpha, \rho\rangle}{\sqrt{\det A_\lambda}(2\pi)^{m/2} N^{(\dim G)/2}} \Big($$

where the matrix A *is defined above,* m *is the rank of* G *and the inner product* $\langle\cdot,\cdot\rangle$ *is the Killing form.*

Irrep multiplicities in the strong deviations region

We now consider asymptotic multiplicities of irreps with highest weight in the region of strong deviations.

Theorem **8** *Let $\nu \in M_\lambda \cap \overline{C}$ be a dominant weight in the polytope $Q(\lambda)$. Then*

(14)

$$a_N(\lambda; N\nu) = (2\pi N)^{-m/2} e^{N\delta_\lambda(\nu)}$$

$$\left(\frac{|\Pi(G)|\Delta(\tau_\lambda(\nu)/(2\pi i))e^{-\langle \rho, \tau_\lambda(\nu) \rangle}}{\sqrt{\det A_\lambda(\nu)}} + O(N^{-1}) \right),$$

where m is the number of simple roots, $|G_\lambda|$ is the order of a certain finite group.

Example: $U(2)$

Let us see how our results apply to the simplest case, the Clebsch-Gordon problem for $U(2)$ Pick a dominant weight $\lambda = (\lambda_1, \lambda_2), \lambda_1 > \lambda_2 \geq 0$, and set $n_\lambda = \lambda_1 - \lambda_2 > 0$. The weights in the irrep V_λ are:

$$\nu_j := \lambda - j\alpha, \quad j = 0, \ldots, n_\lambda, \tag{15}$$

where α is the unique positive (simple) root $\alpha = (1, -1)$. All weights have multiplicity one: $m_1(\lambda; \nu_j) = 1$. Therefore, the multiplicity for the high tensor power $V_\lambda^{\otimes N}$ is given by

$$m_N(\lambda; \mu) = \#\{(j_1, \ldots, j_N)\,;\, 0 \leq j_k \leq n_\lambda,$$

$$\mu = N\lambda - (j_1 + \cdots j_N)\alpha\}.$$

Asymptotic formula

Then the multiplicity $a_N(\lambda; N\nu_j)$ of $V_{N\nu_j}$ in $V_\lambda^{\otimes N}$ has the following asymptotic formula:

$$a_N(\lambda; N\nu) = (2\pi N)^{-1/2} e^{-N(n_\lambda - 2j)}$$

$$\left(\frac{\sinh(n_\lambda+1)\tau_j}{\sinh \tau_j}\right)^N \left(a_\lambda(j) + O(N^{-1})\right),$$

where the positive constant $a_\lambda(j)$ is given by

$$a_\lambda(j) = 2e^{-\tau_j} \sqrt{\frac{2\sinh^4 \tau_j \sinh^2(n_\lambda + 1)\tau_j}{\sinh^2(n_\lambda + 1)\tau_j - (n_\lambda + 1)^2 \sinh^2 \tau_j}}.$$

The leading term a_j vanishes if and only if n_λ is even and $j = n_\lambda/2$. In this case, the dominant weight ν_j ($j = n_\lambda/2$) is in the unique wall of the Weyl chamber C.

Multiplicities and lattice paths

The proofs of the multiplicity asymptotics are based on the relatation betweeen

- multiplicities of weights and irreps to multiplicities ;
- combinatorics and multiplicities of lattice paths with steps in the convex polytope $Q(\lambda)$.

What are lattice paths?

Given a set $S \subset \mathbf{N}^m$ of *allowed steps*, an S- lattice path of length N from 0 to β is a sequence $(v_1, \dots, v_N) \in S^N$ such that $v_1 + \cdots + v_N = \beta$. We define the multiplicity (or partition) function of the lattice path problem by

(16)

$$\mathcal{P}_N(\gamma) = \#\{(v_1, \dots, v_N) \in S^N : v_1 + \cdots + v_N = \gamma\}.$$

The set of possible endpoints of an S- path of length N forms a set $\mathcal{P}_{S,N}$, and we may ask how the numbers $\mathcal{P}_N(\gamma)$ are distributed as γ varies over $\mathcal{P}_{S,N}$.

Weighted lattice paths

Let X be a real vector space and let and $L \subset X$ be a lattice, with duals X^* and L^*. Let $S \subset L^*$ ($\#S \geq 2$) be a finite set, and let P be the convex hull of the finite set S. Let c be a strictly positive function c on S, and define the *weighted multiplicity of lattice paths* $\mathcal{P}_N^c$ *of length* N *with weight* c and *the set of the allowed steps* S by

(17)

$$\mathcal{P}_N^c(\gamma) = \sum_{\beta_1, \ldots, \beta_N \in S\,;\, \gamma = \beta_1 + \cdots + \beta_N} c(\beta_1) \cdots c(\beta_N).$$

If $c \equiv 1$, then $\mathcal{P}_N^c(\gamma) = \mathcal{P}_N(\gamma)$.

Examples

Example 1 Let $S = p\Sigma \cap \mathbb{N}^m$, where Σ is the standard simplex and p is a positive integer, and let $c(\beta) = \frac{p!}{\beta!(p-|\beta|)!} = \binom{p}{\beta}$. Then the weighted multiplicity function $\mathcal{P}_N^c(\gamma)$ is given by $\mathcal{P}_N^c(\gamma) = \binom{Np}{\gamma}$.

Example 2 Let $S_\lambda = \{\mu - \lambda\,;\, \mu \in M_\lambda\}$, and let P_λ be its convex hull. Then $P_\lambda = Q(\lambda) - \lambda$. Let $c_\lambda(\beta) := m_1(\lambda; \mu), \quad \beta = \mu - \lambda \in S_\lambda$. Then:

$$m_N(\lambda; \mu) = \mathcal{P}_N^{c_\lambda}(\mu - N\lambda)$$

for every $\mu \in NQ(\lambda)$. Further,

$$a_N(\lambda; \mu) = \Sigma_{w \in W} \mathrm{sgn}(w) \mathcal{P}_N^{c_\lambda}(\mu - N\lambda + \rho - w\rho).$$

Multiplicities of lattice paths

It follows that asymptotics of multiplicities of weights and irreps reduces to lattice path asymptotics. How do we study these?

Multiplicities of lattice paths = Fourier coefficients of powers $k(w)^N$ of a complex exponential sum of the form

$$k(w) = \sum_{\beta \in P} c(\beta) e^{\langle \beta, w \rangle}, \quad w \in \mathbb{C}^n \tag{18}$$

with positive coefficients $c(\beta)$, where P is a convex lattice polytope.

Multiplicities and lattice paths

One obtain the precise asymptotics of the Fourier coefficients of $k(w)^N$ by a complex stationary phase (or steepest descent) argument. It is necessary to deform the contour of the Fourier integral to pick up the relevant complex critical points and to study the geometry of the complexified phase

Final remarks and open problems

As mentioned at the outset, there are three parameters in the multiplicity $m_N(\lambda, \mu)$ for $V_\lambda^{\otimes N}$ in $SU(k)$ and it is interesting to analyze the asymptotic behavior in all three parameters. Some problems are:

- Asymptotic log convexity of multiplicities.

- Joint asymptotics in (N, λ)? The asymptotics in λ are semi-classical and the limit distribution of multiplicities in this aspect for $N = 1$ is given by the Duistermaat-Heckmann measure on $Q(\lambda)$. The simplest problem is the center of mass problem for the joint asymptotics of $m_N(N\lambda_0 N, \mu)$. One could also consider pointwise asymptotics.

More problems

- Letting the dimension of the group vary raises new problems, associated with Kerov-Vershik, Olshansky, Okounkov, Borodin and others.

- Finite groups such as the symmetric group S_d. Joint asymptotics of V_λ^N in (N, d). Large d asymptotics often involve free probability theory.

Entropy problems in spectral geometry

Let $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ denote the *distinct* eigenvalues of Δ and let $m(\lambda_j)$ denote the multiplicity.

- Suppose g is a metric on S^n with the eigenvalue multiplicities $m_0(\lambda_j^0)$ of the standard metric g_0. Is $g = g_0$?

- Define the (quantum statistical entropy) of $e^{-t\Delta}$ as follows: $Z(t) = Tre^{-t\Delta}$ and put the probability measure

$$p_t(j) = \frac{m(\lambda_j)e^{-t\lambda_j}}{Z(t)}$$

on $\mathbb{N}$. Let $S(t) = \sum_{j=0}^{\infty} p_t(j) \log p_t(j)$ be the entropy of this measure. What are the metrics of maximal entropy? It is easy to see that g_0 is a critical point for $S(t)$ for all t. Is it the maximum?