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Statistics of critical points in complex geometry and supersymmetric vacua in string/M theory

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Joint Work with
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Related: Ashok-Douglas, Denef-Douglas, Bleher

Our topics

- (Mathematics) The statistics of critical points of random holomorphic sections of line bundles over Kähler manifolds.

- (Physics) The vacuum selection problem of string/M theory. M. R. Douglas' program of statistics of vacua.

- (Other physics applications) The statistics of supersymmetric black holes (Ferrara-Gibbons-Kallosh, Strominger).

Small extra dimensions in string/M theory

According to string/M theory, our universe is 10- (or 11-) dimensional. In the simplest model, it has the form $M^{3,1} \times X$ where X is a complex 3-dimensional *Calabi-Yau* manifold.

A CY manifold is a complex manifold with a nowhere vanishing $(3,0)$-form Ω, i.e. type $dz_1 \wedge dz_2 \wedge dz_3$. In each Kähler class it has a unique Ricci flat metric.

Reference: P. Candelas, G. T. Horowitz, A. Strominger, E. Witten, Vacuum configurations for superstrings, Nucl. Ph. B 258 (1985), 46-74.

The vacuum selection problem

The **vacuum selection problem**: Which CY manifold (X, τ) forms the ‘small’ or ‘extra’ dimensions of our universe? How to select the right vacuum? Here, τ is the complex structure on X.

Popular references: Bousso-Polchinski (Sci Am) or B. Greene, Elegant Universe;

Technical: M. R. Douglas, The statistics of string/M theory vacua. J. High Energy Phys. 2003, no. 5, 046.

The string theory landscape

Landscape (L. Susskind) = graph of the *vacuum energy* of a string theory, plotted as a function on the parameter space $\mathcal{M}$ of the 6-dimensional X giving the small dimensions.

$\mathcal{M}$ = moduli space of Calabi-Yau (Ricci-flat Kähler) metrics on X. Often fix Kähler class, then $\mathcal{M}$ = moduli space of complex structures on X.

The string/M vacua are the local minima in the landscape, i.e the local minima of the energy.

Enter Complex geometry

The moduli space of CY metrics on X of fixed Kähler class = moduli space of complex structures on X.

It is a complex manifold of dimension $b = b_{2,1}(X) = \dim H^{2,1}(X)$, i.e. dimension of holomorphic $(2,1)$-forms on X.

It has a Kähler metric, the Weil Petersson metric ω_{WP}. There is a line bundle $\mathcal{L} \to \mathcal{M}$ with $c_1(\mathcal{L}) = -\omega_{WP}$.

The setting of string/M theory (or effective supergravity) is

$$(\mathcal{M}, \mathcal{L}, \omega_{WP}).$$

Physics versus complex geometry

- Vacuum energy at $\tau \in \mathcal{M}$

  $$= ||\nabla_{WP} W(\tau)||^2_{WP} - 3||W(\tau)||^2_{WP}.$$

- W = superpotential, usually flux superpotential $W = \widehat{\gamma}(\tau) = \int_\gamma \Omega_\tau$

- W is a holomorphic section of a line bundle $\mathcal{L} \to \mathcal{M}$.

- $|| \cdot ||_{WP}$ = Weil-Petersson hermitian metric on $\mathcal{L}$; ∇_{WP} = WP connection.

Thus, the setting for the vacuum selection problem (or SUSY black holes) is hermitian holomorphic differential geometry of line bundles over Kähler manifolds.

$\mathcal{L}$ = dual of Hodge bundle

Given a complex structure τ on X, let $H^{3,0}(X_\tau)$ be the space of holomorphic $(3,0)$ forms on X, i.e. type $dz_1 \wedge dz_2 \wedge dz_3$.

On a Calabi-Yau 3-fold, $\dim H^{3,0}(X_\tau) = 1$. Hence, $H^{3,0}(X_\tau) \to \mathcal{M}$ is a (holomorphic) line bundle.

The formula: $\widehat{\gamma}(\Omega_\tau) = \int_\gamma \Omega_\tau$ defines a linear functional on $H^{3,0}(X_\tau)$, so $\widehat{\gamma}$ is a holomorphic section of the line bundle $\mathcal{L}$ dual to $H^{3,0} \to \mathcal{M}$.

Quantized and general flux superpotentials

A quantized flux superpotential is $\widehat{\gamma}(\Omega_\tau) = \int_\gamma \Omega_\tau$ where $\gamma \in H_3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$, i.e. γ is an integral cycle.

Complex superpotentials are complex linear combinations $W = \sum_\alpha N_\gamma \widehat{\gamma}$ of quantized flux superpotentials.

Define the space of flux superpotentials by:

$$\mathcal{F} \subset H^0(\mathcal{M},) = \text{span } \{\widehat{\gamma} : \gamma \in \mathcal{H}_\ni(\mathcal{X}, \mathbb{Z})\}.$$

Vacua as critical points

Recall that the small dimensions must be local mimima of the energy landscape.

The supersymmetric vacua solve

$$\nabla_{WP} W(\tau) = 0, \quad \tau \in \mathcal{M}$$

where W is a flux superpotential. Moreover, the Hessian must be negative definite.

Here, ∇_{WP} is the covariant derivative on $H^0(\mathcal{M}, \mathcal{L})$ arising from ω_{WP}.

The discretuum

- Candidate CY's for small dimension forms a discrete set: union over topological types of CY, union over quantized flux superpotentials, union over critical points of each.

- The the possible small dimensions X thus form a 'discretuum'.

- This term is also used for the values $||W(\tau)||_{WP}$ of the Weil-Petersson norm at these local minima ($=$ cosmological constants).

Tadpole constraint

Additional constraint on flux superpotentials:

$$Q[W] \leq L$$

where Q is a quadratic form, the Hodge-Riemann form corresponding to the intersection form on 3-cycles. It is an indefinite quadratic form on all of complex flux space. However, 'special geometry' shows that if W has a critical point, then $Q[W] \geq 0$.

M. R. Douglas' statistical program

1. Count the number of critical points (local minima) of all flux superpotentials $\widehat{\gamma}$ with $Q[\widehat{\gamma}] \leq L$.

2. Find out how they are distributed in $\mathcal{M}$.

3. How many are consistent with the standard model and the known cosmological constant?

More generally: endow the space $\mathcal{F} \subset H^0(\mathcal{M}, \mathcal{L})$ of superpotentials W with a physically relevant measure and study the statistics of critical points.

Douglas and Denef-Douglas conjecture

Let $\mathcal{N}_{SUSY}(L \leq L_*)$ denote the number of supersymmetric vacua with tadpole constraint $W\eta W \leq L$.

Conjecture **1** *Let* $K = \dim \mathcal{F}$. *Then,*

$$\mathcal{N}_{SUSY}(L \leq L_*) \sim \frac{L_*^{K/2}}{K/2} \mathcal{N}(1),$$

with

$$\mathcal{N}(1) \simeq \int_{\mathcal{M}} d^{2m} z \int d^K W e^{-\frac{1}{2} W \eta W} \delta^{2m}(DW) |\det D^2 W|.$$

Here, η is the intersection form. A more precise version will be stated later.

Discrete vs continuous ensembles of superpotentials

Discrete shell ensembles

Embed $H_3(X, \mathbb{Z}) \to \mathcal{F} \subset H^0(\mathcal{M}, \mathcal{L})$ under $\gamma \to \widehat{\gamma}$ = Lattice of quantized flux superpotentials.

Discrete shell ensemble: Given $L > 0$, put delta-functions at the lattice points $\widehat{\gamma} \in \mathcal{F}_{\mathbb{Z}} \subset \mathcal{F}$ with $Q[\gamma] \leq L$.

Approximation by continuous ensembles

Analysis problem: for large L, this discrete ensemble may be approximated by Lebesgue measure in $\{Q[W] \leq L\} \subset \mathcal{F}$. Further, this may be approximated by a Gaussian ensemble defined by Q on $\mathcal{F}$.

Mathematical problems

The counting of stable string/M vacua has two parts:

1. Prove that the continuous shell or Gaussian ensembles are good approximations (a lattice point problem).

2. Prove statistical results for the continuous ensembles.

At this time, the statistical results for continuous ensembles are mostly done. The approximation is in progress. We concentrate on (2).

General results on critical points of Gaussian random holomorphic sections

Setting:

- A holomorphic line bundle $L \to M$;
- A hermitian metric h on L;
- The Chern connection ∇_h of h;
- The curvature Θ_h of ∇_h.
- An inner product $\langle , \rangle$ on the space $H^0(M, L)$ of holomorphic sections (or on a subspace).
- The Gaussian measure γ associated to $\langle , \rangle$.

Metrics, connections, curvature

A Hermitian metric on L is a family of h_z of hermitian inner products on the lines L_z over $z \in M$. In a local frame $e(z)$, h_z is specified by the positive function $h(z) = ||e(z)||_h$.

Definition: the metric (Chern) connection $\nabla = \nabla_h$ of h is the unique connection preserving the metric h and satisfying $\nabla'' s = 0$ for any holomorphic section s. Here, $\nabla = \nabla' + \nabla''$ is the splitting of the connection into its $L \otimes T^{*1,0}$ resp. $L \otimes T^{*0,1}$ parts.

We denote by Θ_h the curvature of h:

$$\Theta_h = \partial \bar{\partial} K, \qquad K = -\log h.$$

Critical point

Definition: Let $(L, h) \to M$ be a Hermitian holomorphic line bundle over a complex manifold M, and let $\nabla = \nabla_h$ be its Chern connection.

A critical point of a holomorphic section $s \in H^0(M, L)$ is defined to be a point $z \in M$ where $\nabla s(z) = 0$, or equivalently, $\nabla' s(z) = 0$.

We denote the set of critical points of s with respect to the Chern connection ∇ of a Hermitian metric h by $Crit(s, h)$.

Critical points depend on the metric

The set of critical points $Crit(s,h)$ of s, and even its number $\#Crit(s,h)$, depends on ∇_h or equivalently on the metric h.

In a local frame e critical point equation for $s = fe$ reads:

$$\partial f + f \partial K = 0.$$

Recall that $K = -\log h$.

The critical point equation is only C^∞ and not holomorphic since K is not holomorphic.

An equivalent definition of critical point

An essentially equivalent definition: $w \in Crit(s,h)$ if

$$d|s(w)|_h^2 = 0. \tag{1}$$

Since

$$d|s(w)|_h^2 = 0 \iff 0 = \partial|s(w)|_h^2 = h_w(\nabla' s(w), s(w))$$

it follows that $\nabla' s(w) = 0$ as long as $s(w) \neq 0$. So this notion of critical point is the union of the zeros and critical points.

The Morse theory of connection critical points $\nabla s(w) = 0$ is equivalent to the Morse theory of $|s(w)|_h^2$.

Gaussian random holomorphic sections

Now for the statistics. The simplest measures on $H^0(M,L)$ are Gaussian measures. A Gaussian measure γ is induced by an inner product on $H^0(M,L)$.

By definition,

$$(2) \qquad d\gamma(s) = \frac{1}{\pi^d} e^{-\|c\|^2} dc\,, \qquad\qquad s = \sum_{j=1}^{d} c_j e_j,$$

where dc is Lebesgue measure and $\{e_j\}$ is an orthonormal basis basis. We denote the expected value of a random variable X on with respect to γ by $\mathbf{E}_\gamma$.

Hermitian Gaussian measures

These are determined entirely by a Hermitian metric h. The inner product $\langle , \rangle_h$ is induced by a hermitian metric h on L:

$$(3) \qquad \langle s_1, s_2 \rangle_h = \int_M h(s_1(z), s_2(z)) dV(z)$$

on $H^0(M, L)$, where $dV = \frac{\Theta_h^m}{m!}$.

The relevant Gaussian measure for string/M theory is not Hermitian. But Hermitian Gaussian measures are simple models for geometry of critical points.

Two point function

It is the invariant of the Gaussian measure γ on the space $\mathcal{S}$ defined by:

$$\Pi(z,w) = \mathbf{E}_{\gamma}(s(z) \otimes \overline{s}(w)).$$

In the Hermitian Gaussian case, it is the Szegö kernel of $H^0(M,L)$, i.e. the orthogonal projection on the space of holomorphic sections.

In the string/M case, it is $\int_X \Omega_z \wedge \overline{\Omega}_w$ where $z,w \in \mathcal{M}$.

Example: Random $SU(m+1)$ complex polynomials

Definition: $\mathcal{P}_N^m :=$ holomorphic homogeneous polynomials

$$F(z_0, z_1, \ldots, z_m) = \sum_{\alpha \in \mathbb{N}^m : |\alpha| = N} \lambda_\alpha z_0^{\alpha_0} z_1^{\alpha_1} \cdots z_m^{\alpha_m},$$

of degree N in m complex variables with $c_\alpha \in \mathbb{C}$.

Random polynomial: a probability measure on the coefficients λ_α.

Gaussian random:

$$f = \sum_{|\alpha| = N} \lambda_\alpha \sqrt{\binom{N}{\alpha}} z^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \bar{\lambda}_\beta) = \delta_{\alpha\beta}.$$

In coordinates λ_α:

$$d\gamma(f) = \frac{1}{\pi^{k_N}} e^{-|\lambda|^2} d\lambda \text{ on } \mathcal{P}_N^m.$$

Polynomials as sections of powers of the hyperplane section line bundle over $\mathbb{CP}^m$

Recall: $\mathcal{O}(1) \to \mathbb{CP}^m$ is the line bundle whose fiber at a line in $\mathbb{C}^{m+1}$ = linear functions on that line.

Its tensor powers are $\mathcal{O}(N) = \mathcal{O}(1)^N \to \mathbb{CP}^m$.

Holomorphic sections $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ can be identified with space of homogeneous holomorphic polynomials $\mathcal{P}_N^m$: of degree N on $\mathbb{C}^{m+1}$.

Gaussian measure $\iff$ inner product

The Gaussian measure above on polyomials is the Fubini-Study inner product on $\mathcal{P}_N^{\mathbb{C}}$ viewed as sections of $\mathcal{O}(N)$. Indeed,

$$||z^\alpha||_{FS} = \binom{N}{\alpha}^{-1/2}, \;\; \langle z^\alpha, z^\beta \rangle = 0, \; \alpha \neq \beta.$$

Namely,
$||F||_{FS}^2 = \int_{S^{2m+1}} |F|^2 \, d\sigma$, (Haar measure).

Thus, the same ensemble could be written:

$$F = \sum_{|\alpha|=N} \;\lambda_\alpha \;\; \frac{z^\alpha}{||z^\alpha||_{FS}},$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \overline{\lambda}_\beta) = \delta_{\alpha\beta}.$$

Statistics of critical points I: density

Distribution of critical points of a fixed section s with respect to a connection ∇ is the measure

$$C_s^{\nabla} := \sum_{z \in Crit(s,\nabla)} \delta_z, \tag{4}$$

where δ_z is the Dirac point mass at z. When $\nabla = \nabla_h$ we write C_s^h.

Definition: The (expected) density of critical points of $s \in \mathcal{S} \subset H^0(M,L)$ with respect to ∇ and a Gaussian measure γ is defined by

$$K^{\mathrm{crit}}(z)\, dV(z) = \mathbf{E}_{\gamma} C_s^{\nabla}\ ,$$

i.e.,

$$\int_M \varphi(z) K^{\mathrm{crit}}(z)\, dV(z) = \int_{\mathcal{S}} \left[\sum_{z: \nabla s(z) = 0} \varphi(z) \right] d\gamma(s).$$

Expected number of critical points

Our key new invariant is:

Definition: The expected number of critical points of a Gaussian random section is defined by

$$\begin{aligned}\mathcal{N}^{crit}(\nabla,\gamma) &= \int_M K^{\text{crit}}(z)\,dV(z)\\ &= \int_{\mathcal{S}} \#Crit(s,\nabla) d\gamma(s).\end{aligned}$$

For Hermitian Gaussian measures, where γ comes from the inner product $\langle,\rangle_h$, $\mathcal{N}^{crit}(h,\gamma)$ is a purely metric invariant of a line bundle.

Our problems, precisely stated

We would like to estimate $\mathcal{N}^{crit}(h, \gamma)$ in the string/M problem. But we know little about it a priori, even for the Hermitian Gaussian measure:

1. Even for the Hermitian Gaussian measure $\gamma = \gamma_h$, how does $\mathcal{N}^{crit}(h)$ depend on h? Does it in fact depend on h, or is it a topological invariant?

2. If $\mathcal{N}^{crit}(h)$ depends on h, which h gives 'lots' of critical points to average sections? Which gives the fewest?

3. How are local minima distributed?

General formula for density critical points

We denote by Sym$(m,\mathbb{C})$ the space of complex $m\times m$ symmetric matrices. In well-chosen local coordinates $z=(z_1,\dots,z_m)$, in a local frame e, we have:

Theorem **1** *Fix* ∇,γ. *Then there exist positive-definite Hermitian matrices*

$A(z):\mathbb{C}^m\to\mathbb{C}^m$,

$\Lambda(z):\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}\to \mathit{Sym}(m,\mathbb{C})\oplus\mathbb{C}$, *s.th.*

$$\mathcal{K}^{\mathrm{crit}}_{\nabla,\gamma}(z)=\frac{1}{\det A(z)\det\Lambda(z)}\times\int_{\mathbb{C}}\int_{\mathrm{Sym}(m,\mathbb{C})}$$

$$|\det\begin{pmatrix} H' & x\,\Theta(z)\\ \bar{x}\,\bar{\Theta}(z) & \bar{H}'\end{pmatrix}|\,e^{-\langle\Lambda(z)^{-1}(H'\oplus x),\,H'\oplus x\rangle}\,dH'\,dx\,.$$

Formulae for $A(z)$ and $\Lambda(z)$

$A(z)$ and $\Lambda(z)$ depend only on ∇ and on the two-point function $\Pi_{\mathcal{S}}(z,w)$ of γ. Let $F_{\mathcal{S}}(z,w)$ be the local expression for $\Pi_{\mathcal{S}}(z,w)$ in the frame e_L. Then $\Lambda = C - B^* A^{-1} B$, where

$$\begin{array}{l}
A = \left(\frac{\partial^2}{\partial z_j \partial \bar{w}_{j'}} F_{\mathcal{S}}(z,w)|_{z=w} \right), \\
B = \left[\left(\frac{\partial^3}{\partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}}) F_{\mathcal{S}}|_{z=w} \right) \quad \left((\frac{\partial}{\partial z_j} F_{\mathcal{S}}|_{z=w} \right) \right], \\
C = \left[\begin{array}{cc} \left(\frac{\partial^4}{\partial z_q \partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}}|_{z=w} \right) & \left(\frac{\partial^2}{\partial z_j \partial z_q} F_{\mathcal{S}} \right) \\ \left(\frac{\partial^2}{\partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}} \right) |_{z=w} & F_{\mathcal{S}}(z,z) \end{array} \right], \\
1 \le j \le m\,, 1 \le j \le q \le m\,, 1 \le j' \le q' \le m\,.
\end{array}$$

In the above, A, B, C are $m \times m$, $m \times n$, $n \times n$ matrices, respectively, where $n = \frac{1}{2}(m^2 + m + 2)$.

Comments

- This follows from general results of the authors with P. Bleher.

- For real Gaussian random functions on $\mathbb{R}^n$ for $n \leq 3$, this kind of formula was stated by Rice (1940), Halperin (1960), Hammersley (1965), Szalay et al (1985). There are also formulae for correlations between critical points.

- The absolute value $|\det \begin{pmatrix} H' & x\,\Theta(z) \\ \bar{x}\,\bar{\Theta}(z) & \bar{H}' \end{pmatrix}|$ makes this a difficult formula. Wick's formula doesn't apply. But there is an Itzykson-Zuber type version which simplifies it to a contour integral.

Positive/negative line bundles

These are the simplest bundles. The string/M line bundle is negative.

In a local frame e, the hermitian metric is a positive function $h(z) = ||e||_z$.

The curvature form is defined locally by

$$\Theta_h = \partial\bar{\partial} K, \qquad K = -\log h.$$

The bundle is called **positive** (resp. **negative**) if Θ_h is a positive (resp. negative) $(1,1)$ form.

Given one positive metric h_0 on L, the other metrics have the form $h_\varphi = e^\varphi h$ and $\Theta_h = \Theta_{h_0} - \partial\bar{\partial}\varphi$, with $\varphi \in C^\infty(M)$.

Positive/Negative line bundles

In these cases, we can simplify a bit:

Corollary **2** *Let* $(L, h) \to M$ *denote a positive or negative holomorphic line bundle. Give* M *the volume form* $dV = \frac{1}{m!}\left(\pm\frac{i}{2}\Theta_h\right)^m$ *induced from the curvature of* L*. Let* $\nabla = \nabla_h$*. Then*

$$K^{\text{crit}}_{\nabla,\mathcal{S}}(z) = \frac{1}{\det A \det \Lambda} \int_{\text{Sym}(m,\mathbb{C})\times\mathbb{C}} \left|\det(H'H'^* - |x|^2 I)\right|$$

$$|e^{-\langle \Lambda(z)^{-1}(H',x),(H',x)\rangle}\, dH'\, dx\,.$$

Here, $H' \in Sym(m, \mathbb{C})$ is a complex symmetric matrix, and the matrix Λ is a Hermitian operator on the complex vector space $Sym(m, \mathbb{C}) \times \mathbb{C}$.

Density of critical points on Riemann surfaces

Put:

$$Q = \begin{pmatrix} -1 & 0 \\ \\ 0 & 1 \end{pmatrix}$$

and denote the eigenvalues of $\Lambda(z)Q$ by μ_1, μ_2. We observe that μ_1, μ_2 have opposite signs since $\det Q\Lambda = -\det \Lambda < 0$. Let $\mu_2 < 0 < \mu_1$.

Theorem **3** *let $(L, h) \to M$ be a positive or negative Hermitian line bundle on a (possibly non-compact) Riemann surface M with volume form $dV = \pm \frac{i}{2}\Theta_h$. Then:*

$$K_h^{\mathrm{crit}}(z) = \frac{1}{\pi A(z)} \frac{\mu_1^2 + \mu_2^2}{|\mu_1| + |\mu_2|}\ ,.$$

Hermitian Gaussian measure on positive/negative line bundle

In this case:

$$\mathcal{N}^{\mathrm{crit}}(h) = \int_M \{ \frac{1}{\det A \det \Lambda} \int_{\mathrm{Sym}(m,\mathbb{C}) \times \mathbb{C}}$$

$$\left| \det(H' H'^* - |x|^2 I) \right| e^{-\langle \Lambda(z)^{-1}(H',x),(H',x) \rangle} \, dH' \, dx \} dV_h \, .$$

Here, Λ, A depend only h, in fact on the Szegö kernel (orthogonal projector to $H^0(M, L)$.

This is the simplest setting to explore the geometry of critical points.

Semi-classical asymtotics

The formula for $\mathcal{N}(h)$ simplifies in the semi-classical limit. Here, we replace L by its tensor power $L^{\otimes N}$ and let $N \to \infty$.

We define $K_N^{\text{crit}}(z)$ to be the density of critical points of Gaussian random sections in $H^0(M, L^N)$ w.r.t. the metric h^N.

In the case of projective space, this amounts to studying the expected number of critical points of a polynomial of degree N (in the metric Fubini-Study sense!) and letting $N \to \infty$.

Semiclassical asymptotics of the critical point density

Theorem **4** *For any positive Hermitian line bundle $(L, h) \to (M, \omega)$ over any Kähler manifold, the critical point density relative to the curvature volume form has an asymptotic expansion of the form*

$$N^{-m} K_N^{\text{crit}}(z) \sim \Gamma_m^{\text{crit}} + a_1(z)N^{-1} + a_2(z)N^{-2} + \cdots ,$$

where Γ_m^{crit} is a universal constant depending only on the dimension m of M, and the a_j are curvature invariants of h.

Thus, critical points are uniformly distributed relative to the curvature volume form in the $N \to \infty$ limit. [Curvature causes sections to oscillate more rapidly, so critical points concentrate where the curvature concentrates.]

Universal limit theorem

The leading coefficient depends only in the dimension:

Corollary **5** *The expected total number of critical points on M is*

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!}\Gamma_m^{\mathrm{crit}}\; c_1(L)^m\, N^m + O(N^{m-1})\,.$$

The leading constant in the expansion is given by the integral formula

$$\Gamma_m^{\mathrm{crit}} = \left(2\pi^{\frac{m+3}{2}}\right)^{-m} \int_0^{+\infty} \int_{\mathrm{Sym}(m,\mathbb{C})} |\det(SS^* - tI)|$$

$$e^{-\frac{1}{2}\|S\|^2_{\mathrm{HS}} - t}\, dS\, dt\,,$$

As we will see, the leading order constant is larger than 1, so positive curvature causes polynomials of degree N to have substantially more critical points than in the classical flat sense of $dF = 0$.

Number of critical points on Riemann surfaces

Corollary **6** *For the case where M is a Riemann surface, we have $\Gamma_1^{\text{crit}} = \frac{5}{3\pi}$, and hence the expected number of critical points is $\mathcal{N}(h^N) = \frac{5}{3}c_1(L)N + O(\sqrt{N})$. The expected number of saddle points is $\frac{4}{3}N$ while the expected number of local maxima is $\frac{1}{3}N$.*

There are $\sim N$ critical points of a polynomial of degree N in the classical sense, all of which are saddle points. There are an extra $\frac{1}{3}N$ saddles cancelled by an extra $\frac{1}{3}N$ local maxima.

Exact formula on $\mathbb{CP}^1$

Theorem **7** *The expected number of critical points of a random section* $s_N \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$ *(with respect to the Gaussian measure on* $H^0(\mathbb{CP}^1, \mathcal{O}(N$ *induced from the Fubini-Study metrics on* $\mathcal{O}(N)$ *and* $\mathbb{CP}^1$*) is*

$$\frac{5N^2 - 8N + 4}{3N - 2} = \frac{5}{3}N - \frac{14}{9} + \frac{8}{27}N^{-1} \cdots .$$

Of course, relative to the flat connection d/dz the number is $N - 1$.

Asymptotic expansion for number of critical points

We can calculate the first three terms in the expansion of the number of critical points for (L^N, h^N):

Theorem **8**

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!}\Gamma_m^{\text{crit}}\, c_1(L)^m\, N^m + \int_M \rho dV_\omega N^{m-1}$$

$$+C_m \int_M \rho^2 dV_\Omega N^{m-2} + O(N^{m-3})\,.$$

The first two terms are topological invariants of a positive line bundle, i.e. independent of the metric! (Both are Chern numbers of L).

But $C_m > 0$ (by a difficult computer calculation. In fact, we only proved $C_m > 0$ for dimensions ≤ 5 but we expect the same in all dimensions. The proof is just a matter of computer time).

Asymptotically minimal number of critical points

Question Which hermitian metrics minimize the expected number of critical points? These would be ideal for vacuum selection.

I.e. let $L \to (M, [\omega])$ have $c_1(L) = [\omega]$, and consider the space of Hermitian metrics h on L for which the curvature form is a positive $(1,1)$ form:

$$P(M, [\omega]) = \{h : \frac{i}{2}\Theta(h) \text{ is a positive } (1,1)- \text{ form } \}.$$

Definition: We say that $h \in P([\omega])$ is asymptotically minimal if

(5)

$$\exists N_0 : \forall N \geq N_0, \ \mathcal{N}(h^N) \leq \mathcal{N}(h_1^N), \quad \forall h_1 \in P([\omega]).$$

Calabi extremal metrics are asymptotic minimizers

Theorem **9** *Let $L \to M$ be a positive line bundle. Then the Calabi extremal hermitian metrics on L are the unique minimizers of the metric invariant $\mathcal{N}(h^N)$ = average number of critical points for K^N.*

From the expansion

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!}\Gamma_m^{\text{crit}}\, c_1(L)^m\, N^m + \int_M \rho dV_\omega N^{m-1}$$

$$+C_m \int_M \rho^2 dV_\Omega N^{m-2} + O(N^{m-3})\,.$$

we see that the metric with asymptotically minimal $\mathcal{N}(h^N)$ is the one with minimal $\int_M \rho^2 dV_\omega$.

E.g. for the canonical bundle, Kähler -Einstein metrics are asymptotic minimizers of the functional $\mathcal{N}(h^N)$.

Applications to string/M theory

In the string/M theory problem:

- $M = \mathcal{M}$, the moduli space of CY metrics on X.

- $L = \mathcal{L}$, the dual of $H^{3,0}X$.

- h is the Weil-Petersson hermitian metric on $\mathcal{L}$, $h_\tau(\Omega, \Omega) = \int_X \Omega_\tau \wedge \overline{\Omega}_\tau$.

- We restrict to the flux superpotential subspace $\mathcal{F} \subset H^0(\mathcal{M}, \mathcal{L})$ spanned by $\widehat{\gamma}, \gamma \in H_3(X, \mathbb{Z})$.

- The 'inner product' is the Hodge-Riemann form $Q(\varphi, \psi) = i^3 \int_X \varphi \wedge \overline{\psi}$ on $H^3(X, \mathbb{C})$.

Problem

Problem *Count the total number* $\mathcal{N}_{susy}(L)$ *of critical points* $\nabla_{WP} N(J) = 0$ *in* $\mathcal{M}$ *as* N *ranges over quantized flux superpotentials satisfying the tadpole constraint: i.e.*

$$\mathcal{F}_{\mathbb{Z},L} = \{N \in \mathcal{F}_{\mathbb{Z}} : Q[N] \leq L\}$$

and J *ranges over* $\mathcal{M}$*, or the number* $\mathcal{N}_{susy}(L; B)$ *in a given compact subset* $B \subset \mathcal{M}$*. Find the density of such critical points in* $\mathcal{M}$*.*

Discrete shell ensemble

Let $d\mu_L =$ (un-normalized) measure on $H^3(X, \mathbb{C})$ obtained by putting delta-functions (point masses) at the lattice points $N \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$ satisfying $0 \leq H[N] \leq L$ (= the *discrete shell ensemble of height* L).

We are interested in:

$\int_{\mathcal{M}} \psi(\tau) \mathcal{K}^{crit}_{\mu_L}(\tau)$:

$= \sum_{N \in H^3(X, \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}) : H[N] \leq L} \langle C_N, \psi \rangle.$

Here,

$$\langle C_N, \psi \rangle = \sum_{\tau : \nabla N(\tau) = 0} \psi(\tau).$$

Lebesgue shell ensemble

Let $d\mu_L^c$ = Lebesgue measure on $W \in \{0 \leq H[W] \leq L\} \subset \mathcal{F}$.

Definition: The distribution of critical points with respect to $d\mu_L^c$ *is defined by*

$$\int_{\mathcal{M}} \psi \mathcal{K}^{\text{crit}}_{\mu_L^c}(\tau) :$$

$$= \int_{0 \leq H[W] \leq L} \left\{ \sum_{\tau \in \mathcal{M} : \nabla_{WP} W(\tau) = 0} \psi(\tau) \right\} dW.$$

Conjecture

Conjecture **10** *Let $\Omega \subset \mathcal{M}$ be a bounded smooth domain with $\partial\Omega \subset int(\mathcal{M})$. Let $\psi \in C(\Omega)$ Then, asymptotically as $L \to \infty$, there is a limit density k_{∞}^{crit} (computable from the Gaussian density) such that*

$$\int_{\Omega} \psi \mathcal{K}^{\text{crit}}_{\mu_L^c} = L^{b_3} \int_{\Omega} \psi \mathcal{K}^{crit}_{\infty}$$

$$\int_{\Omega} \psi \mathcal{K}^{\text{crit}}_{\mu_L} = \int_{\Omega} \psi \mathcal{K}^{\text{crit}}_{\mu_L^c} + O(L^{b_3 - 1}), \text{ as } L \to \infty.$$

Here, $b = \dim H^3(X, \mathbb{C})$.

Also need: dependence of O on geometry. Really estimate number of physically realistic vacua.

See F. Denef and M. R. Douglas, Distributions of flux vacua, hep-th/0404116 for the conjecture, examples, calculations...

Difficulties

This approximates lattice point sums in dilating domains by volume measure. Problems:

1. Q is indefinite.

2. We are summing a non-smooth function over lattice points. The function $f_\psi(W) = \sum_{\tau:\nabla W(\tau)=0} \psi(\tau)$ is not even continuous, and the integral over W involves $|\det D\nabla W(\tau)|$, which is not smooth.

Status of conjecture

- Indefiniteness of Q is cured by special geometry: Q is positive on each subspace $\mathcal{F}_\tau = \{W : \nabla W(\tau) = 0\}$. I.e. as τ varies over $\mathcal{M}$, $\mathcal{F}_\tau$ varies in $Q > 0$. But Q might degenerate as $\tau \to \partial\mathcal{M}$.

- $k^{crit}(\tau) = \mathbf{E}[|\det D\nabla W(\tau)| \; \nabla W(\tau) = 0]$, and the conditional ensembles for fixed τ are nice Gaussian ensembles.

- For black hole counting, the integrand of k^{crit} is smooth, so the conjecture is probably true (in progress). Estimate in progress in string/M theory.