31/B - Final - Solutions

December 9, 2011

1. (20 points) Calculate $g(1)$ and $g'(1)$, where $g(x)$ is the inverse of $f(x) = x + \ln x$.

Solution First we solve, $1 = x + \ln x$, and we see that $x = 1$. Thus, $g(1) = 1$. Now, $f'(x) = 1 + \frac{1}{x}$. So,

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{1 + \frac{1}{g(1)}} = \frac{1}{1 + \frac{1}{1}} = \frac{1}{2}.$$

2. (20 points) Evaluate the integral

$$\int x\sqrt{9 - x^2}\, dx$$

using trigonometric substitution.

Solution We substitute $x = 3\sin\theta$. Then, $dx = 3\cos\theta\, d\theta$. So, the integral becomes

$$\begin{aligned} \int x\sqrt{9 - x^2}\, dx &= \int 3\sin\theta\sqrt{9 - 9\sin^2\theta} 3\cos\theta\, d\theta \\ &= 27\int \sin\theta\cos^2\theta\, d\theta. \end{aligned}$$

Now we substitute $u = \cos\theta$. Then, $du = -\sin\theta\, d\theta$, so the integral becomes

$$\begin{aligned} 27\int \sin\theta\cos^2\theta\, d\theta &= -27\int u^2\, du \\ &= -27\frac{u^3}{3} \\ &= -9\cos^3\theta. \end{aligned}$$

Using triangles, we see that

$$\cos\theta = \frac{\sqrt{9 - x^2}}{3}.$$

Thus, the final answer is

$$\int x\sqrt{9-x^2}\,dx = -\frac{9\left(9-x^2\right)^{3/2}}{27} = -\frac{(9-x^2)^{3/2}}{3}.$$

3. (20 points) Evaluate the integral

$$\int \frac{x^5+2}{x^2(x+1)}\,dx.$$

Solution First, dividing x^5+2 by x^3+x^2 we see that $x^5+2 = (x^2-x+1)(x^3+x^2)-x^2+2$. So,

$$\begin{aligned}\int \frac{x^5+2}{x^2(x+1)}\,dx &= \int \frac{(x^2-x+1)(x^3+x^2)-x^2+2}{x^3+x^2}\,dx \\ &= \int \left(x^2-x+1\right)\,dx - \int \frac{x^2-2}{x^3+x^2}\,dx \\ &= \frac{x^3}{3} - \frac{x^2}{2} + x - \int \frac{x^2-2}{x^2(x+1)}\,dx.\end{aligned}$$

We solve for A, B, and C in the equation

$$\frac{x^2-2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

by multiplying across by $x^2(x+1)$ to obtain

$$\begin{aligned}x^2-2 &= Ax(x+1) + B(x+1) + Cx^2 \\ &= Ax^2 + Ax + Bx + B + Cx^2 \\ &= (A+C)x^2 + (A+B)x + B.\end{aligned}$$

Equating coefficients, we see that $B=-2$, $A=2$, and $C=-1$. Therefore,

$$\begin{aligned}\int \frac{x^2-2}{x^2(x+1)}\,dx &= \int \frac{2\,dx}{x} - \int \frac{2\,dx}{x^2} - \int \frac{dx}{x+1} \\ &= 2\ln|x| + \frac{2}{x} - \ln|x+1| + C.\end{aligned}$$

Thus, the final answer is

$$\int \frac{x^5+2}{x^2(x+1)}\,dx = \frac{x^3}{3} - \frac{x^2}{2} + x - 2\ln|x| - \frac{2}{x} + \ln|x+1| + C.$$

4. (20 points) Evaluate the integral

$$\int \sin(\ln x)\,dx.$$

Solution There are two ways to do this problem. One, you may simply start with integration by parts with $u = \sin(\ln x)$. Two, you may first substitute. I show the second way here. First, we must substitute $w = \ln x$. Then, $dw = \frac{dx}{x}$, or $dx = xdw = e^w dw$. Thus, the integral becomes

$$\int \sin(\ln x)\,dx = \int e^w \sin(w)\,dw.$$

Second, we do integration by parts twice both times with $u = e^w$ to obtain

$$\int e^w \sin(w)\,dw = -e^w \cos(w) + \int e^w \cos(w)\,dw = -e^w \cos(w) + e^w \sin(w) - \int e^w \sin(w)\,dw$$

Therefore,

$$\int \sin(\ln x)\,dx = \int e^w \sin(w)\,dw = \frac{e^w}{2}\left(\sin(w) - \cos(w)\right) = \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right).$$

5. (20 points) Determine whether or not the improper integral

$$\int_1^2 \frac{dx}{x \ln x}$$

converges.

Solution We do the substitution $u = \ln x$, $du = \frac{dx}{x}$ to obtain

$$\int_1^2 \frac{dx}{x \ln x} = \int_0^{\ln 2} \frac{du}{u} = \lim_{R \to 0} \ln u|_R^{\ln 2},$$

which diverges.

6. (20 points) Use the error bound for Taylor polynomials to find a value of n for which

$$|\ln 2 - T_n(2)| \leq 10^{-6},$$

where T_n is the nth Taylor polynomial for $f(x) = \ln x$ with center 1.

Solution We know that the nth derivative of $\ln x$ is

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}.$$

On The interval $[1, 2]$, the function $|f^{(n)}(x)|$ is decreasing, so we can take $K_n = |f^{(n)}(1)| = (n-1)!$. The error bound then gives,

$$|\ln 2 - T_n(2)| \leq \frac{K_{n+1}(2-1)^{n+1}}{(n+1)!} = \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

So, we need

$$\frac{1}{n+1} \le \frac{1}{1\,000\,000},$$

or $n \ge 999\,999$.

7. (20 points) Determine whether or not

$$\sum_{n=1}^{\infty} \frac{5^{(n^2)}}{n!}$$

converges.

Solution First, note that $5^{n^2} = (5^n)^n$. Second, note that $n! \le n^n$ for all $n \ge 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{5^{(n^2)}}{n!} \ge \sum_{n=1}^{\infty} \frac{5^{(n^2)}}{n^n}.$$

If we show the right-hand series diverges, then we will have shown that the left-hand series diverges by the comparison test. The root test gives

$$L = \lim_{n\to\infty} \left(\frac{(5^n)^n}{n^n} \right)^{1/n} = \lim_{n\to\infty} \frac{5^n}{n} = +\infty.$$

So,

$$\sum_{n=1}^{\infty} \frac{5^{(n^2)}}{n!}$$

diverges.

8. (20 points) Find the interval of convergence of the power series

$$F(x) = \sum_{n=1}^{\infty} \frac{n(2x)^{2n}}{5n+4}.$$

Solution The ratio test produces

$$\begin{aligned} \rho(x) &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)(2x)^{2n+2}}{5(n+1)+4}}{\frac{n(2x)^{2n}}{5n+4}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(2x)^{2n+2}}{(2x)^{2n}} \right| \frac{5n+4}{5n+9} \frac{n+1}{n} \\ &= (2|x|)^2 = 4|x|^2. \end{aligned}$$

Therefore, $\rho(x) < 1$ when $|x|^2 < \frac{1}{4}$. That is, when $|x| < \frac{1}{2}$. Thus, the radius of convergence is $R = \frac{1}{2}$. When $x = -\frac{1}{2}$ or $x = \frac{1}{2}$ the limit of the sequence is not zero, so the divergence test says that the series diverges. Thus, the interval of convergence is

$$\left(-\frac{1}{2}, \frac{1}{2}\right).$$

9. (20 points) Approximate using Taylor series the integral

$$S = \int_0^1 \cos(x^3)\, dx$$

with an error of at most 10^{-4}.

Solution The Taylor series for $\cos(x^3)$ is

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}.$$

Thus,

$$\int \cos(x^3)\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(6n+1)(2n!)}.$$

So,

$$S = \int_0^1 \cos(x^3)\, dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(6n+1)(2n!)} \right) \Big|_0^1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(6n+1)(2n!)}.$$

This is an alternating sum, and we know that if

$$S_N = \sum_{n=0}^{N} (-1)^n \frac{1}{(6n+1)(2n!)},$$

then

$$|S - S_N| < \frac{1}{(6(N+1)+1)(2(N+1))!}.$$

So, we need to find N such that

$$(6N+7)(2N+2)! > 10\,000.$$

If $N = 1$, we have $(13)(4)! = 13 \cdot 24 = 312$. If $N = 2$, we have $(19)(6)! = 13680$. So, $N = 2$ works. That is,

$$S_2 = 1 - \frac{1}{7 \cdot 2!} + \frac{1}{13 \cdot 4!}$$

approximates the integral with an error of at most 10^{-4}.

10. (20 points) Find the terms through degree 7 of the Taylor series $T(x)$ centered at $c = 0$ of $f(x) = \sin(x)\cos(x)$.

Solution We know that the $T(x)$ is the product of the Taylor series centered at 0 of $\sin(x)$ and $\cos(x)$. That is,

$$\begin{aligned}
T(x) &= \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right) - \frac{x^2}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) + \frac{x^4}{4!}\left(x - \frac{x^3}{3!}\right) - \frac{x^6}{6!}(x) + \cdots \\
&= x - \left(\frac{1}{3!} + \frac{1}{2!}\right)x^3 + \left(\frac{1}{5!} + \frac{1}{2!\cdot 3!} + \frac{1}{4!}\right)x^5 - \left(\frac{1}{7!} + \frac{1}{2!\cdot 5!} + \frac{1}{3!\cdot 4!} + \frac{1}{6!}\right)x^7 + \cdots
\end{aligned}$$