ICM 2014: The Structure and Meaning of Ricci Curvature

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Outline of Talk

- Background of Ricci Curvature and Limit Spaces
- Structure of Spaces with Lower Ricci Curvature
- Regularity of Spaces with Bounded Ricci Curvature
- Characterizing Ricci Curvature

Background: Curvatures

- (M^n, g, x) n-dimensional pointed Riemannian Manifold.
- Curvature: $Rm(X, Y)Z \equiv \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z$
- - Should be interpretted as *Hessian* of the metric
- Ricci: $Rc(X, Y) \equiv \sum\langle Rm(E_a, X)Y, E_a\rangle = tr(Rm)$
- - Should be interpretted as *Laplacian* of the metric
- Normalized Volume Measure: $\nu \equiv \frac{dv_g}{Vol(B_1(x))}$.

Background: Limit Spaces

- $(M_i^n, g_i, \nu_i, x_i) \xrightarrow{mGH} (X, d, \nu, x)$, convergence in measured Gromov-Hausdorff topology.

Theorem (Gromov 81')

Let (M_i^n, g_i, x_i) be a sequence of Riemannian manifolds with $Rc_i \geq -\lambda g$. Then there exists a metric space (X, d, x), such that after passing to a subsequence we have that

$$(M_i^n, g_i, x_i) \xrightarrow{GH} (X, d, x)\,. \tag{1}$$

- Initiated study of Ricci limit spaces.
- Generalized by Fukaya and others to say that $(M_i^n, g_i, \nu_i, x_i) \xrightarrow{mGH} (X, d, \nu, x)$.
- Question: What is the structure of X?

Structure of Limit Spaces, Lower Ricci Curvature
Background:

Theorem (Cheeger-Colding 96')

Let $(M^n_i, g_i, \nu_i, x_i) \xrightarrow{GH} (X, d, \nu, x)$ where $Rc_i \geq -\lambda g$. Then for ν-a.e. $x \in X$ the tangent cone at x is unique and isometric to $\mathbb{R}^{k_x}$ for some $0 \leq k_x \leq n$.

Conjecture (Cheeger-Colding 96')

There exists $0 \leq k \leq n$ such that $k_x \equiv k$ is independent of $x \in X$. In particular, X has a well defined dimension.

Conjecture (96')

The isometry group of X is a Lie Group.

Structure of Limit Spaces, Lower Ricci Curvature: Geometry of Geodesics

- Proof requires new understanding of the geometry of geodesics:

Theorem (Colding-Naber 10')

Let $(M^n_i, g_i, \nu_i, x_i) \xrightarrow{GH} (X, d, \nu, x)$ where $Rc_i \geq -\lambda g$, and let $\gamma : [0,1] \to X$ be a minimizing geodesic. Then there exists $C(n,\delta,\lambda), \alpha(n) > 0$ such that for every $s,t \in [\delta, 1-\delta]$ and $r \leq 1$ we have

$$d_{GH}\Big(B_r(\gamma(s)), B_r(\gamma(t))\Big) \leq \frac{C}{\delta}|t-s|^{\alpha}\, r\,. \tag{2}$$

- In words, the geometry of balls along geodesics can change at most at a Hölder rate.
- Corollary: By taking $r \to 0$ we see that tangent cones change continuously along geodesics.

Structure of Limit Spaces, Lower Ricci Curvature: Applications

Theorem (Colding-Naber 10')

Let $(M^n_i, g_i, \nu_i, x_i) \xrightarrow{GH} (X, d, \nu, x)$ where $Rc_i \geq -\lambda g$. Then there exists $0 \leq k \leq n$ and a full measure subset $R(X) \subseteq X$ such that the tangent cone at each point is unique and isometric to $\mathbb{R}^k$.

- Can define $dim(X) \equiv k$
- Proof through the example of the trumpet space:

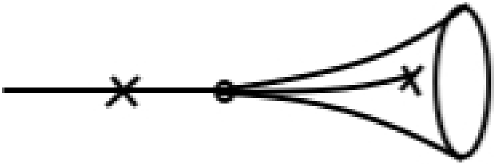
- Pick a geodesic γ connecting the x-points.

Structure of Limit Spaces, Lower Ricci Curvature: Applications

Theorem (Colding-Naber 10')

Let $(M^n_i, g_i, \nu_i, x_i) \xrightarrow{GH} (X, d, \nu, x)$ where $Rc_i \geq -\lambda g$. Then there exists $0 \leq k \leq n$ and a full measure subset $R(X) \subseteq X$ such that the tangent cone at each point is unique and isometric to $\mathbb{R}^k$.

- Can define $dim(X) \equiv k$
- Proof through the example of the trumpet space:



- Tangent cones along $\gamma(t)$ discontinuous at o, hence the trumpet space cannot arise as a Ricci limit space.

Structure of Limit Spaces, Lower Ricci Curvature: Applications

- Hölder continuity of tangent cones allow for further refinements about geometry of the regular set:

Theorem (Colding-Naber 10')

The regular set $R(X)$ is weakly convex. In particular, $R(X)$ is connected.

- The convexity of $R(X)$ is the key point in showing the Isometry group conjecture:

Theorem (Colding-Naber 10')

The isometry group of X is a Lie Group.

- Proof by contradiction. Push *small subgroups* into a regular tangent cone, namely $\mathbb{R}^k$.
- Produces small subgroups of the isometry group of $\mathbb{R}^k$, contradiction.

Structure of Limit Spaces, Lower Ricci Curvature Open Questions:

Conjecture

$\dim(X)$ is equal to the Hausdorff dimension of X.

Open Question

Does the singular set $S(X) \equiv X \setminus R(X)$ have $\dim(X) - 1$-Hausdorff dimension?

Open Question

Is there an open dense subset of X which is homeomorphic to a manifold?

Open Question

Is there an open dense subset of X which is bilipschitz to a manifold?

Structure of Limit Spaces, Bounded Ricci

Background:

Theorem (Anderson,Bando,Kasue,Nakajima,Tian 89')

Let $(M^4_i, g_i) \overset{GH}{\to} (X, d)$ satisfy $diam(M_i) \leq D$, $|Rc_i| \leq 3$, $Vol(M^4) > v > 0$ and $|b_2(M_i)| \leq A$, then X is a Riemannian orbifold with isolated singularities.

Conjecture (Codimension Four Conjecture 89')

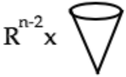
Let $(M^4_i, g_i, p_i) \overset{GH}{\to} (X, d, p)$ satisfy $|Rc_i| \leq n - 1$ and $Vol(B_1(p_i)) > v > 0$, then X is a Riemannian manifold away from a set of codimension four.

Conjecture (Anderson 94')

Let (M^4, g) satisfy $diam(M_i) \leq D$, $|Rc_i| \leq 3$, $Vol(M^4) > v > 0$, then M^4 is one of $C(D, v)$-diffeomorphism types.

Structure of Limit Spaces
Bounded Ricci, codimension two singularities:

- The key to solving the codimension four conjecture is to rule out codimension two singularities.
- That is, if an Einstein manifold M is close to $\mathbb{R}^{n-2} \times C(S^1_r)$:



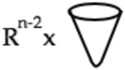
- Then M is actually smoothly close to Euclidean space $\mathbb{R}^n$.
- Key new idea is the slicing theorem:

Theorem (Cheeger-Naber 14')

For each $\epsilon > 0$ there exists $\delta > 0$ such that if $u : B_2(x) \to \mathbb{R}^{n-2}$ is a harmonic δ-splitting, then there exists $G_\epsilon \subseteq B_1(0^{n-2})$ with $|B_1 \setminus G_\epsilon| < \epsilon$ such that: if $s \in G_\epsilon$ and $x \in u^{-1}(s) \cap B_1(x)$ with $0 < r < 1$, then $\exists\, A_r \in GL(n-2)$ such that $Au : B_r(x) \to \mathbb{R}^{n-2}$ is an ϵ-splitting.

Structure of Limit Spaces
Bounded Ricci, codimension two singularities:

- Proof of the slicing theorem is by far the most involved aspect of the proof.
- The level sets $u^{-1}(s)$ are approximations of the cone factors.
- Slicing theorem allows one to pick *good* cone slices and reblow up to arrive at a new Ricci limit space which is a smooth cone:



- From this picture it is easy to contradict the existence of the upper Ricci bound, and thus the existence of codimension two singularities.

Structure of Limit Spaces
Bounded Ricci, ϵ – *regularity*:

- An easier argument works for codim 3 singularities. Combining yields:

Theorem (Cheeger-Naber 14')

Let (M^n, g, p) satisfy $Vol(B_1(p)) > v > 0$. Then there exists $\epsilon(n, v) > 0$ such that if $|Rc| \leq \epsilon$ and if

$$d_{GH}\Big(B_2(p), B_2(y)\Big) < \epsilon\,,$$

where $y \in \mathbb{R}^{n-3} \times C(Y)$, then the harmonic radius satisfies $r_h(x) \geq 1$. If M^n is Einstein then we further have

$$sup_{B_1(p)}|Rm| \leq 1\,. \tag{3}$$

Structure of Limit Spaces, Bounded Ricci:

- By combining the previous ϵ-regularity theorem with the stratification theory one obtains:

Corollary (Cheeger-Naber 14')

Let $(M_i^4, g_i, p_i) \stackrel{GH}{\to} (X, d, p)$ satisfy $|Rc_i| \leq n-1$ and $Vol(B_1(p_i)) > v > 0$, then there exists $S(X) \subseteq X$ with $dim S(X) \leq n-4$ such that $R(X) \equiv X \setminus S(X)$ is a Riemannian manifold.

Structure of Limit Spaces, Bounded Ricci:

- More generally, by combining the previous ϵ-regularity theorem with the quantitative stratification theory one obtains the effective estimates:

Theorem (Cheeger-Naber 14')

Let (M^n, g, p) satisfy $|Rc| \leq n-1$ and $Vol(B_1(p)) > v > 0$. Then for every $\epsilon > 0$ there exists $C_\epsilon(n, v, \epsilon)$ such that

$$\fint_{B_1(p)} |Rm|^{2-\epsilon} \leq C_\epsilon \,. \qquad (4)$$

Furthermore, for every $r \leq 1$ we have the harmonic radius estimate

$$Vol(B_r\{x : r_h(x) \leq r\} \cap B_1(p)) \leq C_\epsilon r^{4-\epsilon} \,. \qquad (5)$$

Structure of Limit Spaces, Bounded Ricci Dimension Four:

- These results may be pushed further in dimension four.
- As a starting point consider the following:

Theorem (Cheeger-Naber 14')

If $(M^4_i, g_i, p_i) \xrightarrow{GH} (X, d, p)$ where $|Rc_i| \leq 3$ and $Vol(B_1(p_i)) > v > 0$, then X is a Riemannian orbifold with isolated singularities.

- This in turn may be used to prove Anderson's conjecture:

Theorem (Cheeger-Naber 14')

If (M^4, g) such that $|Rc| \leq 3$, $Vol(B_1(p)) > v > 0$, and $diam(M) \leq D$. Then there exists $C(v, D)$ such that M has at most one of C-diffeomorphism types.

Structure of Limit Spaces, Bounded Ricci Dimension Four:

- A local version of the finite diffeomorphism type theorem, when combined with the Chern-Gauss-Bonnet, gives rise to the following:

Theorem (Cheeger-Naber 14')

Let (M^4, g, p) satisfy $|Rc| \leq 3$ and $Vol(B_1(p)) > v > 0$. Then there exists $C(v)$ such that

$$\fint_{B_1(p)} |Rm|^2 \leq C\,. \tag{6}$$

- This estimate is sharp

Structure of Limit Spaces, Bounded Ricci Open Questions:

Conjecture

Let (M^n, g, p) satisfy $|Rc| \leq n-1$ and $Vol(B_1(p)) > v > 0$. Then there exists $C(n,v)$ such that $⨍_{B_1(p)} |Rm|^2 \leq C$.

Corollary

Let $(M^n_i, g_i, p_i) \xrightarrow{GH} (X, d, p)$ with $|Rc_i| \leq n-1$ and $Vol(B_1(p_i)) > v > 0$, then the singular set $S(X)$ is $n-4$-rectifiable with $H^{n-4}(S(X) \cap B_1(p)) \leq C(n,v)$.

Conjecture

Let $(M^n_i, g_i, p_i) \xrightarrow{GH} (X, d, p)$ with $|Rc_i| \leq n-1$ and $Vol(B_1(p_i)) > v > 0$, then X is bilipschitz to a real analytic variety.

Meaning of Ricci Curvature, Background:

- Future directions in Ricci curvature will involve more than a regularity theory.
- Many ways to interpret the *meaning* of Ricci curvature bounds.
- Each new method leads to new understanding.
- At this stage there are many methods for characterizing lower Ricci curvature (see next slide).
- Want to characterize and understand the meaning of *bounded* Ricci curvature.

Characterizing Ricci Curvature, Background: Lower Ricci Curvature

Theorem (Bakry-Emery-Ledoux 85')

Let (M^n, g) be a complete manifold, then the following are equivalent:

1. $Ric \geq -\kappa g$.
2. $|\nabla H_t u|(x) \leq e^{\frac{\kappa}{2}t} H_t|\nabla u|(x)\ \forall x$.
3. $\lambda_1(-\Delta_{x,t}) \geq \kappa^{-1}\left(e^{\kappa t} - 1\right)$
4. $\int_M u^2 \ln u^2 \rho_t(x, dy) \leq 2\kappa^{-1}\left(e^{\kappa t} - 1\right)\int_M |\nabla u|^2 \rho_t(x, dy)$ *if* $\int_M u^2 \rho_t = 1$.

- H_t heat flow operator, ρ_t heat kernel, $\Delta_{x,t} = \Delta + \nabla ln\rho_t \cdot \nabla$ heat kernel laplacian.
- Lower Bounds on Ricci $\Leftrightarrow$ Analysis on M
- More recently: lower ricci $\Leftrightarrow$ convexity of the entropy functional (Lott,Villani, Sturm, Ambrosio, Gigli, Saviere').

Characterizing Ricci Curvature: Bounded Ricci Curvature

- Characterizations of bounded Ricci curvature will require estimates on the path space $P(M)$ of the manifold.
- There will be a 1-1 correspondence between the B-E-L estimates and the new estimates on path space.
- In fact, for each estimate on path space, we will see how when it is applied to the *simplest* functions on path space we recover the BEL estimates. Namely, $F(\gamma) = u(\gamma(t))$.
- We will see that Bounded Ricci Curvature $\Leftrightarrow$ Analysis on Path Space of M.

Characterizing Bounded Ricci Curvature: Path Space Basics

- $P(M) \equiv C^0([0,\infty), M)$
- $P_x(M) \equiv \{\gamma \in P(M) : \gamma(0) = x\}$.

- For a partition $\mathbf{t} \equiv \{0 \le t_1 < \ldots < t_k < \infty\}$ denote by $e_{\mathbf{t}} : P(M) \to M^k$ the evaluation mapping given by

$$e_{\mathbf{t}}(\gamma) = (\gamma(t_1), \ldots, \gamma(t_k))\,.$$

- For $x \in M$ let Γ_x be the associated Wiener measure on $P(M)$. Defined by its pushforwards:

$$e_{\mathbf{t},*}\Gamma_x = \rho_{t_1}(x, dy_1)\rho_{t_2-t_2}(y_1, dy_2)\cdots\rho_{t_k-t_{k-1}}(y_{k-1}, dy_k)\,.$$

Characterizing Bounded Ricci Curvature: Gradients on Path Space

- If $F : P(M) \to \mathbb{R}$ we define the Parallel Gradient:

$$|\nabla_0 F|(\gamma) = sup\{D_V F : |V|(0) = 1 \text{ and } |\nabla_{\dot\gamma} F| \equiv 0\}\,.$$

- If $F : P(M) \to \mathbb{R}$ we define the t-Parallel Gradient:

$$|\nabla_t F|(\gamma) = sup\{D_V F : |V|(s) = 0 \text{ for } s < t,\ |V|(t) = 1 \text{ and } |\nabla_{\dot\gamma} F|(s) \equiv 0 \text{ for } s > t\}\,.$$

Characterizing Bounded Ricci Curvature: First Characterization, Gradient Bounds:

- Given $F : P(M) \to \mathbb{R}$ let us construct a function on M by

$$\int_{P(M)} F d\Gamma_x : M \to \mathbb{R}\,.$$

- If $F \in C(P(M))$ then $\int F d\Gamma_x \in C(M)$.
- What about gradient bounds? Do gradient bounds on F give rise to gradient bounds on $\int F d\Gamma_x$? In fact:

$$|\nabla \int_{P(M)} F d\Gamma_x| \leq \int_{P(M)} |\nabla_0 F| d\Gamma_x$$

$$\Updownarrow$$

$$Rc \equiv 0$$

Characterizing Bounded Ricci Curvature: First Characterization, Example:

- Let us apply this to the simplest functions on path space.
- For $t > 0$ fixed and $u : M \to \mathbb{R}$ let $F(\gamma) = u(\gamma(t))$.
- Let us compute $|\nabla \int_{P(M)} F d\Gamma_x| \leq \int_{P(M)} |\nabla_0 F| d\Gamma_x$:

1. $\int_{P(M)} F d\Gamma_x \equiv H_t u(x)$.
2. $|\nabla_0 F|(\gamma) = |\nabla u|(\gamma(t))$.
3. Thus

$$|\nabla \int_{P(M)} F d\Gamma_x| \leq \int_{P(M)} |\nabla_0 F| d\Gamma_x \quad \forall\, F \in L^2(P(M))$$

$$\Downarrow$$

$$|\nabla H_t u|(x) \leq H_t |\nabla u|(x) \quad \forall\, u \in L^2(M)$$

- Recover Bakry-Emery, hence $\mathrm{Ric} \geq 0$.

Characterizing Bounded Ricci Curvature: Second Characterization:

- Recall $L^2(P(M))$ comes naturally equipped with a one-parameter family of closed nested subspaces $L^2_t \subseteq L^2(P(M))$.
- $F \in L^2_t$ if $F(\gamma) = F(\sigma)$ whenever $\gamma|_{[0,t]} = \sigma|_{[0,t]}$.
- Given F can construct a family of functions $F_t \in L^2_t \subseteq L^2(P(M))$ by projection.
- F_t is a martingale. As a curve in L^2, F_t is precisely $C^{1/2}$.

Characterizing Bounded Ricci Curvature: Second Characterization:

- To understand $C^{1/2}$-derivative define Quadratic Variation

$$[F_t] \equiv \lim_{\mathbf{t}\subseteq[0,t]} \sum \frac{(F_{t_{a+1}} - F_{t_a})^2}{t_{a+1} - t_a}$$

- Can we control the derivative of $[F]_t$? In fact:

$$\frac{d}{dt}[F_t](\gamma) \leq \int_{P_{\gamma(t)}(M)} |\nabla_t F|$$

$$\Updownarrow$$

$$Rc \equiv 0\,.$$

- Similar statements for $|Rc| \leq k$, metric measure spaces, and dimensional versions.

Characterizing Bounded Ricci Curvature: Third Characterization:

- Recall that the Ornstein-Uhlenbeck operator $L_x : L^2(P_x(M)) \to L^2(P_x(M))$ is a self adjoint operator on based path space.
- Arises from the Dirichlet Form $E[F] \equiv \int_{P_x(M)} |\nabla_{H^1} F|^2 d\Gamma_x = \int_{P_x(M)} \int_0^\infty |\nabla_s F|^2 d\Gamma_x$, where $\nabla_{H^1} F$ is the Malliavin gradient.
- Acts as an infinite dimensional laplacian. Spectral gap first proved by Gross in $\mathbb{R}^n$, and Aida and K. D. Elworthy for general compact manifolds. Fang and Hsu first proved estimates using Ricci curvature.

Characterizing Bounded Ricci Curvature: Third Characterization:

- More generally one can define the time restricted Dirichlet energies $E_{t_0}^{t_1}[F] \equiv \int_{P_x(M)} \int_{t_0}^{t_1} |\nabla_s F|^2 d\Gamma_x$.
- Thus $E_0^\infty \equiv E$, and in general $E_{t_0}^{t_1}$ is the part of the Dirichlet energy which only sees the gradient on the time range $[t_0, t_1]$.
- From these energies one can define the induced Ornstein-Uhlenbeck operators $L_{t_0}^{t_1} : L^2(P_x(M)) \to L^2(P_x(M))$ with $L_0^\infty \equiv L_x$.

Characterizing Bounded Ricci Curvature: Third Characterization:

- Is the spectrum of the operators $L_{t_0}^{t_1}$ controlled or characterized by Ricci Curvature? In fact:

$$\int_{P(M)} \left|F_{t_1} - F_{t_0}\right|^2 \leq \int_{P(M)} \langle F, L_{t_0}^{t_1} F \rangle$$

$$\Updownarrow$$

$$Rc \equiv 0\,.$$

  In particular, we have the spectral gap $\lambda(L_x) \geq 1$ for the standard Ornstein-Uhlenbeck operator.

- More generally there are log-Sobolev versions of this result, as well as similar statements for $|Rc| \leq k$, metric measure spaces, and dimensional versions.

Characterizing Bounded Ricci Curvature:

- Below is a partial list of the main results, see [N] for the complete statement:

Theorem (Naber 13')

Let (M^n, g) be a smooth Riemannian manifold, then the following are equivalent:

1. $-\kappa g \leq Ric \leq \kappa g$.
2. $|\nabla \int_{P(M)} F\, d\Gamma_x| \leq \int_{P(M)} \Big(|\nabla_0 F| + \int_0^\infty \frac{\kappa}{2} e^{\frac{\kappa}{2}t} |\nabla_t F| dt \Big) d\Gamma_x$.
3. $\int_{P(M)} \big|F_{t_1} - F_{t_0}\big|^2 \leq e^{\frac{\kappa}{2}(T-t_0)} \int_{P(M)} \langle F, L^{t_1}_{t_0,\kappa} F\rangle$, *in particular* $\lambda^1(L^T_x) \geq \frac{2}{e^{\kappa T}+1}$ *for the standard Ornstein-Uhlenbeck operator.*
4. $\frac{d}{dt}[F_t](\gamma) \leq e^{\kappa(T-t)} \int_{P_{\gamma(t)}(M)} |\nabla_t F| + \int_t^T \frac{\kappa}{2} e^{\frac{\kappa}{2}s} |\nabla_s F|^2 d\Gamma_{\gamma(t)}$

where F is an $\mathcal{F}^T$-measurable function.