HW 8.

(l) $V = M_{2\times 2}(C)$, $S = \left\{ \begin{pmatrix} 1-i & -2-3i \\ 2+2i & 4+i \end{pmatrix}, \begin{pmatrix} 8i & 4 \\ -3-3i & -4+4i \end{pmatrix}, \begin{pmatrix} -25-38i & -2-13i \\ 12-78i & -7+24i \end{pmatrix} \right\}$, and $A = \begin{pmatrix} -2+8i & -13+i \\ 10-10i & 9-9i \end{pmatrix}$

(m) $V = M_{2\times 2}(C)$, $S = \left\{ \begin{pmatrix} -1+i & -i \\ 2-i & 1+3i \end{pmatrix}, \begin{pmatrix} -1-7i & -9-8i \\ 1+10i & -6-2i \end{pmatrix}, \begin{pmatrix} -11-132i & -34-31i \\ 7-126i & -71-5i \end{pmatrix} \right\}$, and $A = \begin{pmatrix} -7+5i & 3+18i \\ 9-6i & -3+7i \end{pmatrix}$

Section 6.2.

3. In R^2, let

$$\beta = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) \right\}.$$

Find the Fourier coefficients of $(3,4)$ relative to β.

4. Let $S = \{(1,0,i),(1,2,1)\}$ in C^3. Compute $S^\perp$.

5. Let $S_0 = \{x_0\}$, where x_0 is a nonzero vector in R^3. Describe $S_0^\perp$ geometrically. Now suppose that $S = \{x_1, x_2\}$ is a linearly independent subset of R^3. Describe $S^\perp$ geometrically.

6. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y\rangle \neq 0$. *Hint:* Use Theorem 6.6.

7. Let β be a basis for a subspace W of an inner product space V, and let $z \in V$. Prove that $z \in W^\perp$ if and only if $\langle z, v\rangle = 0$ for every $v \in \beta$.

8. Prove that if $\{w_1, w_2, \ldots, w_n\}$ is an orthogonal set of nonzero vectors, then the vectors $v_1, v_2, \ldots, v_n$ derived from the Gram–Schmidt process satisfy $v_i = w_i$ for $i = 1, 2, \ldots, n$. *Hint:* Use mathematical induction.

9. Let $W = \text{span}(\{(i,0,1)\})$ in C^3. Find orthonormal bases for W and $W^\perp$.

10. Let W be a finite-dimensional subspace of an inner product space V. Prove that there exists a projection T on W along $W^\perp$ that satisfies $N(T) = W^\perp$. In addition, prove that $\|T(x)\| \le \|x\|$ for all $x \in V$. *Hint:* Use Theorem 6.6 and Exercise 10 of Section 6.1. (Projections are defined in the exercises of Section 2.1.)

11. Let A be an $n \times n$ matrix with complex entries. Prove that $AA^* = I$ if and only if the rows of A form an orthonormal basis for C^n.

12. Prove that for any matrix $A \in M_{m\times n}(F)$, $(R(L_{A^*}))^\perp = N(L_A)$.

13. Let V be an inner product space, S and S_0 be subsets of V, and W be a finite-dimensional subspace of V. Prove the following results.

(a) $S_0 \subseteq S$ implies that $S^\perp \subseteq S_0^\perp$.
(b) $S \subseteq (S^\perp)^\perp$; so $\text{span}(S) \subseteq (S^\perp)^\perp$.
(c) $W = (W^\perp)^\perp$. *Hint:* Use Exercise 6.
(d) $V = W \oplus W^\perp$. (See the exercises of Section 1.3.)

14. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$. (See the definition of the sum of subsets of a vector space on page 22.) *Hint for the second equation:* Apply Exercise 13(c) to the first equation.

15. Let V be a finite-dimensional inner product space over F.

(a) *Parseval's Identity.* Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. For any $x, y \in V$ prove that

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.$$

(b) Use (a) to prove that if β is an orthonormal basis for V with inner product $\langle \cdot, \cdot\rangle$, then for any $x, y \in V$

$$\langle \phi_\beta(x), \phi_\beta(y)\rangle' = \langle [x]_\beta, [y]_\beta\rangle' = \langle x, y\rangle,$$

where $\langle \cdot, \cdot\rangle'$ is the standard inner product on F^n.

16. **(a)** *Bessel's Inequality.* Let V be an inner product space, and let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal subset of V. Prove that for any $x \in V$ we have

$$\|x\|^2 \ge \sum_{i=1}^{n} |\langle x, v_i\rangle|^2.$$

Hint: Apply Theorem 6.6 to $x \in V$ and $W = \text{span}(S)$. Then use Exercise 10 of Section 6.1.

(b) In the context of (a), prove that Bessel's inequality is an equality if and only if $x \in \text{span}(S)$.

17. Let T be a linear operator on an inner product space V. If $\langle T(x), y\rangle = 0$ for all $x, y \in V$, prove that $T = T_0$. In fact, prove this result if the equality holds for all x and y in some basis for V.

$T_0 = 0$ linear operator.

18. Let $V = C([-1,1])$. Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. (See Exercise 22

(a) $V = R^3$, $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$
(b) $V = C^2$, $g(z_1, z_2) = z_1 - 2z_2$
(c) $V = P_2(R)$ with $\langle f, h\rangle = \int_0^1 f(t)h(t)\,dt$, $g(f) = f(0) + f'(1)$

Section 6.3.

3. For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.
(a) $V = R^2$, $T(a, b) = (2a + b, a - 3b)$, $x = (3, 5)$.
(b) $V = C^2$, $T(z_1, z_2) = (2z_1 + iz_2, (1 - i)z_1)$, $x = (3 - i, 1 + 2i)$.
(c) $V = P_1(R)$ with $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)\,dt$, $T(f) = f' + 3f$, $f(t) = 4 - 2t$

4. Complete the proof of Theorem 6.11.

5. (a) Complete the proof of the corollary to Theorem 6.11 by using Theorem 6.11, as in the proof of (c).
(b) State a result for nonsquare matrices that is analogous to the corollary to Theorem 6.11, and prove it using a matrix argument.

6. Let T be a linear operator on an inner product space V. Let $U_1 = T + T^*$ and $U_2 = TT^*$. Prove that $U_1 = U_1^*$ and $U_2 = U_2^*$.

7. Give an example of a linear operator T on an inner product space V such that $N(T) \neq N(T^*)$.

8. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

9. Prove that if $V = W \oplus W^\perp$ and T is the projection on W along $W^\perp$, then $T = T^*$. *Hint:* Recall that $N(T) = W^\perp$. (For definitions, see the exercises of Sections 1.3 and 2.1.)

10. Let T be a linear operator on an inner product space V. Prove that $\|T(x)\| = \|x\|$ for all $x \in V$ if and only if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$. *Hint:* Use Exercise 20 of Section 6.1.

11. For a linear operator T on an inner product space V, prove that $T^*T = T_0$ implies $T = T_0$. Is the same result true if we assume that $TT^* = T_0$?

12. Let V be an inner product space, and let T be a linear operator on V. Prove the following results.
(a) $R(T^*)^\perp = N(T)$.
(b) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$. *Hint:* Use Exercise 13(c) of Section 6.2.

13. Let T be a linear operator on a finite-dimensional inner product space V. Prove the following results.
(a) $N(T^*T) = N(T)$. Deduce that $\text{rank}(T^*T) = \text{rank}(T)$.
(b) $\text{rank}(T) = \text{rank}(T^*)$. Deduce from (a) that $\text{rank}(TT^*) = \text{rank}(T)$.
(c) For any $n \times n$ matrix A, $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

14. Let V be an inner product space, and let $y, z \in V$. Define $T\colon V \to V$ by $T(x) = \langle x, y\rangle z$ for all $x \in V$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

The following definition is used in Exercises 15–17 and is an extension of the definition of the *adjoint* of a linear operator.

Definition. *Let $T\colon V \to W$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, respectively. A function $T^*\colon W \to V$ is called an* **adjoint** *of T if $\langle T(x), y\rangle_2 = \langle x, T^*(y)\rangle_1$ for all $x \in V$ and $y \in W$.*

15. Let $T\colon V \to W$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, respectively. Prove the following results.
(a) There is a unique adjoint T^* of T, and T^* is linear.
(b) If β and γ are orthonormal bases for V and W, respectively, then $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$.
(c) $\text{rank}(T^*) = \text{rank}(T)$.
(d) $\langle T^*(x), y\rangle_1 = \langle x, T(y)\rangle_2$ for all $x \in W$ and $y \in V$.
(e) For all $x \in V$, $T^*T(x) = 0$ if and only if $T(x) = 0$.

16. State and prove a result that extends the first four parts of Theorem 6.11 using the preceding definition.

17. Let $T\colon V \to W$ be a linear transformation, where V and W are finite-dimensional inner product spaces. Prove that $(R(T^*))^\perp = N(T)$, using the preceding definition.

18.† Let A be an $n \times n$ matrix. Prove that $\det(A^*) = \overline{\det(A)}$.

19. Suppose that A is an $m \times n$ matrix in which no two columns are identical. Prove that A^*A is a diagonal matrix if and only if every pair of columns of A is orthogonal.

20. For each of the sets of data that follows, use the least squares approximation to find the best fits with both (i) a linear function and (ii) a quadratic function. Compute the error E in both cases.
(a) $\{(-3, 9), (-2, 6), (0, 2), (1, 1)\}$