of dinners into nonoverlapping subsets. To count the number of dinners, we use the Multiplication Principle. To count the number of items available at Kay's Quick Lunch, we just sum the number of items each category since dividing the items by category naturally splits them into nonoverlapping subsets. We are *not* counting the individual items available by constructing them using a step-by-step process. To count the total number of items available, we use the Addition Principle.

The Inclusion-Exclusion Principle (Theorem 6.1.12) is a variant of the Addition Principle that can be used when the sets involved are *not* pairwise disjoint.

Section Review Exercises

1. State the Multiplication Principle and give an example of its use.
2. State the Addition Principle and give an example of its use.
3. State the Inclusion-Exclusion Principle for two sets and give an example of its use.

Exercises

Use the Multiplication Principle to solve Exercises 1–9.

1. How many dinners at Kay's Quick Lunch (Figure 6.1.1) consist of one appetizer and one beverage?
2. How many dinners at Kay's Quick Lunch (Figure 6.1.1) consist of one appetizer, one main course, and an optional beverage?
3. How many dinners at Kay's Quick Lunch (Figure 6.1.1) consist of an optional appetizer, one main course, and an optional beverage?
4. A man has eight shirts, four pairs of pants, and five pairs of shoes. How many different outfits are possible?
5. The options available on a particular model of a car are five interior colors, six exterior colors, two types of seats, three types of engines, and three types of radios. How many different possibilities are available to the consumer?
6. The Braille system of representing characters was developed early in the nineteenth century by Louis Braille. The characters, used by the blind, consist of raised dots. The positions for the dots are selected from two vertical columns of three dots each. At least one raised dot must be present. How many distinct Braille characters are possible?
7. Two dice are rolled, one blue and one red. How many outcomes are possible?
8. How many different car license plates can be constructed if the licenses contain three letters followed by two digits if repetitions are allowed? if repetitions are not allowed?
9. A restaurant chain advertised a special in which a customer could choose one of five appetizers, one of 14 main dishes, and one of three desserts. The ad said that there were 210 possible dinners. Was the ad correct? Explain.

Use the Addition Principle to solve Exercises 10–18.

10. Three departmental committees have 6, 12, and 9 members with no overlapping membership. In how many ways can these committees send one member to meet with the president?
11. In how many ways can a diner choose one item from among the appetizers and main courses at Kay's Quick Lunch (Figure 6.6.1)?
12. In how many ways can a diner choose one item from among the appetizers and beverages at Kay's Quick Lunch (Figure 6.6.1)?
13. How many times are the print statements executed?

```
for i = 1 to m
   println(i)
for j = 1 to n
   println(j)
```

14. How many times is the print statement executed?

```
for i = 1 to m
   for j = 1 to n
      println(i, j)
```

15. Given that there are 32 eight-bit strings that begin 101 and 16 eight-bit strings that begin 1101, how many eight-bit strings begin either 101 or 1101?
16. Two dice are rolled, one blue and one red. How many outcomes give the sum of 2 or the sum of 12?
17. A committee composed of Morgan, Tyler, Max, and Leslie is to select a president and secretary. How many selections are there in which Tyler is president or not an officer?
18. A committee composed of Morgan, Tyler, Max, and Leslie is to select a president and secretary. How many selections are there in which Max is president or secretary?
19. Comment on the following item from *The New York Times:*

> Big pickups also appeal because of the seemingly infinite ways they can be personalized; you need the math skills of Will Hunting to total the configurations. For starters, there are 32 combinations of cabs (standard, Club Cab, Quad Cab), cargo beds (6.5 or 8 feet) and engines (3.9-liter V6, 5.2-liter V8, 5.9-liter V8, 5.9-liter turbo-diesel inline 6, 8-liter V10).

In Exercises 20–27, two dice are rolled, one blue and one red.

20. How many outcomes give the sum of 4?
21. How many outcomes are doubles? (A double occurs when both dice show the same number.)
22. How many outcomes give the sum of 7 or the sum of 11?
23. How many outcomes have the blue die showing 2?
24. How many outcomes have exactly one die showing 2?
25. How many outcomes have at least one die showing 2?
26. How many outcomes have neither die showing 2?
27. How many outcomes give an even sum?

In Exercises 28–30, suppose there are 10 roads from Oz to Mid Earth and five roads from Mid Earth to Fantasy Island.

28. How many routes are there from Oz to Fantasy Island passing through Mid Earth?
29. How many round-trips are there of the form Oz–Mid Earth–Fantasy Island–Mid Earth–Oz?
30. How many round-trips are there of the form Oz–Mid Earth–Fantasy Island–Mid Earth–Oz in which on the return trip we do not reverse the original route from Oz to Fantasy Island?
31. How many eight-bit strings begin 1100?
32. How many eight-bit strings begin and end with 1?
33. How many eight-bit strings have either the second or the fourth bit 1 (or both)?
34. How many eight-bit strings have exactly one 1?
35. How many eight-bit strings have exactly two 1's?
36. How many eight-bit strings have at least one 1?
37. How many eight-bit strings read the same from either end? (An example of such an eight-bit string is 01111110. Such strings are called *palindromes*.)

In Exercises 38–43, a six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

38. How many selections exclude Connie?
39. How many selections are there in which neither Ben nor Francisco is an officer?
40. How many selections are there in which both Ben and Francisco are officers?
41. How many selections are there in which Dolph is an officer and Francisco is not an officer?
42. How many selections are there in which either Dolph is chairperson or he is not an officer?
43. How many selections are there in which Ben is either chairperson or treasurer?

In Exercises 44–51, the letters ABCDE are to be used to form strings of length 3.

44. How many strings can be formed if we allow repetitions?
45. How many strings can be formed if we do not allow repetitions?
46. How many strings begin with *A*, allowing repetitions?
47. How many strings begin with *A* if repetitions are not allowed?
48. How many strings do not contain the letter *A*, allowing repetitions?
49. How many strings do not contain the letter *A* if repetitions are not allowed?
50. How many strings contain the letter *A*, allowing repetitions?
51. How many strings contain the letter *A* if repetitions are not allowed?

Exercises 52–62 refer to the integers from 5 to 200, inclusive.

52. How many numbers are there?
53. How many are even?
54. How many are odd?
55. How many are divisible by 5?
56. How many are greater than 72?
57. How many consist of distinct digits?
58. How many contain the digit 7?
59. How many do not contain the digit 0?
60. How many are greater than 101 and do not contain the digit 6?
61. How many have the digits in strictly increasing order? (Examples are 13, 147, 8.)
62. How many are of the form xyz, where $0 \neq x < y$ and $y > z$?
63. (a) In how many ways can the months of the birthdays of five people be distinct?
    (b) How many possibilities are there for the months of the birthdays of five people?
    (c) In how many ways can at least two people among five have their birthdays in the same month?

Exercises 64–68 refer to a set of five distinct computer science books, three distinct mathematics books, and two distinct art books.

64. In how many ways can these books be arranged on a shelf?
65. In how many ways can these books be arranged on a shelf if all five computer science books are on the left and both art books are on the right?
66. In how many ways can these books be arranged on a shelf if all five computer science books are on the left?
67. In how many ways can these books be arranged on a shelf if all books of the same discipline are grouped together?
★68. In how many ways can these books be arranged on a shelf if the two art books are not together?

69. In some versions of FORTRAN, an identifier consists of a string of one to six alphanumeric characters beginning with a letter. (An *alphanumeric character* is one of A to Z or 0 to 9.) How many valid FORTRAN identifiers are there?

70. If X is an n-element set and Y is an m-element set, how many functions are there from X to Y?

★71. There are 10 copies of one book and one copy each of 10 other books. In how many ways can we select 10 books?

72. How many terms are there in the expansion of

$$(x+y)(a+b+c)(e+f+g)(h+i)?$$

★73. How many subsets of a $(2n+1)$-element set have n elements or less?

74. A Class B Internet address, used for medium-sized networks, is a bit string of length 32. The first bits are 10 (to identify it as a Class B address). The netid is given by the next 14 bits, which identifies the network. The hostid is given by the remaining 16 bits, which identifies the computer interface. The hostid must not consist of all 0's or all 1's. How many Class B addresses are available?

75. A Class C Internet address, used for small networks, is a bit string of length 32. The first bits are 110 (to identify it as a Class C address). The netid is given by the next 21 bits, which identifies the network. The hostid is given by the remaining 8 bits, which identifies the computer interface. The hostid must not consist of all 0's or all 1's. How many Class C addresses are available?

76. Given that the IPv4 Internet address of a computer interface is either Class A, Class B, or Class C, how many IPv4 Internet addresses are available?

77. How many symmetric relations are there on an n-element set?

78. How many antisymmetric relations are there on an n-element set?

79. How many reflexive and symmetric relations are there on an n-element set?

80. How many reflexive and antisymmetric relations are there on an n-element set?

81. How many symmetric and antisymmetric relations are there on an n-element set?

82. How many reflexive, symmetric, and antisymmetric relations are there on an n-element set?

83. How many truth tables are there for an n-variable function?

84. How many binary operators are there on $\{1, 2, \ldots, n\}$?

85. How many commutative binary operators are there on $\{1, 2, \ldots, n\}$?

Use the Inclusion-Exclusion Principle (Theorem 6.1.12) to solve Exercises 86–91.

86. How many eight-bit strings either begin 100 or have the fourth bit 1 or both?

87. How many eight-bit strings either start with a 1 or end with a 1 or both?

In Exercises 88 and 89, a six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

88. How many selections are there in which either Ben is chairperson or Alice is secretary or both?

89. How many selections are there in which either Connie is chairperson or Alice is an officer or both?

90. Two dice are rolled, one blue and one red. How many outcomes have either the blue die 3 or an even sum or both?

91. How many integers from 1 to 10,000, inclusive, are multiples of 5 or 7 or both?

92. Prove the Inclusion-Exclusion Principle for three finite sets:

$$|X \cup Y \cup Z| = |X|+|Y|+|Z|-|X \cap Y|-|X \cap Z|-|Y \cap Z|+|X \cap Y \cap Z|.$$

Hint: Write the Inclusion-Exclusion Principle for two finite sets as

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and let $A = X$ and $B = Y \cup Z$.

93. In a group of 191 students, 10 are taking French, business, and music; 36 are taking French and business; 20 are taking French and music; 18 are taking business and music; 65 are taking French; 76 are taking business; and 63 are taking music. Use the Inclusion-Exclusion Principle for three finite sets (see Exercise 92) to determine how many students are not taking any of the three courses.

94. Use the Inclusion-Exclusion Principle for three finite sets (see Exercise 92) to solve the problem in Example 1.1.20.

95. Use the Inclusion-Exclusion Principle for three finite sets (see Exercise 92) to solve Exercise 55, Section 1.1.

96. Use the Inclusion-Exclusion Principle for three finite sets (see Exercise 92) to compute the number of integers between 1 and 10,000, inclusive, that are multiples of 3 or 5 or 11 or any combination thereof.

★97. Use Mathematical Induction to prove the general Inclusion-Exclusion Principle for finite sets $X_1, X_2, \ldots, X_n$:

$$|X_1 \cup X_2 \cup \cdots \cup X_n| = \sum_{1 \le i \le n} |X_i| - \sum_{1 \le i < j \le n} |X_i \cap X_j|$$
$$+ \sum_{1 \le i < j < k \le n} |X_i \cap X_j \cap X_k| - \cdots$$
$$+ (-1)^{n+1} |X_1 \cap X_2 \cap \cdots \cap X_n|.$$

Hint: In the Inductive Step, refer to the hint in Exercise 92.

98. Using the previous exercise, write the Inclusion-Exclusion Principle for four finite sets.

99. How many integers between 1 and 10,000, inclusive, are multiples of 3 or 5 or 11 or 13 or any combination thereof?

Problem-Solving Corner — Counting

Problem

Find the number of ordered triples of sets X_1, X_2, X_3 satisfying

$$X_1 \cup X_2 \cup X_3 = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$\text{and} \quad X_1 \cap X_2 \cap X_3 = \varnothing.$$

By *ordered triple*, we mean that the order of the sets X_1, X_2, X_3 is taken into account. For example, the triples

$$\{1, 2, 3\}, \quad \{1, 4, 8\}, \quad \{2, 5, 6, 7\}$$

and

$$\{1, 4, 8\}, \quad \{1, 2, 3\}, \quad \{2, 5, 6, 7\}$$

are considered distinct.

Attacking the Problem

It would be nice to begin by enumerating triples, but there are so many it would be hard to gain much insight from staring at a few triples. Let's simplify the problem by making it smaller. Let's replace

$$\{1, 2, 3, 4, 5, 6, 7, 8\}$$

by $\{1\}$. What could be simpler than $\{1\}$? (Well, maybe $\varnothing$, but that's too simple!) We can now enumerate all ordered triples of sets X_1, X_2, X_3 satisfying $X_1 \cup X_2 \cup X_3 = \{1\}$ and $X_1 \cap X_2 \cap X_3 = \varnothing$. We must put 1 in at least one of the sets X_1, X_2, X_3 (so that the union will be $\{1\}$), but we must not put 1 in all three of the sets X_1, X_2, X_3 (otherwise, the intersection would not be empty). Thus 1 will be in exactly one or two of the sets X_1, X_2, X_3. The complete list of ordered triples is as follows:

$$X_1 = \{1\}, \quad X_2 = \varnothing, \quad X_3 = \varnothing;$$
$$X_1 = \varnothing, \quad X_2 = \{1\}, \quad X_3 = \varnothing;$$
$$X_1 = \varnothing, \quad X_2 = \varnothing, \quad X_3 = \{1\};$$
$$X_1 = \{1\}, \quad X_2 = \{1\}, \quad X_3 = \varnothing;$$
$$X_1 = \{1\}, \quad X_2 = \varnothing, \quad X_3 = \{1\};$$
$$X_1 = \varnothing, \quad X_2 = \{1\}, \quad X_3 = \{1\}.$$

Thus there are six ordered triples of sets X_1, X_2, X_3 satisfying

$$X_1 \cup X_2 \cup X_3 = \{1\} \quad \text{and} \quad X_1 \cap X_2 \cap X_3 = \varnothing.$$

Let's step up one level and enumerate all ordered triples of sets X_1, X_2, X_3 satisfying $X_1 \cup X_2 \cup X_3 = \{1, 2\}$ and $X_1 \cap X_2 \cap X_3 = \varnothing$. As before, we must put 1 in at least one of the sets X_1, X_2, X_3 (so that 1 will be in the union), but we must not put 1 in all three of the sets X_1, X_2, X_3 (otherwise, the intersection would not be empty). This time we must also put 2 in at least one of the sets X_1, X_2, X_3 (so that 2 will also be in the union), but we must not put 2 in all three of the sets X_1, X_2, X_3 (otherwise, the intersection would not be empty). Thus each of 1 and 2 will be in exactly one or two of the sets X_1, X_2, X_3. We enumerate the sets in a systematic way so that we can recognize any patterns that appear. The complete list of ordered triples is shown in the table at the bottom of this page. For example, the top left entry,

1 is in	*2 is in*	*1 is in*	*2 is in*	*1 is in*	*2 is in*
X_1	X_1	X_1	X_2	X_1	X_3
X_2	X_1	X_2	X_2	X_2	X_3
X_3	X_1	X_3	X_2	X_3	X_3
X_1, X_2	X_1	X_1, X_2	X_2	X_1, X_2	X_3
X_1, X_3	X_1	X_1, X_3	X_2	X_1, X_3	X_3
X_2, X_3	X_1	X_2, X_3	X_2	X_2, X_3	X_3
X_1	X_1, X_2	X_1	X_1, X_3	X_1	X_2, X_3
X_2	X_1, X_2	X_2	X_1, X_3	X_2	X_2, X_3
X_3	X_1, X_2	X_3	X_1, X_3	X_3	X_2, X_3
X_1, X_2	X_1, X_2	X_1, X_2	X_1, X_3	X_1, X_2	X_2, X_3
X_1, X_3	X_1, X_2	X_1, X_3	X_1, X_3	X_1, X_3	X_2, X_3
X_2, X_3	X_1, X_2	X_2, X_3	X_1, X_3	X_2, X_3	X_2, X_3

counting problems is to determine whether we are counting ordered or unordered items. For example, a line of *distinct* persons is considered ordered. Thus six distinct persons can wait in line in 6! ways; the permutation formula is used. A committee is a typical example of an unordered group. For example, a committee of three can be selected from a set of six distinct persons in $C(6, 3)$ ways; the combination formula is used.

Section Review Exercises

1. What is a permutation of $x_1, \ldots, x_n$?
2. How many permutations are there of an n-element set? How is this formula derived?
3. What is an r-permutation of $x_1, \ldots, x_n$?
4. How many r-permutations are there of an n-element set? How is this formula derived?
5. How do we denote the number of r-permutations of an n-element set?
6. What is an r-combination of $\{x_1, \ldots, x_n\}$?
7. How many r-combinations are there of an n-element set? How is this formula derived?
8. How do we denote the number of r-combinations of an n-element set?

Exercises

1. How many permutations are there of a, b, c, d?
2. List the permutations of a, b, c, d.
3. How many 3-permutations are there of a, b, c, d?
4. List the 3-permutations of a, b, c, d.
5. How many permutations are there of 11 distinct objects?
6. How many 5-permutations are there of 11 distinct objects?
7. In how many ways can we select a chairperson, vice-chairperson, and recorder from a group of 11 persons?
8. In how many ways can we select a chairperson, vice-chairperson, secretary, and treasurer from a group of 12 persons?
9. In how many different ways can 12 horses finish in the order Win, Place, Show?

In Exercises 10–18, determine how many strings can be formed by ordering the letters ABCDE subject to the conditions given.

10. Contains the substring *ACE*
11. Contains the letters *ACE* together in any order
12. Contains the substrings *DB* and *AE*
13. Contains either the substring *AE* or the substring *EA* or both
14. *A* appears before *D*. *Examples*: *BCAED*, *BCADE*
15. Contains neither of the substrings *AB*, *CD*
16. Contains neither of the substrings *AB*, *BE*
17. *A* appears before *C* and *C* appears before *E*
18. Contains either the substring *DB* or the substring *BE* or both
19. In how many ways can five distinct Martians and eight distinct Jovians wait in line if no two Martians stand together?
20. In how many ways can five distinct Martians, ten distinct Vesuvians, and eight distinct Jovians wait in line if no two Martians stand together?
21. In how many ways can five distinct Martians and five distinct Jovians wait in line?
22. In how many ways can five distinct Martians and five distinct Jovians be seated at a circular table?
23. In how many ways can five distinct Martians and five distinct Jovians be seated at a circular table if no two Martians sit together?
24. In how many ways can five distinct Martians and eight distinct Jovians be seated at a circular table if no two Martians sit together?

In Exercises 25–27, let $X = \{a, b, c, d\}$.

25. Compute the number of 3-combinations of X.
26. List the 3-combinations of X.
27. Show the relationship between the 3-permutations and the 3-combinations of X by drawing a picture like that in Figure 6.2.4.
28. In how many ways can we select a committee of three from a group of 11 persons?
29. In how many ways can we select a committee of four from a group of 12 persons?
30. At one point in the Illinois state lottery Lotto game, a person was required to choose six numbers (in any order) among 44 numbers. In how many ways can this be done? The state was considering changing the game so that a person would be required to choose six numbers among 48 numbers. In how many ways can this be done?
31. Suppose that a pizza parlor features four specialty pizzas and pizzas with three or fewer unique toppings (no choosing anchovies twice!) chosen from 17 available toppings. How many different pizzas are there?
32. Suppose that the pizza parlor of Exercise 31 has a special price for four pizzas. How many ways can four pizzas be selected?

Exercises 33–38 refer to a club consisting of six distinct men and seven distinct women.

33. In how many ways can we select a committee of five persons?
34. In how many ways can we select a committee of three men and four women?
35. In how many ways can we select a committee of four persons that has at least one woman?
36. In how many ways can we select a committee of four persons that has at most one man?
37. In how many ways can we select a committee of four persons that has persons of both sexes?
38. In how many ways can we select a committee of four persons so that Mabel and Ralph do not serve together?
39. In how many ways can we select a committee of four Republicans, three Democrats, and two Independents from a group of 10 distinct Republicans, 12 distinct Democrats, and four distinct Independents?
40. How many eight-bit strings contain exactly three 0's?
41. How many eight-bit strings contain three 0's in a row and five 1's?

★42. How many eight-bit strings contain at least two 0's in a row?

In Exercises 43–51, find the number of (unordered) five-card poker hands, selected from an ordinary 52-card deck, having the properties indicated.

43. Containing four aces
44. Containing four of a kind, that is, four cards of the same denomination
45. Containing all spades
46. Containing cards of exactly two suits
47. Containing cards of all suits
48. Of the form A2345 of the same suit
49. Consecutive and of the same suit (Assume that the ace is the lowest denomination.)
50. Consecutive (Assume that the ace is the lowest denomination.)
51. Containing two of one denomination, two of another denomination, and one of a third denomination
52. Find the number of (unordered) 13-card bridge hands selected from an ordinary 52-card deck.
53. How many bridge hands are all of the same suit?
54. How many bridge hands contain exactly two suits?
55. How many bridge hands contain all four aces?
56. How many bridge hands contain five spades, four hearts, three clubs, and one diamond?
57. How many bridge hands contain five of one suit, four of another suit, three of another suit, and one of another suit?
58. How many bridge hands contain four cards of three suits and one card of the fourth suit?
59. How many bridge hands contain no face cards? (A face card is one of 10, J, Q, K, A.)

In Exercises 60–64, a coin is flipped 10 times.

60. How many outcomes are possible? (An *outcome* is a list of 10 H's and T's that gives the result of each of 10 tosses. For example, the outcome

H H T H T H H H T H

represents 10 tosses, where a head was obtained on the first two tosses, a tail was obtained on the third toss, a head was obtained on the fourth toss, etc.)
61. How many outcomes have exactly three heads?
62. How many outcomes have at most three heads?
63. How many outcomes have a head on the fifth toss?
64. How many outcomes have as many heads as tails?

Exercises 65–68 refer to a shipment of 50 microprocessors of which four are defective.

65. In how many ways can we select a set of four microprocessors?
66. In how many ways can we select a set of four nondefective microprocessors?
67. In how many ways can we select a set of four microprocessors containing exactly two defective microprocessors?
68. In how many ways can we select a set of four microprocessors containing at least one defective microprocessor?

★69. Show that the number of bit strings of length $n \geq 4$ that contain exactly two occurrences of 10 is $C(n + 1, 5)$.

★70. Show that the number of n-bit strings having exactly k 0's, with no two 0's consecutive, is $C(n - k + 1, k)$.

★71. Show that the product of any positive integer and its $k - 1$ successors is divisible by $k!$.

72. Show that there are $(2n - 1)(2n - 3) \cdots 3 \cdot 1$ ways to pick n pairs from $2n$ distinct items.

Exercises 73–75 refer to an election in which two candidates Wright and Upshaw ran for dogcatcher. After each vote was tabulated, Wright was never behind Upshaw. This problem is known as the ballot problem.

73. Suppose that each candidate received exactly r votes. Show that the number of ways the votes could be counted is C_r, the rth Catalan number.
74. Suppose that Wright received exactly r votes and Upshaw received exactly u votes, $r \geq u > 0$. Show that the number of ways the votes could be counted is $C(r + u, r) - C(r + u, r + 1)$.
75. Show that if exactly n votes were cast, the number of ways the votes could be counted is $C(n, \lceil n/2 \rceil)$.
76. Suppose that we start at the origin in the xy-plane and take n unit steps (i.e., each step is of length one), where each step is either vertical (up or down) or horizontal (left or right). How many such paths never go strictly below the x-axis?
77. Suppose that we start at the origin in the xy-plane and take n unit steps (i.e., each step is of length one), where each step is either vertical (up or down) or horizontal (left or right). How many such paths stay in the first quadrant ($x \geq 0, y \geq 0$)?

78. Show that the number of ways that $2n$ persons, seated around a circular table, can shake hands in pairs without any arms crossing is C_n, the nth Catalan number.

★79. Show that the print statement in the pseudocode

```
for i1 = 1 to n
  for i2 = 1 to min(i1, n − 1)
    for i3 = 1 to min(i2, n − 2)
      ⋱
        for i_{n−1} = 1 to min(i_{n−2}, 2)
          for i_n = 1 to 1
            println(i1, i2, ..., i_n)
```

is executed C_n times, where C_n denotes the nth Catalan number.

80. Suppose that we have n objects, r distinct and $n - r$ identical. Give another derivation of the formula

$$P(n, r) = r!\,C(n, r)$$

by counting the number of orderings of the n objects in two ways:

- Count the orderings by first choosing positions for the r distinct objects.
- Count the orderings by first choosing positions for the $n - r$ identical objects.

81. What is wrong with the following argument, which purports to show that $4C(39, 13)$ bridge hands contain three or fewer suits?

There are $C(39, 13)$ hands that contain only clubs, diamonds, and spades. In fact, for any three suits, there are $C(39, 13)$ hands that contain only those three suits. Since there are four 3-combinations of the suits, the answer is $4C(39, 13)$.

82. What is wrong with the following argument, which purports to show that there are $13^4 \cdot 48$ (unordered) five-card poker hands containing cards of all suits?

Pick one card of each suit. This can be done in $13 \cdot 13 \cdot 13 \cdot 13 = 13^4$ ways. Since the fifth card can be chosen in 48 ways, the answer is $13^4 \cdot 48$.

83. What is wrong with the following argument, which purports to show that there are $P(n, m)m^{n-m}$ onto functions from the n-element set X to the m-element set Y, $n > m$?

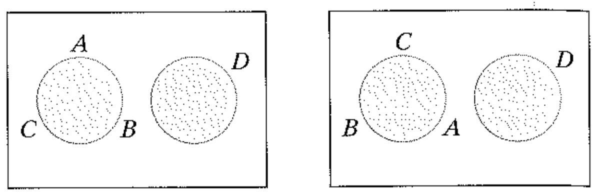
Let $Y = \{y_1, \ldots, y_m\}$. To ensure that a function from X to Y is onto Y, we select an m-permutation of X, say $x_1, \ldots, x_m$, and assign x_1 the value y_1, x_2 the value $y_2, \ldots,$ and x_m the value y_m. We can select the m-permutation in $P(n, m)$ ways. The remainder of the $n - m$ elements in X may be assigned values in Y arbitrarily. The first remaining element in X can be assigned a value in Y in m ways. The next remaining element in X can also be assigned a value in Y in m ways, and so on. Thus the remaining $n - m$ elements in X can be assigned values in Y in m^{n-m} ways. Thus the number of functions from X onto Y is $P(n, m)m^{n-m}$.

84. How many times is the string 10100001 counted in the erroneous argument given in Example 6.2.24?

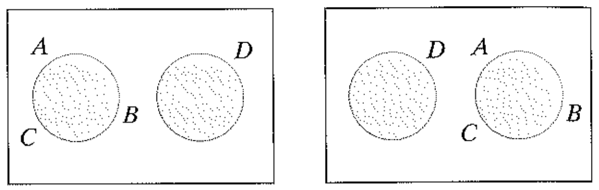
85. How many times is the string 10001000 counted in the erroneous argument given in Example 6.2.24?

86. How many times is the string 00000000 counted in the erroneous argument given in Example 6.2.24?

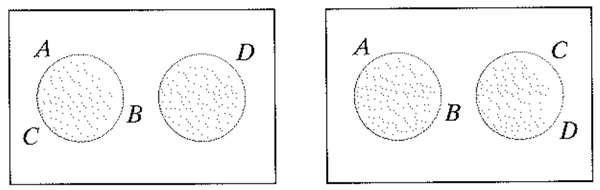
87. Let $s_{n,k}$ denote the number of ways to seat n persons at k round tables, with at least one person at each table. (The numbers $s_{n,k}$ are called *Stirling numbers of the first kind.*) The ordering of the tables is *not* taken into account. The seating arrangement at a table *is* taken into account except for rotations. *Examples:* The following pair is *not* distinct:



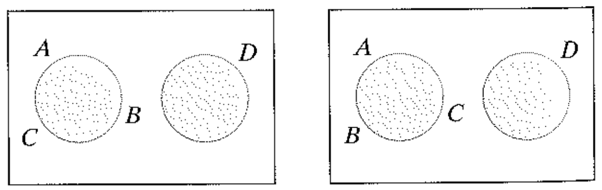
The following pair is *not* distinct:



The following pair *is* distinct:



The following pair *is* distinct:



(a) Show that $s_{n,k} = 0$ if $k > n$.

(b) Show that $s_{n,n} = 1$ for all $n \ge 1$.

(c) Show that $s_{n,1} = (n - 1)!$ for all $n \ge 1$.

(d) Show that $s_{n,n-1} = C(n, 2)$ for all $n \ge 2$.

(e) Show that

$$s_{n,2} = (n-1)!\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right)$$

for all $n \ge 2$.

(f) Show that

$$\sum_{k=1}^{n} s_{n,k} = n! \qquad \text{for all } n \ge 1.$$

(g) Find a formula for $s_{n,n-2}$, $n \ge 3$, and prove it.

88. Let $S_{n,k}$ denote the number of ways to partition an n-element set into exactly k nonempty subsets. The order of the subsets is not taken into account. (The numbers $S_{n,k}$ are called *Stirling numbers of the second kind.*)

(a) Show that $S_{n,k} = 0$ if $k > n$.

(b) Show that $S_{n,n} = 1$ for all $n \ge 1$.

(c) Show that $S_{n,1} = 1$ for all $n \ge 1$.

(d) Show that $S_{3,2} = 3$.

(e) Show that $S_{4,2} = 7$.

(f) Show that $S_{4,3} = 6$.

(g) Show that $S_{n,2} = 2^{n-1} - 1$ for all $n \ge 2$.

(h) Show that $S_{n,n-1} = C(n, 2)$ for all $n \ge 2$.

(i) Find a formula for $S_{n,n-2}$, $n \ge 3$, and prove it.

89. Show that there are

$$\sum_{k=1}^{n} S_{n,k}$$

equivalence relations on an n-element set. [The numbers $S_{n,k}$ are Stirling numbers of the second kind (see Exercise 88).]

90. If X is an n-element set and Y is an m-element set, $n \le m$, how many one-to-one functions are there from X to Y?

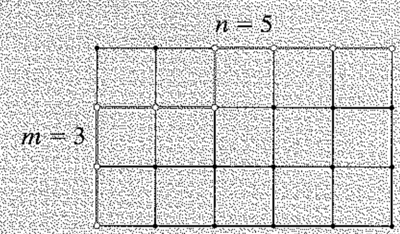
91. If X and Y are n-element sets, how many one-to-one, onto functions are there from X to Y?

92. Show that $(n/k)^k \le C(n, k) \le n^k/k!$.

Problem-Solving Corner — Combinations

Problem

(a) How many routes are there from the lower-left corner to the upper-right corner of an $m \times n$ grid in which we are restricted to traveling only to the right or upward? For example, the following figure is a 3×5 grid and one route is shown.



(b) Divide the routes into classes based on when the route first meets the top edge to derive the formula

$$\sum_{k=0}^{n} C(k + m - 1, k) = C(m + n, m).$$

Attacking the Problem

Example 6.2.22 asked how many paths there were from the lower-left corner to the upper-right corner of an $n \times n$ grid in which we are restricted to traveling only to the right or upward. The solution to that problem encoded each route as a string of n R's (right) and n U's (up). The problem then became one of counting the number of such strings. Any such string can be obtained by selecting n positions for the R's, without regard to the order of selection, among the $2n$ available positions in the string and then filling the remaining positions with U's. Thus the number of strings and number of routes are equal to $C(2n, n)$.

In the present problem, we can encode each route as a string of n R's (right) and m U's (up). As in the previous problem, we must count the number of such strings. Any such string can be obtained by selecting n positions for the R's, without regard to the order of selection, among the $n + m$ available positions in the string and then filling the remaining positions with U's. Thus the number of strings and number of routes are equal to $C(n + m, n)$. We have answered part (a).

In part (b) we are given a major hint: Divide the routes into classes based on when the route first meets the top edge. A route can first meet the top edge at any one of $n + 1$ positions. In the previous figure, the route shown first meets the top edge at the third position from the left. Before reading on, you might think about why we might divide the routes into classes.

Notice that when we divide the routes into classes based on when the route first meets the top edge:

- The classes are *disjoint.*

Section Review Exercises

1. How many orderings are there of n items of t types with n_i identical objects of type i? How is this formula derived?

2. How many unordered, k-element selections are there from a t-element set, repetitions allowed? How is this formula derived?

Exercises

In Exercises 1–3, determine the number of strings that can be formed by ordering the letters given.

1. *GUIDE*
2. *SCHOOL*
3. *SALESPERSONS*
4. How many strings can be formed by ordering the letters *SALESPERSONS* if the four *S*'s must be consecutive?
5. How many strings can be formed by ordering the letters *SALESPERSONS* if no two *S*'s are consecutive?
6. How many strings can be formed by ordering the letters *SCHOOL* using some or all of the letters?

Exercises 7–9 refer to selections among Action Comics, Superman, Captain Marvel, Archie, X-Man, and Nancy comics.

7. How many ways are there to select six comics?
8. How many ways are there to select 10 comics?
9. How many ways are there to select 10 comics if we choose at least one of each book?
10. How many routes are there in the ordinary *xyz*-coordinate system from the origin to the point (i, j, k), where i, j, and k are positive integers, if we are limited to steps one unit in the positive x-direction, one unit in the positive y-direction, or one unit in the positive z-direction?
11. An exam has 12 problems. How many ways can (integer) points be assigned to the problems if the total of the points is 100 and each problem is worth at least five points?
12. A bicycle collector has 100 bikes. How many ways can the bikes be stored in four warehouses if the bikes and the warehouses are considered distinct?
13. A bicycle collector has 100 bikes. How many ways can the bikes be stored in four warehouses if the bikes are indistinguishable, but the warehouses are considered distinct?
14. In how many ways can 10 distinct books be divided among three students if the first student gets five books, the second three books, and the third two books?

Exercises 15–21 refer to piles of identical red, blue, and green balls where each pile contains at least 10 balls.

15. In how many ways can 10 balls be selected?
16. In how many ways can 10 balls be selected if at least one red ball must be selected?
17. In how many ways can 10 balls be selected if at least one red ball, at least two blue balls, and at least three green balls must be selected?
18. In how many ways can 10 balls be selected if exactly one red ball must be selected?
19. In how many ways can 10 balls be selected if exactly one red ball and at least one blue ball must be selected?
20. In how many ways can 10 balls be selected if at most one red ball is selected?
21. In how many ways can 10 balls be selected if twice as many red balls as green balls must be selected?

In Exercises 22–27, find the number of integer solutions of

$$x_1 + x_2 + x_3 = 15$$

subject to the conditions given.

22. $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$
23. $x_1 \ge 1, x_2 \ge 1, x_3 \ge 1$
24. $x_1 = 1, x_2 \ge 0, x_3 \ge 0$
25. $x_1 \ge 0, x_2 > 0, x_3 = 1$
26. $0 \le x_1 \le 6, x_2 \ge 0, x_3 \ge 0$
★27. $0 \le x_1 < 6, 1 \le x_2 < 9, x_3 \ge 0$
★28. Find the number of solutions in integers to

$$x_1 + x_2 + x_3 + x_4 = 12$$

satisfying $0 \le x_1 \le 4$, $0 \le x_2 \le 5$, $0 \le x_3 \le 8$, and $0 \le x_4 \le 9$.

29. Prove that the number of solutions to the equation

$$x_1 + x_2 + x_3 = n, \quad n \ge 3,$$

where x_1, x_2, and x_3 are positive integers, is $(n-1)(n-2)/2$.

30. Show that the number of solutions in nonnegative integers of the inequality

$$x_1 + x_2 + \cdots + x_n \le M,$$

where M is a nonnegative integer, is $C(M+n, n)$.

31. How many integers between 1 and 1,000,000 have the sum of the digits equal to 15?
★32. How many integers between 1 and 1,000,000 have the sum of the digits equal to 20?
33. How many bridge deals are there? (A deal consists of partitioning a 52-card deck into four hands, each containing 13 cards.)

34. In how many ways can three teams containing four, two, and two persons be selected from a group of eight persons?
35. A *domino* is a rectangle divided into two squares with each square numbered one of 0, 1, ..., 6, repetitions allowed. How many distinct dominoes are there?

Exercises 36–41 refer to a bag containing 20 balls—six red, six green, and eight purple.

36. In how many ways can we select five balls if the balls are considered distinct?
37. In how many ways can we select five balls if balls of the same color are considered identical?
38. In how many ways can we draw two red, three green, and two purple balls if the balls are considered distinct?
39. We draw five balls, then replace the balls, and then draw five more balls. In how many ways can this be done if the balls are considered distinct?
40. We draw five balls without replacing them. We then draw five more balls. In how many ways can this be done if the balls are considered distinct?
41. We draw five balls and at least one is red, then replace them. We then draw five balls and at most one is green. In how many ways can this be done if the balls are considered distinct?
42. In how many ways can 15 identical mathematics books be distributed among six students?
43. In how many ways can 15 identical computer science books and 10 identical psychology books be distributed among five students?
44. In how many ways can we place 10 identical balls in 12 boxes if each box can hold one ball?
45. In how many ways can we place 10 identical balls in 12 boxes if each box can hold 10 balls?
46. Show that $(kn)!$ is divisible by $(n!)^k$.
47. By considering

for $i_1 = 1$ to n
  for $i_2 = 1$ to i_1
    println(i_1, i_2)

and Example 6.3.9, deduce

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

★48. Use Example 6.3.9 to prove the formula

$$C(k-1, k-1) + C(k, k-1) + \cdots + C(n+k-2, k-1) = C(k+n-1, k).$$

49. Write an algorithm that lists all solutions in nonnegative integers to

$$x_1 + x_2 + x_3 = n.$$

50. What is wrong with the following argument, which supposedly counts the number of partitions of a 10-element set into eight (nonempty) subsets?

List the elements of the set with blanks between them:

$$x_1 \_ x_2 \_ x_3 \_ x_4 \_ x_5 \_ x_6 \_ x_7 \_ x_8 \_ x_9 \_ x_{10}.$$

Every time we fill seven of the nine blanks with seven vertical bars, we obtain a partition of $\{x_1, \ldots, x_{10}\}$ into eight subsets. For example, the partition $\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_5\}, \{x_6\}, \{x_7, x_8\}, \{x_9\}, \{x_{10}\}$ would be represented as

$$x_1 \mid x_2 \mid x_3\, x_4 \mid x_5 \mid x_6 \mid x_7\, x_8 \mid x_9 \mid x_{10}.$$

Thus the solution to the problem is $C(9, 7)$.

6.4 ➜ Algorithms for Generating Permutations and Combinations

The rock group "Unhinged Universe" has recorded n videos whose running times are

$$t_1, t_2, \ldots, t_n$$

seconds. A tape is to be released that can hold C seconds. Since this is the first tape by the Unhinged Universe, the group wants to include as much material as possible. Thus the problem is to choose a subset $\{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ such that the sum

$$\sum_{j=1}^{k} t_{i_j} \qquad (6.4.1)$$