Structure of Singular Sets of Stationary and Minimizing Harmonic Maps

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- Joint work with Daniele Valtorta.
- Discussing recent results on the singular sets of nonlinear equations.
- Will focus on harmonic maps between Riemannian manifolds, however techniques are very general.
- Main requirement for nonlinear equation is the existence of a monotone quantity.
- Similar results are proven for minimal surfaces, and future papers will deal with the cases of lower Ricci curvature, mean curvature flow, etc...

Outline of Talk

- Preliminaries on Harmonic Maps Between Riemannian Manifolds
- Structure of Singular Sets for Stationary Harmonic Maps
- Regularity Theory for Minimizing Harmonic Maps
- Outline of Proof
- - 1. Quantitative Stratification
- - 2. Energy Covering
- - 3. New Reifenberg-type Theorems
- - 4. L^2-subspace approximation theorem
- - Completion of Proof

Background: Harmonic Maps between Riemannian manifolds

- Consider a mapping $f : B_2 \subseteq M \to N$ between two Riemannian manifolds.
- Since $\nabla f : T_x M \to T_{f(x)} N$ is a linear map we can define the energy $E[f] \equiv \frac{1}{2} \int_{B_2} |\nabla f|^2 dv_g$.
- To say f is harmonic can mean one of three things:

(1) Weakly Harmonic: f solves the Euler Lagrange $\Delta_M f = A(\nabla f, \nabla f)$.

(2) Stationary: f is a critical point of E.

(3) Minimizing: f is a minimizer of E.

- If $N = \mathbb{R}$ then these are all equivalent.
- In general we only have that (3) $\implies$ (2) $\implies$ (1).

Background: Regularity of Harmonic Maps

- In general the regularity theory of a harmonic map depends a great deal on which definition of harmonic map you take.
- Weakly harmonic maps may be everywhere discontinuous (Riviere).
- Stationary harmonic maps are smooth away from a set of codimension two (Bethuel).
- Minimizing harmonic maps are smooth away from a set of codimension three (Schoen-Uhlenbeck).
- Focus of this lecture is on the structure of the singular sets of stationary and minimizing harmonic maps.

Background: Tangent Maps

- Question: What do singular sets look like?
- To answer this let us first recall tangent maps.

Definition (Tangent Maps)

Consider a mapping $f : B_2 \to N$:

1. If $x \in B_1$ and $r < 1$ define $f_{x,r} : B_{r^{-1}}(0) \to N$ by $f_{x,r}(y) = f(x + ry)$.
2. We call $f_x : \mathbb{R}^n \to N$ a tangent map at $x \in B_1$ if there exists $r_j \to 0$ such that $f_{x,r_j} \to f_x$ in L^2.

- $f_{x,r}$ essentially zooms up the map f on $B_r(x)$.
- f_x represents infinitesimal behavior of f at x.
- Remark: If f is stationary then for every $r_i \to 0$ a subsequence of $f_{x_{r_i}} \to f_x$ converges to a tangent map.
- For stationary maps it is better to define tangent maps to include a defect measure, we will ignore this but all the results of this paper are valid in this case.

Background: Symmetries of Maps

- In general a tangent map f_x may not be a constant.
- We will be interested in *stratifying* the singular set based on how many symmetries tangent maps have.

Definition (Symmetries of Maps)

Consider a mapping $f : \mathbb{R}^n \to N$.

1. We say f is 0-symmetric if for each $\lambda > 0$ we have $f(\lambda x) = f(x)$ (radial invariance).
2. We say f is k-symmetric if f is 0-symmetric and there exists a k-plane $V^k \subseteq \mathbb{R}^n$ such that $f(x+v) = f(x)$ for each $v \in V^k$ (translation invariance).

- A k-symmetric function may be identified with a function on the $n-k-1$ sphere S^{n-k-1}.

Background: Examples

- Three most important examples:

- Example 1: A function $f^1 : \mathbb{R}^n \to N$ is n-symmetric iff $f^1 \equiv const$.

- Example 2: Consider $f^2 = \frac{x}{|x|} : \mathbb{R}^3 \to S^2$ obtained by projection to standard S^2.
- f^2 is 0-symmetric, with an isolated singularity at 0.
- $f^2_x = constant$ if $x \neq 0$ and $f^2_x = f$ if $x = 0$.
- In fact, f^2 is a minimizing harmonic map.

- Example 3: Consider $f^3 : \mathbb{R}^{k+3} \to S^2$ obtained by projection to the last three variables.
- f^3 is k-symmetric with respect to $\mathbb{R}^k \times \{0^3\}$.
- f^3 is a minimizing harmonic map.

Background: Stratification of Singular Set

- For a stationary harmonic map we will decompose $Sing(f)$ based on symmetries of tangent cones.

Definition (Stratification)

For a stationary harmonic mapping $f : B_2 \subseteq M \to N$ define

1. $S^k(f) \equiv \{x \in B_1 : \text{no tangent cone at } x \text{ is } k+1\text{-symmetric}\}$.

- Definition frustrating - want to define S^k as those points which look k-dimensional. Instead, define S^k as those points which do not look $k+1$-dimensional.
- Note $S^0(f) \subseteq S^1(f) \subseteq \cdots$.
- In Example 2 $Sing(f^2) = S^0(f^2) = \{0\}$ and in Example 3 $Sing(f^3) = S^k(f^3) = \mathbb{R}^k \times \{0^3\}$.

Background: Known Structural Results

Theorem (Schoen-Uhlenbeck 82')

1. *If $f : B_2 \to N$ is a stationary harmonic map then* $\dim S^k(f) \leq k$.
2. *If $f : B_2 \to N$ is a minimizing harmonic map then $Sing(f) = S^{n-3}(f)$ and hence* $\dim Sing(f) \leq n-3$.

- What about structure of the singular set?

Theorem (Simon 95')

If $f : B_2 \to N$ is a minimizing harmonic map with N an analytic manifold then $Sing(f) = S^{n-3}(f)$ is $n-3$ rectifiable.

- k-rectifiable 'essentially' means a k-manifold away from a set of measure zero. See Federer for precise definition.
- Question: What about general stationary case?
- Question: What about general stratum?
- Question: What about more analytic estimates?

Structure of Stationary Harmonic Maps

- Our first result:

Theorem (NV 15')

If $f : B_2 \subseteq M \to N$ is a stationary harmonic map then

1. *$S^k(f)$ is k-rectifiable for all k.*
2. *In fact, for k-a.e. $x \in S^k(f)$ there exists a unique k-plane $V^k \subseteq \mathbb{R}^n$ such that every tangent map at x is k-symmetric with respect to V^k.*

- In comparison to previous results: f only needs to be stationary, N needs only be C^2, and the rectifiability holds for every stratum $S^k(f)$.
- The second statement tells us that we can define $S^k(f)$ the more intuitive way (points with k-symmetry), and it agrees with the usual definition a.e.

Regularity of Minimizing Harmonic Maps I

- For minimizing harmonic maps we can do better.
- Assume $|sec_M|, |sec_N|, \mathrm{diam}(N), \mathrm{Vol}^{-1}(B_2), \mathrm{Vol}^{-1}(N) \leq K$.
- First result on Hausdorff measure of singular set:

Theorem (NV 15')

Let $f : B_2 \subseteq M \to N$ be a minimizing harmonic map with $\int_{B_2} |\nabla f|^2 \leq \Lambda$. Then there exists $C(n, K, \Lambda) > 0$ such that

$$Vol(B_r Sing(f)) \leq Cr^3\,. \tag{1}$$

In particular, $H^{n-3}(Sing(f)) \leq C$ is uniformly bounded.

- One can even prove that $\mathrm{Sing}(f)$ has effective packing estimates. That is, if $\{B_{r_j}(x_j)\}$ is *any* Vitali covering of $\mathrm{Sing}(f)$ then $\sum r_j^{n-3} < C$.
- Covering estimates: Hausdorff < Minkowski < Packing

Regularity of Minimizing Harmonic Maps II

- We also have more effective analytic estimates:

Theorem (NV 15')

Let $f : B_2 \subseteq M \to N$ be a minimizing harmonic map with $\int_{B_2} |\nabla f|^2 \leq \Lambda$. Then there exists $C(n, K, \Lambda) > 0$ such that

$$Vol(\{|\nabla f| > r^{-1}\}) \leq Vol(B_r\{|\nabla f| > r^{-1}\}) \leq Cr^3 \,. \tag{2}$$

In particular, $|\nabla f| \in L^3_{weak}$ has apriori estimates. Similarly, one can also show $|\nabla^2 f| \in L^{3/2}_{weak}$.

- These estimates are sharp! Example 2 satisfies $|\nabla f|(x) \approx |x|^{-1}$ and thus $|\nabla f| \in L^3_{weak}$ but $|\nabla f| \notin L^3$.

Proof of Main Results: Weak vs. Strong Methods

- Before discussing details let us compare a little weak versus strong methods.

- Main tool of stationary harmonic map: Normalized Dirichlet energy $\theta_r(x) \equiv r^{2-n}\int_{B_r(x)} |\nabla f|^2$ is monotone:

$$\frac{d}{dr}\theta_r(x) = 2r^{2-n}\int_{S_r(x)} \left|\frac{\partial f}{\partial r}\right|^2 \geq 0\,. \tag{3}$$

- Note: $\theta_r(x)$ independent of $r \iff f$ is 0-symmetric.

- Weak Methods: Only aspect of a harmonic map which is exploited is above monotonicity.
- Strong Methods: Anything else (i.e. Lojasiewich inequalities, tangent cone uniqueness methods, etc...).

Proof of Main Results: Four Points

- Proof requires four relatively new ideas.
- Two have been introduced in the last three years, two are introduced in this paper.

- Idea 1: Quantitative Stratification.
- Introduced in [CN] to prove effective estimates on Einstein manifolds.
- The stratification is *never* directly estimated. One must break it into more manageable pieces.

- Idea 2: Energy Covering.
- Introduced in [NV] to prove estimates similar to those in this paper on critical sets of elliptic equations.
- Crucial for effective estimates:
- Note: even for minimizers Simon never proves hausdorff measure estimates. Using the energy covering alone combined with (suitable generalizations of) his techniques one could accomplish this (for analytic targets).

Proof of Main Results: Four Points

- Idea 3: New Reifenberg-Type Theorems.
- Classic Reifenberg gives criteria to determine when a set $S \subseteq \mathbb{R}^n$ is $C^{0,\alpha}$-bihölder to a ball $B_1(0^k) \subseteq \mathbb{R}^k$.
- Three new types of Reifenberg theorems introduced.
- First uses L^2 closeness criteria to determine when a set is $W^{1,p}$-equivalent to a ball in $\mathbb{R}^k$. ($\implies$ gradient control)
- Second weakens the closeness criteria in exchange for only showing a set is rectifiable. (e.g. allows holes)
- Third is a discrete version which proves volume control on appriopriate discrete measures.

- Idea 4: New L^2-subspace approximation theorems.
- New Reifenberg results only important if we can show the stratum of the (quantitative) singular sets satisfy the criteria.
- New approximation theorems give 'very' general criteria under which we can relate how close the quantitative stratifications can be approximated in L^2 by a k-dimensional subspace.

Idea 1: Quantitative Stratification: Definition

- Stratification separates points based on actual symmetries on infinitesimal scales.
- Quantitative Stratification separates points based on almost symmetries on balls of definite sizes.

Definition (Almost Symmetries)

Given $f : B_2 \to N$ we say $B_r(x) \subseteq B_2$ is (k, ϵ)-symmetric if there exists an actual k-symmetric $h : \mathbb{R}^n \to N$ such that $⨍_{B_r(x)} |f - h|^2 < \epsilon$.

Definition (Quantitative Stratification)

1. $S^k_{\epsilon,r}(f) \equiv \{x \in B_1 : \text{for no } r < s \leq 1 \text{ is } B_s(x)\ (k+1, \epsilon)\text{-symmetric}\}$
2. $S^k_{\epsilon}(f) \equiv \{x \in B_1 : \text{for no } 0 < s \leq 1 \text{ is } B_s(x)\ (k+1, \epsilon)\text{-symmetric}\}$

- Exercise: Show that $S^k(f) = \bigcup_\epsilon S^k_\epsilon(f)$.
- Thus $S^k_\epsilon(f)$ are those points x for which no ball at x is ever close to having $k+1$-symmetries.

Idea 1: Quantitative Stratification: Results

- The real main result in the paper is the following:

Theorem (NV 15')

Let $f : B_2 \subseteq M \to N$ be a stationary harmonic map with $\int_{B_2} |\nabla f|^2 \leq \Lambda$. Then for each $\epsilon > 0$ we have

$$Vol(B_r S^k_\epsilon(f)) \leq C_\epsilon r^{n-k} \,. \tag{4}$$

In particular, $H^k(S^k_\epsilon(f)) < C_\epsilon$. Further, the set $S^k_\epsilon(f)$ is k-rectifiable.

- Note: Since $S^k(f) = \bigcup_\epsilon S^k_\epsilon(f)$ this proves the main result on stationary maps.
- Note: If f is minimizing then there exists $\epsilon(n, K, \Lambda)$ such that $\text{Sing}(f) \subseteq S^{n-3}_\epsilon(f)$. This proves the main results for minimizing maps.

Idea 2: Energy Covering

- In fact, even the Quantitative Stratification needs to be broken down into more manageable pieces.
- Covering scheme introduced in [NV] to study critical sets of elliptic equations.
- Good aspect of the scheme is that it gives rise to very effective estimates (packing estimates).
- Bad aspect is that it requires comparing balls of arbitrarily different sizes.
- In critical set context this was handled by proving effective tangent cone uniqueness statements. Allowed us to relate balls of arbitrarily different sizes.
- In this context we will need the new Reifenberg.

Idea 2: Energy Covering:Packing

- Rough Idea for Packing Estimate for S^k_ϵ:
- Let $\{B_{r_j}(x_j)\} \subseteq B_2$ be a collection of balls such that
  1. $\{B_{r_j/5}(x_j)\}$ disjoint (Vitali condition).
  2. $x_j \in S^k_\epsilon \cap B_1$.
- Then we want to conclude $\sum r_j^k < C(n, \epsilon, \Lambda, K)$.
- First consider the weaker statement:

Lemma (Main Lemma)

Let $\{B_{r_j}(x_j)\} \subseteq B_2$ satisfy

1. $\{B_{r_j/5}(x_j)\}$ *disjoint (Vitali condition).*
2. $x_j \in S^k_\epsilon \cap B_1$.
3. $|\theta_1(x_j) - \theta_{r_j}(x_j)| < \eta(n, K, \Lambda, \epsilon)$.

Then $\sum r_j^k \leq C(n)$.

- We can prove the packing estimate by inductively applying the Main Lemma $\Lambda\eta^{-1}$ times.
- Our main goal is therefore to prove the Main Lemma.

Idea 2: Energy Covering:Packing II

- We can prove the packing estimate for $\{B_{r_j}(x_j)\}$ by inductively applying the Main Lemma.
- Indeed: Build a new Vitali cover $S^k_\epsilon \cap B_1 \subseteq \bigcup B_{s_j}(y_j)$ with $E \equiv \sup_{B_1} \theta_1(y)$ such that
  1. For each ball we have $\sup_{B_{s_j}(y_j)} \theta_{s_j}(y) > E - \eta$.
  2. If $B_{r_j}(x_j)$ satisfies $\sup_{B_{r_j}(x_j)} \theta_{r_j}(y) > E - \eta$, then $B_{r_j(x_j)} \subseteq \{B_{s_j}(y_j)\}$ is a ball in our new cover.
  3. For all other $B_{s_j}(y_j)$ we have $\sup_{B_{s_j}(y_j)} \theta_{s_j}(y) = E - \eta$.
- Applying the Main Lemma we have that $\sum s_j^k \leq C(n)$.
- Now we can look at each ball $B_{s_j}(y_j)$ which is not a ball in $\{B_{r_j}(x_j)\}$ and repeat the above recovering process on $B_{s_j}(y_j)$.
- We need only repeat this process $\Lambda\eta^{-1}$ since the energy drops by η each time, at this stage we must have our original covering.
- This shows $\sum r_j^k \leq C(n)^{\Lambda\eta^{-1}} = C(n, K, \Lambda)$, as claimed.
- Thus our main goal is to prove the Main Lemma.

Idea 2: Energy Covering:Rectifiable

- One can modify the procedure to handle not just the effective estimates but the rectifiable structure itself.
- To do this build a cover $S^k_\epsilon \cap B_1 \subseteq U_0 \cup U_+ = U_0 \cup \bigcup B_{s_j}(y_j)$ such that
  1. U_0 is k-rectifiable with $\lambda^k(U_0) < C_\epsilon$. (proof done here)
  2. $\sum s_j^k < C_\epsilon$. (bad balls have k-content bound)
  3. If $E = \sup_{B_1} \theta_1(y)$ then for each ball we have $\sup_{B_{s_j}(x_j)} \theta_{s_j}(y) < E - \eta$. (definite energy drop on bad balls).
- To build such a cover the inductive scheme is the same as before, but to control U_0 we will need a version of the Main Lemma which includes U_0.
- In the proof of the Main Lemma this will come down to applying the new rectifiable-Reifenberg, not just the new discrete-Reifenberg. (See paper for details on this.)

Idea 3: New Reifenberg Type Theorems

- Strategy of proving Main Lemma is by proving and applying some new type of Reifenberg results.
- Recall standard Reifenberg:

Theorem (Reifenberg)

For each $\epsilon > 0$ and $\alpha < 1$ there exists $\delta(n, \epsilon, \alpha) > 0$ such that if $S \subseteq B_2$ is a closed set such that for all $B_r(x) \subseteq B_2$ there exists a k-plane L^k such that $d_H(S \cap B_r, L \cap B_r) < \delta r$, then there exists a $1 + \epsilon$ bi-$C^{0,\alpha}$ homeomorphism $\phi : B_1(0^k) \to S \cap B_1$.

- Issues: No gradient control, no volume control, no rectifiable structure.
- Various generalizations in the literature, but requires too many assumptions to get the desired gradient control.
- There are three generalizations of the Reifenberg we will consider in the paper.

Idea 3: New Reifenberg Type Theorems: Displacement

- To describe the results we begin with the following:

Definition

Let $\mu \subseteq B_2$ be a measure, define the k-displacement $D^k_\mu(x,r) \equiv \inf_{L^k} r^{-2-k} \int_{B_r} d^2(x,L^k)d\mu$ if $\mu(B_r(x)) \geq \epsilon_n r^k$ and $D^k_\mu(x,r) \equiv 0$ if $\mu(B_r(x)) < \epsilon_n r^k$.

Definition

If $S \subseteq B_2$ define $D^k_S(x,r) \equiv D^k_\mu(x,r)$ where $\mu \equiv \lambda^k|_S$ is the k-dimensional Hausdorff measure on S.

- Thus $D^k(x,r)$ measures how closely, in the L^2-sense, μ is contained in a k-dimensional plane.
- If one replaces d^2 with d^p, then for $p > 2$ the results fail.

Idea 3: New Reifenberg Type Theorems: $W^{1,p}$-Reifenberg

- Our first new Reifenberg result is the following:

Theorem ($W^{1,p}$-Reifenberg)

For each $\epsilon > 0$ and $p < \infty$ there exists $\delta(n, \epsilon, p) > 0$ such that if $S \subseteq B_2$ is a closed set such that for all $B_r(x) \subseteq B_2$ we have

1. *There exists a k-plane L^k such that $d_H(S \cap B_r, L \cap B_r) < \delta r$.*
2. $r^{-k} \int_{B_r} \int_0^r D^k_S(y, s) d\mu \frac{ds}{s} < \delta.$

then there exists a $1 + \epsilon$ bi-$W^{1,p}$ homeomorphism $\phi : B_1(0^k) \to S \cap B_1$. In particular for $p > n$, we have the estimates:

1. *$S \cap B_1$ is k-rectifiable.*
2. $A(n)^{-1} \leq \lambda^k(S \cap B_1) \leq A(n)$
3. $A(n)^{-1} r^k \leq \lambda^k(S \cap B_r(x)) \leq A(n) r^k.$

- (3) above holds by constructing $\phi : B_r(0^k) \to S \cap B_r(x)$ and applying (2).

Idea 3: New Reifenberg Type Theorems: rectifiable-Reifenberg

- Next we drop the assumption that S is close to a k-plane.
- In this case we cannot get a homeomorphic structure on S, e.g. take $S \subseteq L^k$ to be an arbitrary subset, then $D^k(y,s) \equiv 0$.
- We therefore see that the most we can hope is that S is rectifiable with upper volume estimates.
- In fact this is true:

Theorem (rectifiable-Reifenberg)

There exists $\delta(n) > 0$ such that if $S \subseteq B_2$ is a closed set such that for all $B_r(x) \subseteq B_2$ we have

$$r^{-k}\int_{B_r}\int_0^r D^k_S(y,s)\,d\mu\,\frac{ds}{s} < \delta\,, \tag{5}$$

then we have the estimates:

1. $S \cap B_1$ *is k-rectifiable.*
2. $\lambda^k(S \cap B_1) \leq A(n)$
3. $\lambda^k(S \cap B_r(x)) \leq A(n)r^k$.

Idea 3: New Reifenberg Type Theorems: discrete-Reifenberg

- Let us now even drop that S is a set and discuss the case of a general discrete measure μ.
- We cannot expect any rectifiable structure, the most we might hope is that the upper volume estimates survive.
- In fact this is true:

Theorem (discrete-Reifenberg)

Let $\{B_{r_j}(x_j)\} \subseteq B_2$ be a Vitali set with $\mu \equiv \sum r_j^k \delta_{x_j}$, then there exists $\delta(n) > 0$ such that if for all $B_r(x) \subseteq B_2$ we have

$$r^{-k} \int_{B_r} \int_0^r D^k_\mu(y,s)d\mu \frac{ds}{s} < \delta\,, \tag{6}$$

then we have the estimate $\sum r_j^k \leq A(n)$.

- This result is used to provide all of the effective estimates in the paper, in particular the Main Lemma from before.
- We can replace condition (6) with $r^{-k} \int_{B_r} \sum_{r_\alpha \leq r} D^k_\mu(y,s)d\mu < \delta$, where $r_\alpha \equiv 2^{-\alpha}$.

Proving the new Reifenberg results I

- Let us first recall the rough outline of the proof of the standard Reifenberg:
- One builds an inductive sequence of approximating submanifolds $S_\beta \to S$ by the following scheme:
  1. Cover S with a Vitali collection $\{B_{s_\beta}(y_{\beta,j})\}$, where $s_\beta = 2^{-\beta}$.
  2. On each ball pick a best approximating k-plane $L^k_{\beta,j}$.
  3. Use a partition of unity to glue these together to form S_β.
- There is a natural projection $\phi_\beta : S_\beta \to S_{\beta+1}$, and composing gives $\phi : S_1 \to S$ by $\phi = \cdots \circ \phi_2 \circ \phi_1$, where $S^1 \approx B_1(0^k)$.
- Carefully keeping track of the errors shows this is the desired bi-Holder map.
- The $W^{1,p}$-Reifenberg is the most natural of the generalizations, and one would like to prove it in precisely the same manner.
- In fact, if we assumed apriori that $\lambda^k\big|_S$ satisfied the Alhfors regular condition (3), then this would work exactly.
- By far the most challenging aspect of the proof is therefore to remove this Alhfors assumption, and indeed seeing it is a conclusion.

Proving the new Reifenberg results II

- Let out roughly outline this in a little more detail.
- In our descriptions the natural and intuitive ordering was to first explain the $W^{1,p}$-Reifenberg, and then the rectifiable-Reifenberg, and then finally the discrete Reifenberg.
- In the proof we must go the other direction. We must proceed by discrete Reifenberg $\implies$ rect-Reifenberg $\implies$ $W^{1,p}$-Reifenberg.
- As sketched the standard Reifenberg involves an argument which starts at the top scale and inducts downward.
- We will see that we need a form of double induction which begins at the bottom scale going up, and then at each induction stage requires a separate downward induction.
- This double induction is what will allow us to regain the Ahlfors-regularity which is otherwise lost.

Proving the discrete Reifenberg I

- We will first consider the discrete Reifenberg.
- There is no loss in assuming $r_j = 2^{-n_j}$ is a power of two and the collection $\{B_{r_j}(x_j)\}$ is a finite collection.
- Now let us focus on proving the stronger result:

  For every x_ℓ and $r_\ell \le s \le 2$ we have $\mu(B_s(x_\ell)) \le A(n)s^k$. $(\star)$

- We will prove $(\star)$ inductively on $s = s_\beta = 2^{-\beta}$, i.e. this is our upward induction.
- In particular, for $s_\beta \approx \min r_j$ the result is clear by the definition of μ.
- Thus given that $(\star)$ holds for some $s_{\beta+1}$, we need to prove it for s_β.

Proving the discrete Reifenberg II

- Note first that by a covering argument we can obtain the worse estimate $\mu(B_{s_\beta}(x_\ell)) \leq B(n)s^k_\beta$, with $B >> A$.
- Also wlog $\mu(B_{s_\beta}(x_\ell)) > 10^{-1}As^k_\beta$, since otherwise we are done.

- Now similar to the proof of the Reifenberg we build a sequence of approximate submanifolds S_γ with $\gamma \geq \beta$, i.e. our downward induction.
- An important difference is that if for some $B_{s_\gamma}(y)$ we have either
  - (a) $B_{s_\gamma}(y) = B_{r_j}(x_j)$ (a final ball)
  - (b) $\mu(B_{s_\gamma}(y)) < \epsilon_n s^k_\gamma$ (a small volume ball)

  then we let $S_{\gamma'} \cap B_{s_\gamma}(y) = S_{\gamma-1} \cap B_{s_\gamma}(y)$ for all $\gamma' \geq \gamma$.
- Note by (a) that for large γ we have that $S_\gamma = S_{\gamma+1} \equiv S_\infty$ stabilizes.
- Note that (b) gives an Alhfors regularity condition.
- Therefore as previously suggested we can estimate $\phi_\gamma : S_\gamma \to S_{\gamma+1}$ by (6) and show that $\phi : S_\beta \to S_\infty$ has $W^{1,p}$ estimates for $p > n$.
- Using (a) and (b) this shows $\mu(B_{s_\beta}(x_\ell)) = \sum_{x_j \in B_s} r^k_j \leq As^k_\beta$, which finishes the inductive step and hence proof.

Proving the Rectifiable and $W^{1,p}$-Reifenberg

- To show the rect-Reifenberg we first show S must be sigma-finite, and thus we can restrict to a subset $\tilde{S}$ to assume $\lambda^k(\tilde{S}) < \infty$.
- Now find a cover $\{B_{r_j}(x_j)\}$ such that $B_{r_j}(x_j)$ are H^k-density balls, i.e. $\lambda^k\big|_S\big(B_{r_j}(x_j)\big) \approx r_j^k$.
- We can now apply the discrete Reifenberg to show $\sum r_j^k \leq Ar^k$.
- Since $B_r(x)$ was arbitrary this proves the upper volume estimates for $\tilde{S}$.
- Since $\tilde{S}$ was arbitrary, this shows the upper volume estimate for S.

- To see S is rectifiable now pick a ball B_{s_β} with $\lambda^k\big|_S(B_{s_\beta}) > (1-\epsilon_n)\omega_k s_\beta^k$.
- Consider $S \cap B_{s_\beta}$ and return to the construction of S_γ and ϕ_γ as in the discrete case, still under the condition (b).
- As before we can limit to a $W^{1,p}$ map $\phi : S_\beta \to S_\infty$ with $p > n$. Note $S = S_\infty$ for each $x \in S$ such that $\lambda^k(S \cap B_r(x)) > \epsilon_n r^k$ for all $r < s_\beta$.
- A small argument shows $\lambda^k(B_{s_\beta} \cap S \cap S_\infty) > \frac{1}{2}\omega_k s_\beta^k$.
- In particular, the rectifiability of S is easy to conclude from this.

- Finally, to prove the $W^{1,p}$-Reifenberg one observes that the lower volume estimate $\lambda^k(S \cap B_r(x)) > \epsilon_n r^k$ automatically holds.
- Thus $S_\infty \equiv S$ and we get our desired $W^{1,p}$-equivalence.

Idea 4: L^2-subspace approximation theorem

- Our goal is now to prove the Main Lemma by applying the discrete-Reifenberg.
- To accomplish this we need to understand how to approximate an essentially arbitrary measure μ by a k-plane, using only the properties of a stationary map.
- In general this is crazy. Surprisingly, if your stationary harmonic map is *not* $(k+1, \epsilon)$-symmetric, then it is true:

Theorem (L^2-subspace approximation theorem)

Let $f : B_8 \to N$ be a stationary harmonic map with $\int_{B_8} |\nabla f|^2 \leq \Lambda$. Let μ be an arbitrary measure supported on B_1. Then there exists $\epsilon(n, \Lambda, K), C(n, \Lambda, K)$ such that if B_8 is not $(k+1, \epsilon)$-symmetric, then we can estimate

$$D^k_\mu(x,1) \equiv \inf_{L^k} \int_{B_1} d^2(x, L^k)\, d\mu \leq C \int_{B_1} |\theta_8(y) - \theta_1(y)|\, d\mu\,. \quad (7)$$

Idea 4: L^2-subspace approximation: fake proof

- The proof of the L^2-approximation theorem is too involved to discuss in detail here, but let us discuss an important special case which helps build an intuition for why the theorem is true:
- Indeed: let us assume we have a measure μ and that $\int_{B_1} |\theta_8(y) - \theta_1(y)| d\mu = 0$.
- Then the theorem should imply there exists a k-plane L^k such that $\text{supp}\mu \subseteq L^k$. Let us show this:
- Thus note that if $x \in \text{supp}\mu$ then f is 0-symmetric at x, that is f is radially invariant with respect to x as the center point.
- Thus if there exists $j+1$ linearly independent points $\{x_0, \ldots, x_j\} \in \text{supp}\mu$ then f is j-symmetric with respect to the j-plane spanned by $\{x_0, \ldots, x_j\}$.
- Since f is not $k+1$ symmetric we must then get that there exists at most $k+1$ linearly independent points in $\text{supp}\mu$.
- This is precisely the statement that $\text{supp}\mu$ is contained in some k-plane L^k.

Proving the Main Lemma

- Let us now return to the Main Lemma.
- Recall that we have reduced the proof of the main theorems to the proof of the quantitative stratification estimates.
- We have further reduced the proof of the quantitative stratification estimates to the proof of the main lemma.
- Thus we must tackle this using the new Reifenberg results and the L^2-approximation theorems.

- Recall the setup. We have a Vitali collection $\{B_{r_j}(x_j)\} \subseteq B_2$ such that
  1. $x_j \in S^k_\epsilon \cap B_1$.
  2. $|\theta_1(x_j) - \theta_{r_j}(x_j)| < \eta$.
- From this we want to prove $\sum r_j^k \leq A(n)$.

Proving the Main Lemma

- Define $\mu \equiv \sum r_j^k \delta_{x_j}$. Thus we want to prove $\mu(B_1) \leq A(n)$.
- Instead, let us prove the stronger result:

  For every x_j and $r_j \leq s \leq 2$ we have that $\mu(B_s(x_j)) \leq As^k$. $(\star)$

- We will prove $(\star)$ inductively on $s = s_\beta = 2^{-\beta}$.
- In particular, for $s_\beta \approx \min r_j$ the result is clear.
- Thus imagine we have proved $(\star)$ for some $s_{\beta+1}$, and let us prove it for s_β.

Proving the Main Lemma

- Point 1: By covering $B_{10s_\beta}(x_j)$ by a controlled number of balls of radius $s_{\beta+1}$, we can use the inductive hypothesis to get that $\mu(B_{10s_\beta}(x_j)) \leq B(n)s_\beta^k$ where potentially $B(n) >> A(n)$.

- Point 2: Now since $x_j \in S^k_\epsilon$ we have that $B_s(x)$ is not $(k+1,\epsilon)$-symmetric. By the L^2-approximation theorem we have for every $s \leq 10s_\beta$ that

$$D^k_\mu(x_j, s) \leq Cs^{-k}\int_{B_s} |\theta_{8s} - \theta_s| d\mu\,. \tag{8}$$

- This gives us for each $s \leq 10s_\beta$ that

$$s_\beta^{-k}\int_{B_{s_\beta}} D_\mu(y,s)d\mu[y] \leq Cs^{-k}s_\beta^{-k}\int_{B_{s_\beta}}\int_{B_s}|\theta_{8s}-\theta_s|d\mu\, d\mu$$

$$\leq Cs^{-k}s_\beta^{-k}\int_{B_{s_\beta}}\mu(B_s(y))|\theta_{8s}-\theta_s|d\mu \leq Cs_\beta^{-k}\int_{B_{s_\beta}}|\theta_{8s}-\theta_s|d\mu$$

Proving the Main Lemma

- Summing over $s = s_\gamma \leq s_\beta$ we get the estimate

$$s_\beta^{-k}\int_{B_{s_\beta}}\sum_{s_\gamma\leq s_\beta} D_\mu(y,s_\gamma)d\mu \leq Cs_\beta^{-k}\int_{B_{s_\beta}}\sum|\theta_{8s_\gamma}-\theta_{s_\gamma}|d\mu$$
$$\leq Cs_\beta^{-k}\int_{B_{s_\beta}}|\theta_{8s_\beta}-\theta_{r_y}|(y)d\mu \leq C\eta s_\beta^{-k}\mu(B_{s_\beta}(x_j))$$
$$\leq C(n,\Lambda,K)\eta < \delta\,.$$

- where we have used the estimate $\mu(B_{s_\beta}(x_j)) < Bs_\beta^k$ and have chosen $\eta << C^{-1}(n,\Lambda,K)\delta$.
- Thus we may apply the discrete Reifenberg we can conclude that $\mu(B_{s_\beta}(x_j)) \leq A(n)s_\beta^k$, which completes the induction stage of the proof, and hence the theorem.

Integral Varifolds with Bounded Mean Curvature

Theorem (NV 15')

If I^m be an integral varifold in $B_2 \subseteq M$ with bounded mean curvature and finite mass. Then

1. $S^k(I)$ *is* k*-rectifiable for all* k.
2. *In fact, for* k*-a.e.* $x \in S^k(I)$ *there exists a unique* k*-plane* $V^k \subseteq \mathbb{R}^n$ *such that every tangent map at* x *is* k*-symmetric with respect to* V^k.

Theorem (NV 15')

If $I^m = I^{n-1}$ be a minimizing integral varifold of codimension one with mass bound $|I|(B_2) \leq \Lambda$. Then there exists $C(n, K, \Lambda) > 0$ such that

$$\begin{aligned} |I|(\{|A| > r^{-1}\}) &\leq |I|(B_r\{|A| > r^{-1}\}) \leq Cr^7\,, \\ |I|(B_r Sing(I)) &\leq Cr^7\,. \end{aligned} \tag{9}$$

In particular, $Sing(I)$ is $m-7$ rectifiable with $H^{m-7}(Sing(I)) \leq \Lambda$, and $|A| \in L^7_{weak}$ has apriori estimates.