Can you hear the shape of an analytic drum: higher dimensions.

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Oh Say, Can you Hear...?

Main result: Bounded analytic domains $\Omega \subset \mathbb{R}^n$ with $\pm$ mirror symmetries across all coordinate axes and with one marked invariant bouncing ball orbit are spectrally determined among other such domains.

The spectrum could be Dirichlet or Neumann.

(The symmetry condition is that the maps σ_j : $(x_1, \dots, x_n) \to (x_1, \dots, -x_j, x_{j+1}, \dots, x_n)$ are isometries of the domain).

Prior Results: Counterexamples

- Urakawa (1982): Non-convex non-analytic ounterexamples in dimensions $\geq$ 4: fundamental domains C for finite reflection groups W acting on R^n intersected with S^{n-1}.

- Gordon-Webb- Wolpert (1992): non-convex non-analytic counterexamples in dimension 2;

- Gordon-Webb (1994): Improvement of Urakawa to convex non-analytic examples in dimensions $\geq$ 4.

Some relevant prior Results: Positive

- Balls are spectral determined (isoperimetric inequality).

- Analytic domains with symmetry of ellipse are spectrally rigid (Colin de Verdi'ere, 1985).

- Analytic domains with symmetry of ellipse are spectrally determined in that class (SZ, 1998). Analytic domains with one symmetry are spectrally determined in that class (SZ, 2005).

- Analytic potentials with $(\mathbb{Z}_2)^n$ symmetry are spectrally determined, Guillemin/Uribe/Paul (2007); see also Hezari (2007) for a new proof and fewer symmetry assumptions.

Wave invariants at a billiard orbit

Main tool: explicit formulae for the wave invariants associated to bouncing orbits.

A bouncing ball orbit γ is a 2-link periodic trajectory of the billiard flow, i.e. a reversible periodic billiard trajectory that bounces back and forth along a line segment orthogonal to the boundary at both endpoints. Our main assumption is that the endpoints of the bouncing ball orbit are fixed points of the isometries σ_j for $j = 1, \ldots, n-1$ and are reversed by σ_n.

For a convex symmetric domain, the minimal periodic orbit is an invariant bouncing ball orbit (Ghomi).

Picture

We locally express $\partial\Omega = \partial\Omega^+ \cup \partial\Omega^-$ as the union of two graphs over a ball $B_\epsilon(0)$ around 0 in the x'-hyperplane, namely

$$\partial\Omega^+ = \{x^n = f_+(x'), \quad |x'| \leq \epsilon\},$$

$$\partial\Omega^- = \{x^n = f_-(x'), \quad |x'| \leq \epsilon\}.$$

We assume

$$(1) \quad f_+(x') = -f_-(x'), \; f_\pm(\sigma_j(x')) = f_\pm(x'),$$

where σ_j denotes the reflections in the coordinate hyperplanes of $\mathbb{R}^{n-1}$.

Precise statement of result

We denote by $\mathcal{D}_L$ consists of convex real-analytic domains $\Omega \subset \mathbb{R}^n$ satisfying:

- (i) $\sigma_j : \Omega \to \Omega$ is an isometry for all $j = 1, \dots, n$;

- (ii) There exists a non-degenerate bouncing ball orbit γ which is invariant under all σ_j (and with orientation reversed by σ_n);

- (iii) The lengths $2rL$ of all iterates γ^r ($r = 1, 2, 3, \dots$) have multiplicity one in $Lsp(\Omega)$.

- (iv) In the elliptic case, if $\{e^{\pm i\alpha_1}, ... e^{\pm i\alpha_{n-1}}\}$ are the eigenvalues of the Poincare map $\mathcal{P}_\gamma$, we require that $\{\alpha_1, ..., \alpha_{n-1}\}$ are linearly independent over $\mathbb{Q}$. We assume the same condition in the Hyperbolic case or mixed cases.

Let $\mathrm{Spec}_B(\Omega)$ denote the spectrum of the Laplacian Δ_Ω of the domain Ω with boundary conditions B (Dirichlet or Neumann).

Theorem **1** *For Dirichlet (or Neumann) boundary conditions B, the map $Spec_B : \mathcal{D}_{1,L} \mapsto \mathbb{R}_+^{\mathbf{N}}$ is 1-1.*

Wave invariants

The resolvent of the Laplacian Δ_Ω on Ω with Dirichlet boundary conditions is the operator on $L^2(\Omega)$ defined by

$$R_\Omega(k+i\tau) = -(\Delta_\Omega + (k+i\tau)^2)^{-1}, \quad \tau > 0.$$

Theorem **2** *(Andersson-Melrose; Guillemin-Melrose) Assume that γ is a non-degenerate periodic reflecting ray, and let $\hat{\rho} \in C_0^\infty(L_\gamma - \epsilon, L_\gamma + \epsilon)$, equal to one on $(L_\gamma - \epsilon/2, L_\gamma + \epsilon/2)$ and with no other lengths in its support. Then $Tr 1_\Omega R_\rho(k+i\tau)$ admits a complete asymptotic expansion of the form*

(2)

$$Tr 1_\Omega R^\Omega_{B\rho}(k+i\tau) \sim \mathcal{D}_{B,\gamma}(k+i\tau) \sum_{j=0}^\infty B_{\gamma,j} k^{-j}, \quad k \to \infty,$$

where

- $\mathcal{D}_{B,\gamma}(k+i\tau)$ *is the symplectic pre-factor*

$$\mathcal{D}_{B,\gamma}(k+i\tau) = C_0 \, \epsilon_B(\gamma) \frac{e^{i(k+i\tau)L_\gamma} e^{i\frac{\pi}{4} m_\gamma}}{\sqrt{|\det(I - P_\gamma)|}}$$

- P_γ *is the Poincaré map associated to* γ*;*

- $\epsilon_B(\gamma)$ *is the signed number of intersections of* γ *with* $\partial\Omega$ *(the sign depends on the boundary conditions;* ± 1 *for each bounce for Neumann/Dirichlet boundary conditions);*

- m_γ *is the Maslov index of* γ*;*

- C_0 *is a universal constant (e.g. factors of* 2π*).*

The coefficients $B_{\gamma;j}$ *are the wave invariants.*

Notation

The remainder $R_{2r}(\mathcal{J}^{2j-2}f_+(0), \mathcal{J}^{2j-2}f_-(0))$ is a polynomial in the designated jet of $f_\pm$.

Let

$$[h^{ij,pq}_{+,2r}]_{1\leq i,j\leq n-1}$$

denote the matrix elements of the inverse of the Hessian of the length functional $\mathcal{L}_\pm$ on polygonal paths with $2r$ bounces at the bouncing ball orbit.

Let $\vec{\gamma} = (\gamma_1, ...\gamma_{n-1})$, $|\vec{\gamma}| = \gamma_1 + ... + \gamma_n$, , $\vec{X}^{\vec{\gamma}} = X_1^{\gamma_1}...X_{n-1}^{\gamma_{n-1}}$. Let $\overrightarrow{h^{pq}_{+,2r}}$ be the vector $(h^{11,pq}_{+,2r}, h^{22,pq}_{+,2r}, ..., h^{(n-1,n-1),pq}_{+,2r})$, corresponding to the $(n-1)\times(n-1)$ diagonal matrix $[h^{ij,pq}_{+,2r}]_{1\leq i,j\leq n-1}$.

Formula for wave invariants $B_{\gamma^r,j}$

If the domain is symmetric,

$$B_{\gamma^r,j} \equiv 4rL\mathcal{A}_0(r)$$

$$\{2(w_{\mathcal{G}_{1,j}^{2j,0}})\Big(\sum_{|\gamma|=j+1}(\overrightarrow{h_{+,2r}^{11}})^{\vec{\gamma}} D_{2\vec{\gamma}}^{2j+2} f_+(0)\}$$

$$+R_{2r}(\mathcal{J}^{2j-2}f_+(0), \mathcal{J}^{2j-2}f_-(0))$$

In general:

$$B_{\gamma^r,j} = 2rL\{2(w_{\mathcal{G}^{2j,0}_{1,j}})\Big(\sum_{|\gamma|=j+1}(\overrightarrow{h^{11}_{+,2r}})^{\vec{\gamma}} D^{2j+2}_{2\vec{\gamma}} f_+(0)$$

$$-\sum_{|\gamma|=j+1}(\overrightarrow{h^{11}_{-,2r}})^{\vec{\gamma}} D^{2j+2}_{2\vec{\gamma}} f_-(0)\Big)$$

$$+4(w_{\mathcal{G}^{2j-1,3,0}_{2,j+1}})\sum_{|\vec{\gamma}|=j,|\vec{\beta}|=|\vec{\delta}|=1}\sum_{p,q=1}^{2r}$$

$$(\overrightarrow{h^{pp}_{+,2r}})^{\vec{\gamma}}(\overrightarrow{h^{pq}_{+,2r}})^{\vec{\beta}}(\overrightarrow{h^{qq}_{+,2r}})^{\vec{\delta}} w_+(p)w_+(q)$$

$$\Big(D^{2j+1}_{2\vec{\gamma}+\vec{\beta}} f_{w_+(p)}(0).\, D^{3}_{\vec{\beta}+2\vec{\delta}} f_{w_+(q)}(0)\Big)$$

$$+4(w_{\widehat{\mathcal{G}}^{2j-1,3,0}_{2,j+1}})\sum_{|\vec{\gamma}|=j-1,|\vec{\beta}|=3,}\sum_{p,q=1}^{2r}$$

$$(\overrightarrow{h^{pp}_{+,2r}})^{\vec{\gamma}}(\overrightarrow{h^{pq}_{+,2r}})^{\vec{\beta}} w_+(p)w_+(q)$$

$$\Big(D^{2j+1}_{2\vec{\gamma}+\vec{\beta}} f_{w_+(p)}(0).\, D^{3}_{\vec{\beta}} f_{w_+(p)}(0)\Big)$$

$$+R_{2r}(\mathcal{J}^{2j-2} f_+(0), \mathcal{J}^{2j-2} f_-(0)),$$

Determining Taylor coefficients from wave invariants

We determine Taylor coefficients inductively in

the degree. The $2j$th Taylor coefficients occur for the first time in the term k^{-2j+2}. We use iterates γ^r to decouple Taylor coefficients. This is where the independence over $\mathbb{Q}$ of the eigenvalues of the second fundamental form is used.

Determining Taylor coefficients from wave invariants

We now solve the inverse problem. The top derivative term with $2k$ derivatives is

$$\sum_{\gamma:|\gamma|=k} \prod_{j=1}^{n-1} \left(\sum_{\ell=0}^{2r-1} \frac{1}{\cos\frac{\alpha_j}{2}+\cos\frac{\pi\ell}{r}} \right)^{\gamma_j}$$

$$\left(\prod_{j=1}^{n-1} \frac{\partial^{2\gamma_j}}{\partial (x_1^j)^{2\gamma_j}} \right) f(x')|_{x'=0}$$

We wish to determine each Taylor coefficient $\left(\prod_{j=1}^{n-1} \frac{\partial^{2\gamma_j}}{\partial (x_1^j)^{2\gamma_j}} \right) f(x')|_{x'=0}$ from this sum (by induction) as r varies. This is possible if and only if, for each k, the functions

(3)

$$G_\gamma(r;\alpha_1,\ldots,\alpha_{n-1}) = \prod_{j=1}^{n-1} \left(\sum_{\ell=0}^{2r-1} \frac{1}{\cos\frac{\alpha_j}{2} + \cos\frac{\pi\ell}{r}} \right)^{\gamma_j}$$

are independent functions on $r \in \mathbb{Z}$ as γ varies over $\{\gamma \in \mathbb{N}^{n-1}, |\gamma| = k\}$. This is the pure elliptic case, and one has $\cosh \frac{\alpha_j}{2}$ in the hyperbolic case.

In fact,
(4)
$$G_\gamma(r; \alpha_1, \dots, \alpha_{n-1}) = \Pi_{j=1}^{n-1} \left(\frac{1}{\sin \alpha_j/2} \cot \frac{r\alpha_j}{2} \right)^{\gamma_j}$$
and the question is whether the functions

$$g_\gamma(r; \alpha_1, \dots, \alpha_{n-1}) = \Pi_{j=1}^{n-1} \left(\cot \frac{r\alpha_j}{2} \right)^{\gamma_j} \tag{5}$$

are independent as functions on $\mathbb{Z}$. It is easy to prove that they are.

Note that the formulae for the wave invariants is very similar to the one for Schrödinger operators in Hezari.