Complex geometry and vacua: Counting universes in string theory.

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Thursday 4: 15 - 5: 15 January 19, 2006

Joint work with M. R. Douglas and B. Shiffman
Also based on joint work with P. Bleher

Outline of talk

1. Introduce vacuum selection problem in string/M theory.

2. Statistics of vacua program (M.R. Douglas et al).

3. Boiling down vacuum statistics to classical analysis problems about equidistribution of radial projections of lattice points and about critical points of Gaussian random holomorphic sections.

4. Rigorous vacuum counting results.

5. Open analysis problems on vacua and string theory.

Brief overview

- String/M theory is the only (currently known) consistent theory unifying quantum field theory and gravity.

- The various types of string theory are all unified in M theory– there is only one string/M theory.

- This gives hope that string/M theory will determine basic properties of the universe– e.g. cosmological constant, masses of elementary particles, and so on.

- ALAS! It has *many* vacua. There is just one 'theory' but it has many solutions. This is the problem.

10^{500}

This is the number of possible vacua that one most often hears.

It was proposed by Bousso-Polchinski and is quoted a lot in the physics literature.

Mathematical Problems:

- What are vacua and how does one count them?

- Is the figure of 10^{500} accurate?

- Do there exist any vacua consistent with known values of the cosmological constant/mass of elementary particles etc.?

References

1. M. R. Douglas, B. Shiffman and S. Zelditch, Critical points and supersymmetric vacua I, II, III (all on archive. I has appeared in CMP; II will appear in JDG, III in CMP).

2. Physics articles by: Denef-Douglas, Ashok-Douglas, J. Distler, S. Kachru et al.

3. L. Susskind, The Cosmic Landscape (Just published).

4. B. Greene, Elegant Universe.

5. R. Bousso and J. Polchinski, The String Theory Landscape, Scientific American, September 2004 issue.

6. IAS volume on Quantum Fields and Strings (especially articles by A. Strominger).

The vacuum selection problem in string/M theory

According to string/M theory, our universe is 10- (or 11-) dimensional, and its vacuum state has the form

$$M^{3,1} \times X$$

where $M^{3,1}$ is Minkowski space and where X is a complex 3-dimensional *Calabi-Yau* (= Ricci flat Kähler) manifold.

(See P. Candelas, G. Horowitz, A. Strominger, E. Witten, Vacuum configurations for superstrings. Nuclear Phys. B 258 (1985), no. 1, 46–74)

The **vacuum selection problem**: Which X forms the ‘small’ or ‘extra’ dimensions of our universe? How to select the right vacuum?

How many Calabi-Yau 3-folds are there

A CY d-fold is a complex d-dimensional manifold with trivial canonical bundle = vanishing first Chern class. By Yau: it has a Ricci-flat Kähler metric in each Kähler class).

1. Not yet classified. Even topological types not classified. See CY website.

2. Examples: Quintic hypersurfaces $P_5[z_0, \cdots, z_5]$ in $\mathbb{CP}^4$. 101 complex moduli parameters.

3. Conifold: $xy - zt = \mu\ (x, y, z, t) \in \mathbb{C}^4$ is CY for any $\mu \in \mathbb{C}$. When $\mu = 0$ get 'conifold singularity'.

Are all CY 3-folds possible vacua?

NO! At least, not in the string theory we will discuss (type IIb compactifications on a CY manifold with flux).

Otherwise there would be a continuum of vacua!

There are several further constraints which make the possible vacua a discrete set of CY 3-folds.

The vacua form a “discretuum.”

Vacua = CY which minimize an energy functional

Roughly speaking, an energy functional will be defined on the moduli space of CY threefolds. Vacua = local minima of the energy functional.

Again roughly: the graph of the energy is called the “string theory landscape” and vacua are the valleys.

Effective supergravity

The energy functional comes from the low energy approximation to string/M theory, called effective supergravity theory.

See articles of A. Strominger in IAS volume for background on how this works. Roughly, one "integrates out" the massive modes in string theory.

Specifying an effective supergravity theory?

It consists of $(\mathcal{M}, \mathcal{L}, W)$ where:

1. $\mathcal{M} = \mathcal{M}_{\mathbb{C}} \times \mathcal{H}/SL(2,\mathbb{Z})$, where $\mathcal{M}_{\mathbb{C}}$ = moduli space of complex structures on X, $\mathcal{H}$ = upper half plane. $\mathcal{M}$ = "Configuration space". Identify with a fundamental domain in Teichmuller space $\mathcal{T} \times \mathcal{H}$ under modular group.

2. $\mathcal{L} \to \mathcal{M}$ is a holomorphic line bundle (next slide);

3. the "superpotential" W is a holomorphic section of $\mathcal{L}$.

$\mathcal{L}$ = dual of $H^{3,0} \otimes H^{1,0} \to \mathcal{M}$ line bundle

Given a complex structure z on X, let $H_z^{3,0}(X)$ be the space of holomorphic $(3,0)$ forms on X, i.e. type $dw_1 \wedge dw_2 \wedge dw_3$.

On a Calabi-Yau 3-fold, dim $H_z^{3,0}(X) = 1$. Hence, $H_z^{3,0}(X) \to \mathcal{M}$ is a (holomorphic) line bundle, known as the Hodge bundle.

On the product $\mathcal{M}_{\mathbb{C}} \times \mathcal{H}$, the Hodge bundle is $H_z^{3,0} \otimes H_\tau^{1,0}$ where $(z, \tau) \in \mathcal{M}_{\mathbb{C}} \times \mathcal{H}$.

$\mathcal{L}$ is the dual line bundle to the Hodge bundle.

$\mathcal{M}_{\mathbb{C}}$ versus moduli space of CY metrics on X

Although we are interested in counting all vacua, we are going to fix:

- The topological type of $X-$ later we can sum the results over all topological types (at this time, not known to be finite!);
- The Kähler class of the Ricci flat metric. Once this is fixed, there is a unique Ricci flat metric for each complex structure, hence $\mathcal{M}_{\mathbb{C}}$ parametrizes CY metrics with a fixed Kähler class.
- (It is not good to fix the Kähler class, but dealing with Kähler moduli is an open problem even for physicists).

Vacua are local minima of the energy functional

Given $(\mathcal{M}, \mathcal{L}, W)$, the energy functional is defined by

$$V_W(Z) = ||\nabla W(Z)||^2 - 3||W(Z)||^2.$$

Here, ∇ is the connection on $\mathcal{L}$ determined by the Weil-Petersson metric (definitions later).

The vacua of string/M theory (compactified on a CY manifold) are the local minima Z of this energy functional.

Thus: We want to count critical points of V_W.

Supersymmetric vacua

A supersymmetric vacuum is a special critical point Z where

$$\nabla W(Z) = 0.$$

There are also non-supersymmetric vacua, but for simplicity we only consider SUSY vacua.

Thus, we only count solutions of $\nabla W(Z) = 0$.

What determines the superpotential W?

So far we have specified the low energy SUGRA theory as $(\mathcal{M}, \mathcal{L}, W)$, where W is a holomorphic section of L.

But W is not uniquely specified by string theory. So actually we have to count all critical points of V_W for all W can be superpotentials.

How many such W are there?

W lies in the lattice of quantized (= integral) flux superpotentials

There is a quantization condition that constrains superpotentials: they correspond to integral co-cycles ('fluxes')

$$G = F + iH \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}).$$

Such a G defines a section W_G of $\mathcal{L} \to \mathcal{M}$ by:

$$\langle W_G(z, \tau), \Omega_z \rangle = \int_X [F + \tau H] \wedge \Omega_z.$$

Thus, $G \to W_G$ maps

$$H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}) \to H^0(\mathcal{M}, \mathcal{L}).$$

Here, for $z \in \mathcal{M}_{\mathbb{C}}$, Ω_z is a holomorphic family of holomorphic $(3, 0)$ forms on X.

Spaces of superpotentials

The lattice of flux superpotentials thus ha the form

$$\mathcal{F}_{\mathbb{Z}} = \{W_G, G \in H_3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})\}.$$

The lattice spans the space of complex flux superpotentials by

$$\mathcal{F} = \mathcal{F}_{\mathbb{Z}} \otimes \mathbb{C}.$$

It is a complex vector space of dimension $b_3 = \dim H^3(X, \mathbb{C})$.

Thus,

$$\mathcal{F}_{\mathbb{Z}} \subset \mathcal{F} \subset H^0(\mathcal{M}, \mathcal{L}).$$

Tadpole constraint

There is one more constraint on the flux superpotentials, called the tadpole constraint. It has the form:

$$(1) \qquad \int_X F \wedge H \leq L \iff Q[F + iH] = \leq L$$

where Q is the indefinite quadratic form on $H^3(X, \mathbb{C})$ defined by

$$Q(\varphi_1, \varphi_2) = \int_X \varphi_1 \wedge \overline{\varphi}_2.$$

Q is basically the Hodge-Riemann bilinear form. L is fixed by the topology of X, but we will just think of it as a large parameter.

Important observation: Q is indefinite.

Hence, the relevant superpotentials are lattice points in the hyperbolic shell (1).

Precise statement of the vacuum counting problem

In summary: we want to count all (G, Z) such that:

- $\nabla_{WP} W_G(Z) = 0$, where $\nabla = \nabla_{WP}$ is the Weil-Petersson covariant derivative on $H^0(\mathcal{M}, \mathcal{L})$ arising from ω_{WP}.

- $G = F + iH$ with F, H a pair of integral 3-cycles;

- F, H satisfy the tadpole constraint $\int F \wedge H \leq L$, where L is a parameter called the tadpole constant.

Is the number of vacua finite?

Not yet proved!

Actually we are going to count vacua (G, Z) when Z lies in a fixed compact subset $\mathcal{K} \subset \mathcal{M}$ of the configuration space.

M. R. Douglas' statistical program (Ashok, Denef, Shiffman, Z and others)

Clearly, 10^{500} is just a vague estimate. How many vacua are there actually? How does one count them?

1. Count the number of critical points of all integral flux superpotentials W_G with $Q[G] \leq L$.

2. Find out how they are distributed in $\mathcal{M}$.

3. How many are consistent with the standard model and the known cosmological constant?

Mathematical problem

Given $L > 0$, consider lattice points

$$G = F + iH \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$$

in the hyperbolic shell

$$0 \leq Q[G] \leq L.$$

Let $\mathcal{K} \subset \mathcal{M}$ be a compact subset of moduli space. Count number of critical points in $\mathcal{K}$ for G in shell:

$$\mathcal{N}^{\text{crit}}_{\mathcal{K}}(L) = \sum_{Q[G] \leq L} \#\{(z, \tau) \in \mathcal{K} : \nabla W_G(z, \tau) = 0\}.$$

Problem Determine $\mathcal{N}^{\text{crit}}_{\mathcal{K}}(L)$ where L is the tadpole number of the model. Easier: determine its asymptotics as $L \to \infty$.

Distribution of vacua

More generally, define

$$N_\psi(L) = \sum_{G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}) : H[G] \leq L} \langle C_G, \psi \rangle,$$

where

$$\langle C_G, \psi \rangle = \sum_{(z,\tau) : \nabla G(z,\tau) = 0} \psi(G, z, \tau).$$

Here, $\psi(G, z, \tau)$ is a smooth function with compact support in $(z, \tau) \in \mathcal{M}$ and polynomial growth in G.

Example: Cosmological constant $\psi(G, z, \tau) = \{|\nabla W_G(\tau, z)|^2 - 3|W_G(\tau, z)|^2\}\chi(z, \tau)$, with $\chi \in C_0^\infty(\mathcal{M})$.

Problem Find $N_\psi(L)$ as $L \to \infty$.

Main result

Theorem **1** *If ψ vanishes near the discriminant variety of sections with a degenerate critical point, then*

$$\mathcal{N}_{\psi}(L) = L^{b_3}\left[\int_{\{Q[W]\leq 1\}}\langle C_W,\psi\rangle\, dW + O\left(L^{-\frac{2b_3}{2b_3+1}}\right)\right].$$

Here, $b_3 = \dim H_3(X,\mathbb{C})$, *integral is hyperbolic shell in* $\mathcal{F}$*.*

Further results:

1. Let $\psi = \chi_{\mathcal{K}}$. Same principal term, but $O(L^{b_3-1})$ remainder.

2. Similarly if we drop assumption $\operatorname{Supp}\psi \cap \mathcal{D} = \emptyset$.

Principal term

The principal coefficient $\int_{\{Q[W]\leq 1\}}\langle C_W, \psi\rangle\, dW$ can be rewritten as:

$$\int_{\mathcal{M}}\int_{\{Q_{z,\tau}[W]\leq 1\}}|\det D\nabla W(z,\tau)|\psi(W,z,\tau)dW\,dV(z,\tau)$$

Here, $Q_{z,\tau} = Q|_{\mathcal{F}_{z,\tau}}$ where

$$\mathcal{F}_{z,\tau} = \{W : \nabla W(z,\tau) = 0\}.$$

$Q_{z,\tau} >> 0$ by *special geometry* of $\mathcal{M}$. $dV(z,\tau)$ is a certain volume form on $\mathcal{M}$.

Example: $\mathcal{M}_{\mathbb{C}} = \{pt\}$

Then $\mathcal{M} = \mathcal{H}$. A flux is then $A + \tau B$ with $A = a_1 + ia_2,\ B = b_1 + ib_2 \in \mathbb{Z} + \sqrt{-1}\,\mathbb{Z}$. The critical point equation is

$$\nabla W_{A,B} = 0 \iff A + \tau B = 0 \iff \tau = -\frac{A}{B}.$$

Each flux superpotential $W_{A,B}$ has a unique critical point in $\mathcal{H}$, which may or may not lie in the fundamental domain $\mathcal{M}$. Each $SL(2,\mathbb{Z})$-orbit of fluxes (or superpotentials) contains a unique element whose critical point lies in $\mathcal{M}$.

Thus, counting critical points is equivalent to counting $SL(2,\mathbb{Z})$ orbits of superpotentials satisfying the tadpole constraint.

Example: Counting solutions in $\mathcal{H}/SL(2,$

The pair (A, B) corresponds to the element $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in GL(2, \mathbb{Z})$ and the Hodge-Riemann form quadratic form may be identified with the indefinite quadratic form

$$Q[(A, B)] = a_1 b_2 - b_2 a_1$$

on $\mathbb{R}^4$. Thus, the set of superpotentials satisfying the tadpole constraint is parametrized by:

$$\left\{ \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in GL(2, \mathbb{Z}) : 0 < \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \leq L \right\},$$

and we want to count the number of $SL(2, \mathbb{Z})$-orbits in this set = determining the average order of the classical divisor function $\sigma(m)$:
(2)

$$\mathcal{N}^{\text{crit}}(L) = \sum_{m=1}^{L} \sum_{k|m} k = \sum_{m=1}^{L} \sigma(m) \sim \frac{\pi^2}{12} L^2 + O(L \log L$$

The critical points are uniformly distributed relative to the hyperbolic area form.

Lattice points + Critical points

Mathematically, the vacuum statistics problem is thus a hybrid problem combining

- Counting fluxes satisfying tadpole constraint = equidistribution of lattice points in hyperbolic shells;

- Counting and distribution of critical points of holomorphic sections.

The physicist's 10^{500} mainly comes from counting fluxes. But counting critical points per flux is also very important.

Why isn't the number of critical points of a holomorphic section a topological invariant?

Let $(L, h) \to M$ be a Hermitian holomorphic line bundle over a complex manifold M, and let $\nabla = \nabla_h$ be its Chern connection.

A critical point of a holomorphic section $s \in H^0(M, L)$ is defined to be a point $z \in M$ where $\nabla s(z) = 0$, or equivalently, $\nabla' s(z) = 0$.

In a local frame e critical point equation for $s = fe$ reads:

$$\partial f(w) + f(w) \partial K(w) = 0,$$

where $K = -\log ||e(z)||_h$.

Critical points depend on the metric and section!

The critical point equation is only C^∞ and not holomorphic since K is not holomorphic. Hence it cannot be computed as a Chern class.

Moreover, the number of critical points depends on the section.

The number of critical points of a section s is a very non-trivial random variable.

How we count critical points

Recall that we want to determine

$$N_\psi(L) = \sum_{G \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}) : H[G] \leq L} \langle C_G, \psi \rangle,$$

where

$$\langle C_G, \psi \rangle = \sum_{(z,\tau) : \nabla G(z,\tau) = 0} \psi(G, z, \tau)$$

as $L \to \infty$.

Strategy:

1. First show that the sum over lattice points is asymptotically equivalent as $L \to \infty$ to integrating over the continuum hyperbolic shell $Q[G] \leq L$ of a function with compact support.

2. The integral is similar to the expected number of critical points of a Gaussian random holomorphic section.

Step 1: Radial projection of lattice points

Since $\langle C_G, \psi \rangle$ is homogeneous of degree 0 in G, the lattice point G can be radially projected to the hyperboloid $Q[G] = 1$.

Model problem: take an indefinite quadratic form Q and consider lattice points inside a hyperbolic shell $0 \leq Q \leq L$ which lie inside a proper subcone of the lightcone $Q = 0$. Project onto the hyperboloid $Q = 1$ and measure equidistribution.

Easier (elliptic) model problem: Project lattice points in $\mathbb{R}^m$ in a ball of radius L onto the surface of the sphere. How quickly do they become equidistributed on the sphere (with its usual area form)?

Example of projecting lattice points

Project lattice points $|k| \leq L$ in $\mathbb{R}^3$ onto the surface of the unit sphere S^2. How quickly do they become equidistributed in a region $\Omega \subset S^2$?

Clearly, this involves the number which project over the boundary $\partial\Omega \subset S^2$. E.g. consider a polar cap of angle 45^o. The lattice points projecting to the boundary are Pythagorean triples $a^2+b^2=c^2$ with $|(a,b,c)| \leq L$. It is known since antiquity that there are $O(L \log L)$ of these.

But imagine projecting over the northern hemisphere. Then all lattice points in the $x-y$ plane pile up on the boundary, so there are $O(L^2)$ of these.

We are summing a smooth function over projected lattice points rather than a characteristic function, and get a remainder in between.

Radial projection of lattice points

Let $Q \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded, smooth, strictly convex set with $0 \in Q^\circ$. Let $|X|_Q$ denote the norm of $X \in \mathbb{R}^n$ given by

(3) $$Q = \{X \in \mathbb{R}^n : |X|_Q < 1\}\,.$$

To measure the equidistribution of projections of lattice points, we consider the sums

$$S_f(t) = \sum_{k \in \mathbb{Z}^n \cap tQ \{0\}} f\left(\frac{k}{|k|_Q}\right), \quad \text{with } \ f \in C^\infty(\partial Q),\ t >$$

We extend f to $\mathbb{R}^n$ as a homogeneous function of degree 0, so that $f(k) = f\left(\frac{k}{|k|_Q}\right)$.

Theorem **2**

$$S_f(t) = t^n \int_Q f\, dX + O(t^{n-\frac{2n}{n+1}}),\ \ t \to \infty.$$

Back to flux superpotentials: Discrete to continuous fluxes

Theorem **3** *If ψ vanishes near the discriminant variety of sections with a degenerate critical point, then*

$$\mathcal{N}_\psi(L) = L^{b_3}\left[\int_{\{Q[W]\le 1\}} \langle C_W, \psi\rangle \, dW + O\left(L^{-\frac{2b_3}{2b_3+1}}\right)\right].$$

Here, $b_3 = \dim H_3(X, \mathbb{C})$, *integral is hyperbolic shell in* $\mathcal{F}$.

Further results:

1. Let $\psi = \chi_{\mathcal{K}}$. Same principal term, but $O(L^{b_3-1})$ remainder.

2. Similarly if we drop assumption $\mathrm{Supp}\,\psi \cap \mathcal{D} = \emptyset$.

Discrete to continuous fluxes

Thus, counting critical points of integral flux superpotentials is asymptotically reduced to the continuous analogue:

$$L^{b_3} \int_{\{Q[W] \leq 1\}} \langle C_W, \psi \rangle \, dW$$

where as above

$$(**) \langle C_W, \psi \rangle = \sum_{Z : \nabla W(Z) = 0} \psi(Z).$$

For CY manifolds, $b_3 \sim 300$ quite often and $L \to 100$ and the number of critical points resembles the famous guess of 10^{500}. But before we declare victory and withdraw our troops, we need to understand the coefficient (**).

Gaussian principal term

The principal coefficient $\int_{\{Q[W]\leq 1\}} \langle C_W, \psi\rangle \, dW$ can be rewritten as:

$$\int_{\mathcal{M}} \int_{\{Q_{z,\tau}[W]\leq 1\}} |\det D\nabla W(z,\tau)| \psi(W,z,\tau) dW \, dV(z,\tau)$$

This is like the density of critical points of Gaussian random superpotentials in the space

$$\mathcal{F}_{z,\tau} = \{W : \nabla W(z,\tau) = 0\}$$

with inner product $Q_{z,\tau} = Q|_{\mathcal{F}_{z,\tau}}$.

It is Gaussian because $Q_{z,\tau} >> 0$ by *special geometry* of $\mathcal{M}$. $dV(z,\tau)$ is a certain volume form on $\mathcal{M}$.

[More precisely, the two ensembles are equivalent by writing integrals in polar coordinates].

Estimates of the integral

The integral is very difficult to estimate because it is the integral of a homogeneous function of degree b_3 over a space of dimension b_3 and $b_3 \sim 500$.

Conjecture **4** *If* $L \sim Cb_3$*, then*

$$L^{b_3} \int_{\{Q[W] \leq 1\}} \langle C_W, \psi \rangle \, dW \sim C e^{cb_3}$$

That is, the total number of critical points for all flux superpotentials satisfying the tadpole constraint should have exponential growth in $b_3(X)$, i.e. in $\dim \mathcal{M}$.

The universe as a spin glass

"Spin glasses" are physical systems represented by random Hamiltonians with an exponential growth e^{CN} rate in the expected number of critical points as the dimension N of the configuration space grows. Local minima of the Hamiltonian are called 'metastable states' (Parisi et al).

There is an analogy between counting vacua and counting metastable states, although $b_3(X)$ doesn't have the same meaning as N.

But both theories give rise to very similar integrals (cf. Bray-Moore, Parisi et al, Y. Fyodorov)

Index integral

One piece of evidence for the conjecture: it works if you drop the absolute value. Let $\psi = \chi_K$ and let $\mathcal{I}nd_{\chi_K}(L)$ be like N_ψ but drop the absolute value.

Theorem **5** *(Ashok-Douglas, Douglas-Shiffman-Z) Let* K *be a compact subset of* $\mathcal{C}$ *with piecewise smooth boundary. Then*

$$\mathcal{I}nd_{\chi_K}(L) \;=\; \frac{(\pi L)^{b_3}}{b_3!}[\int_K \det\left(-R - \omega_{WP}\otimes I\right)$$

$$+O\left(L^{-1/2}\right)],$$

where $R = \sum_{ij} R^{\bar{\ell}}_{i\bar{j}k} dz^i \wedge d\bar{z}^{\bar{j}}$ is the curvature $(1,1)$ form of $T^{*(1,0)}(\mathcal{C})$ regarded as an $m \times m$ Hermitian-matrix-valued 2-form (with $m = \dim \mathcal{C} = b_3/2$) and $\omega_{WP} \otimes I$ is a scalar 2-form times the $m \times m$ identity matrix.

Conjectured integrals over moduli space

Tentative Conjecutre (Z. Lu): the Weil-Petersson volume of a ball in moduli space is bounded above by the volume of a ball of the same in $\mathbb{C}^{b_3/2} \sim \frac{1}{(b_3/2)!}$.

This would imply that $\mathcal{I}nd_{\chi_K}(L) \sim \frac{(C_1 L)^{b_3}}{b_3!}$. If we then take $L \sim C b_3$, the number of vacua in K satisfying the tadpole constraint would grow at an exponential rate in b_3.

Does the integral converge over all of $\mathcal{M}$? Unknown at present!

The integral over $\mathcal{M}$ is the expected value of the sum over critical points of W of the Morse index of critical points of W. Since $\mathcal{M}$ is incomplete and has a horrible boundary, this again is not a topological invariant.

It is finite over a compact subset $\mathcal{K} \subset \mathcal{M}$.

See: Zhiqin Lu for integrals over $\mathcal{M}$.

More evidence for exponential growth

The same order of magnitude arises from a probabalistic argument. Roughly, the integral can be re-written as an integral over a subspace of symmetric matrices $\mathcal{H}_Z$ moving with the point $Z \in \mathcal{M}$ (the Hessians $\nabla^2 W(Z)$ of W with a critical point at Z).

If $\mathcal{H}_Z$ moves uniformly around in complex Grassmannian of $b_3/2 - 1$ dimensional complex subspaces of the space of all symmetric matrices, then we can evaluate the integral.

It also has exponential growth in b_3.

How accurate is 10^{500} ?

- It doesn't take into account the number of topological types of CY 3-folds;

- We are ignoring the remainder term in $N_\psi(L)$, even though $L \sim b_3 \sim 500$ is not that large;

- We only have a heuristic estimate that the leading term of $N_\psi(L)$ is of the form e^{Cb_3} in compact subsets of moduli space, without a good estimate of C;

- We do not yet have an estimate of the integral over all of $\mathcal{M}$; possibly physics will find a constraint to a compact subset K.

- We actually want to count vacua consistent with the standard model, and that complicates the coefficient.

Significance of counting results?

In our view, the methods for counting are more useful than hard estimates of the remainders etc. The specifics of the theory could change while one is making hard estimates.

We want to develop counting methods which can be adapted to any string/M theory vacuum counting problem.

Open problems: Existence problems

Problem **6** *Existence problem: Does string theory contain a vacuum consistent with the standard model, and if so, how many? Does the probabalistic method (counting with remainder estimates) prove the existence of such a vacuum? How large does* L *need to be to ensure that there exists a vacuum with*

$$|W_G(Z)|^2 \leq \lambda_* \tag{4}$$

for a specified λ_* *? In that case, how many such vacua are there?*

Open problems: Finiteness problem

Problem **7** *We calculated the asymptotic number of vacua in a compact subset $\mathcal{K} \subset \mathcal{M}$ of moduli space and found that it is finite. Is the total number of vacua in all of moduli space finite?*

Open problems: Tadpole estimates

Our counting results are asymptotic in L but L is determined by the topology of X. In many examples $L \simeq Cb_3$ with $1/3 \leq C \leq 3$.

Problem **8** *How are the order of magnitudes of* $b_3(X)$ *and* L *of related as* X *varies over topologically distinct Calabi-Yau manifolds?*

Open problems: Estimate of leading coefficient

We have already mentioned the importance of obtaining effective estimates in b_3 of the coefficient in the asymptotics.

Problem **9** *Obtain an effective estimate of* $\mathcal{K}^{\mathrm{crit}}(Z)$ *and of its integral over* $\mathcal{C}$ *in* b_3*. Also, obtain such an estimate of the remainder.*