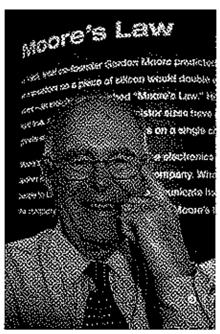
7 EXPONENTIAL FUNCTIONS

Detail of bison and other animals from a replica of the Lascaux cave mural.



Gordon Moore (1929–). Moore, who later became chairman of Intel Corporation, predicted that in the decades following 1965, the number of transistors per integrated circuit would grow "exponentially." This prediction has held up for nearly five decades and may well continue for several more years. Moore has said, "Moore's Law is a term that got applied to a curve I plotted in the mid-sixties showing the increase in complexity of integrated circuits versus time. It's been expanded to include a lot more than that, and I'm happy to take credit for all of it."

This chapter is devoted to exponential functions and their applications. These functions are used to model a remarkably wide range of phenomena, such as radioactive decay, population growth, interest rates, atmospheric pressure, and the diffusion of molecules across a cell membrane. Calculus gives us insight into why exponential functions play an important role in so many different situations. The key, it turns out, is the relation between an exponential function and its derivative.

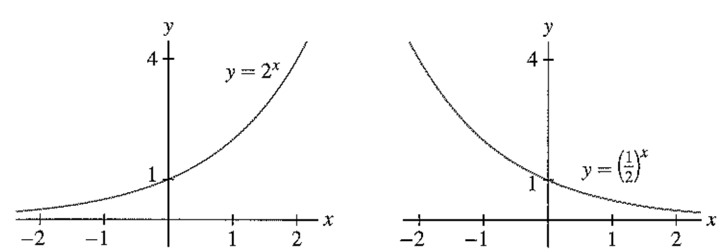
7.1 Derivative of $f(x) = b^x$ and the Number e

An **exponential function** is a function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$. The number b is called the **base**. Some examples are 2^x, $(1.4)^x$, and 10^x. We exclude the case $b = 1$ because $f(x) = 1^x$ is a constant function. Calculators give good decimal approximations to values of exponential functions:

$$2^4 = 16, \quad 2^{-3} = 0.125, \quad (1.4)^3 = 2.744, \quad 10^{4.6} \approx 39{,}810.717$$

Three properties of exponential functions should be singled out from the start (see Figure 1 for the cases $b = 2$ and $b = \frac{1}{2}$):

- *Exponential functions are positive*: $b^x > 0$ for all x.
- The *range* of $f(x) = b^x$ is the set of all positive real numbers.
- $f(x) = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$.



$y = 2^x$ is increasing $\qquad y = \left(\frac{1}{2}\right)^x$ is decreasing

FIGURE 1

If $b > 1$, the exponential function $f(x) = b^x$ is not merely increasing but is, in a certain sense, rapidly increasing. Although the term "rapid increase" is perhaps subjective, the following precise statement is true: $f(x) = b^x$ increases more rapidly than the power function x^n for all n (we will prove this in Section 7.7). For example, Figure 2 shows that $f(x) = 3^x$ eventually overtakes and increases faster than the power functions x^3, x^4, and x^5. Table 1 compares 3^x and x^5.

We now review the laws of exponents. The most important law is

$$\boxed{b^x b^y = b^{x+y}}$$

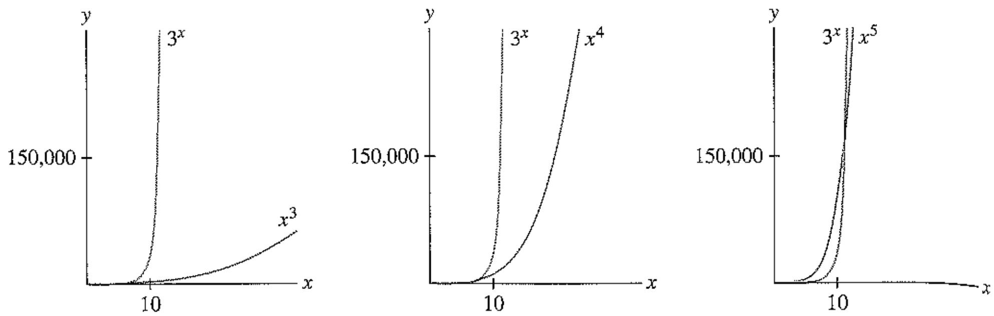


FIGURE 2 Comparison of 3^x and power functions.

In other words, *under multiplication, the exponents add*, provided that the bases are the same. This law does not apply to a product such as $3^2 \cdot 5^4$.

TABLE 1

x	x^5	3^x
1	1	3
5	3125	243
10	100,000	59,049
15	759,375	14,348,907
25	9,765,625	847,288,609,443

Be sure you are familiar with the laws of exponents. They are used throughout this text.

Laws of Exponents ($b > 0$)

	Rule	Example
Exponent zero	$b^0 = 1$	
Products	$b^x b^y = b^{x+y}$	$2^5 \cdot 2^3 = 2^{5+3} = 2^8$
Quotients	$\dfrac{b^x}{b^y} = b^{x-y}$	$\dfrac{4^7}{4^2} = 4^{7-2} = 4^5$
Negative exponents	$b^{-x} = \dfrac{1}{b^x}$	$3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$
Power to a power	$\left(b^x\right)^y = b^{xy}$	$\left(3^2\right)^4 = 3^{2(4)} = 3^8$
Roots	$b^{1/n} = \sqrt[n]{b}$	$5^{1/2} = \sqrt{5}$

EXAMPLE 1 Rewrite as a whole number or fraction:

(a) $16^{-1/2}$ **(b)** $27^{2/3}$ **(c)** $4^{16} \cdot 4^{-18}$ **(d)** $\dfrac{9^3}{3^7}$

Solution

(a) $16^{-1/2} = \dfrac{1}{16^{1/2}} = \dfrac{1}{\sqrt{16}} = \dfrac{1}{4}$

(b) $27^{2/3} = \left(27^{1/3}\right)^2 = 3^2 = 9$

(c) $4^{16} \cdot 4^{-18} = 4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

(d) $\dfrac{9^3}{3^7} = \dfrac{\left(3^2\right)^3}{3^7} = \dfrac{3^6}{3^7} = 3^{-1} = \dfrac{1}{3}$

In the next example, we use the fact that $f(x) = b^x$ is one-to-one. In other words, if $b^x = b^y$, then $x = y$.

EXAMPLE 2 Solve for the unknown:

(a) $2^{3x+1} = 2^5$ **(b)** $b^3 = 5^6$ **(c)** $7^{t+1} = \left(\dfrac{1}{7}\right)^{2t}$

Solution

(a) If $2^{3x+1} = 2^5$, then $3x + 1 = 5$ and thus $x = \frac{4}{3}$.

(b) Raise both sides of $b^3 = 5^6$ to the $\frac{1}{3}$ power. By the "power to a power" rule,

$$b = \left(b^3\right)^{1/3} = \left(5^6\right)^{1/3} = 5^{6/3} = 5^2 = 25$$

(c) Since $\frac{1}{7} = 7^{-1}$, the right-hand side of the equation is $\left(\frac{1}{7}\right)^{2t} = (7^{-1})^{2t} = 7^{-2t}$. The equation becomes $7^{t+1} = 7^{-2t}$. Therefore, $t + 1 = -2t$, or $t = -\frac{1}{3}$. ■

Derivative of $f(x) = b^x$

At this point, it is natural to ask: What is the derivative of $f(x) = b^x$? Our rules of differentiation are of no help because b^x is neither a product, quotient, nor composite of functions with known derivatives. We must go back to the limit definition of the derivative. The difference quotient (for $h \neq 0$) is

$$\frac{f(x+h) - f(x)}{h} = \frac{b^{x+h} - b^x}{h} = \frac{b^x b^h - b^x}{h} = \frac{b^x(b^h - 1)}{h}$$

We shall take for granted that $f(x) = b^x$ is differentiable. Although the proof of this fact is somewhat technical, it is plausible because the graph of $y = b^x$ appears smooth and without corners.

Now take the limit as $h \to 0$. The factor b^x does not depend on h, so it may be taken outside the limit:

$$\frac{d}{dx}b^x = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h\to 0} \frac{b^x(b^h - 1)}{h} = b^x \lim_{h\to 0} \left(\frac{b^h - 1}{h}\right)$$

This last limit (which exists because b^x is differentiable) does not depend on x. We denote its value by $m(b)$:

$$m(b) = \lim_{h\to 0} \left(\frac{b^h - 1}{h}\right)$$

What we have shown, then, is that *the derivative of b^x is proportional to b^x*:

$$\boxed{\frac{d}{dx}b^x = m(b)\, b^x} \tag{1}$$

What is the factor $m(b)$? We cannot determine its exact value at this point (in Section 7.3, we will learn that $m(b)$ is equal to $\ln b$, the natural logarithm of b). To proceed further, let's investigate $m(b)$ numerically.

■ **EXAMPLE 3** Estimate $m(b)$ numerically for $b = 2, 2.5, 3$, and 10.

Solution We create a table of values of difference quotients to estimate $m(b)$:

h	$\frac{2^h - 1}{h}$	$\frac{(2.5)^h - 1}{h}$	$\frac{3^h - 1}{h}$	$\frac{10^h - 1}{h}$
0.01	0.69556	0.92050	1.10467	2.32930
0.001	0.69339	0.91671	1.09921	2.30524
0.0001	0.69317	0.91633	1.09867	2.30285
0.00001	0.69315	0.916295	1.09861	2.30261
	$m(2) \approx 0.69$	$m(2.5) \approx 0.92$	$m(3) \approx 1.10$	$m(10) \approx 2.30$

■

These computations suggest that $m(b)$ is an increasing function of b. In fact, it can be shown that $m(b)$ is both increasing and continuous as a function of b (we shall take these facts for granted). Then, since $m(2.5) \approx 0.92$ and $m(3) \approx 1.10$, there exists a unique number b *between* 2.5 and 3 such that $m(b) = 1$. This is the number e, whose value is approximately 2.718.

Using infinite series (Exercise 87 in Section 10.7), we can show that e is irrational and we can compute its value to any desired accuracy.

Whenever we refer to the *exponential function without specifying the base, the reference is to* $f(x) = e^x$. *In many books,* e^x *is denoted* $\exp(x)$.

Because e is defined by the property $m(e) = 1$, Eq. (1) tells us that $(e^x)' = e^x$. In other words, e^x *is equal to its own derivative.*

The Number e There is a unique positive real number e with the property:

$$\frac{d}{dx}e^x = e^x \qquad (2)$$

The number e is irrational, with approximate value $e \approx 2.718$.

GRAPHICAL INSIGHT The graph of $f(x) = b^x$ passes through (0, 1) for all $b > 0$ because $b^0 = 1$. (Figure 3). The number $m(b)$ is simply the slope of the tangent line at $x = 0$:

$$\left.\frac{d}{dx}b^x\right|_{x=0} = m(b) \cdot b^0 = m(b)$$

These tangent lines become steeper as b increases and $b = e$ is the unique value for which the tangent line has slope 1. In Section 7.3, we will show that $m(b) = \ln b$, the natural logarithm of b.

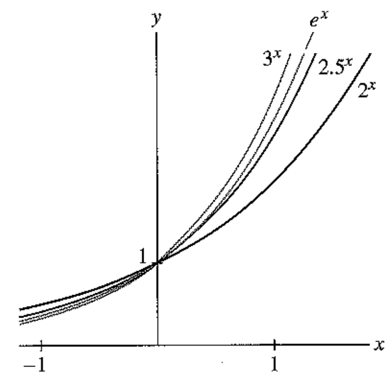


FIGURE 3 The tangent lines to $y = b^x$ at $x = 0$ grow steeper as b increases.

CONCEPTUAL INSIGHT In some ways, the number e is "complicated". It has been computed to an accuracy of more than 100 billion digits, but it is irrational and it cannot be defined without using limits. To 20 places,

$$e = 2.71828182845904523536\ldots$$

However, the elegant formula $\frac{d}{dx}e^x = e^x$ shows that e is "simple" from the point of view of calculus and that e^x is simpler than the seemingly more natural exponential functions such as 2^x or 10^x.

Although written reference to the number π goes back more than 4000 years, mathematicians first became aware of the special role played by e in the seventeenth century. The notation e was introduced around 1730 by Leonhard Euler, who discovered many fundamental properties of this important number.

EXAMPLE 4 Find the equation of the tangent line to the graph of $f(x) = 3e^x - 5x^2$ at $x = 2$.

Solution We compute both $f'(2)$ and $f(2)$:

$$f'(x) = \frac{d}{dx}(3e^x - 5x^2) = 3\frac{d}{dx}e^x - 5\frac{d}{dx}x^2 = 3e^x - 10x$$

$$f'(2) = 3e^2 - 10(2) \approx 2.17$$

$$f(2) = 3e^2 - 5(2^2) \approx 2.17$$

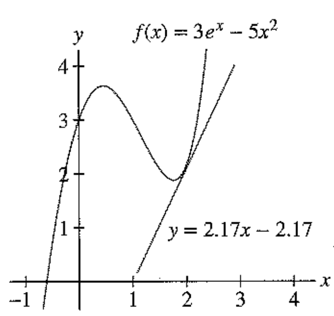


FIGURE 4

The equation of the tangent line is $y = f(2) + f'(2)(x - 2)$. Using these approximate values, we write the equation as (Figure 4)

$$y = 2.17 + 2.17(x - 2) \quad \text{or} \quad y = 2.17(x - 1)$$

EXAMPLE 5 Calculate $f'(0)$, where $f(x) = e^x \cos x$.

Solution Use the Product Rule:

$$f'(x) = e^x \cdot (\cos x)' + \cos x \cdot (e^x)' = -e^x \sin x + \cos x \cdot e^x = e^x(\cos x - \sin x)$$

Then $f'(0) = e^0(1 - 0) = 1$.

To compute the derivative of a function of the form $e^{g(x)}$, write $e^{g(x)}$ as a composite $e^{g(x)} = f(g(x))$ where $f(u) = e^u$, and apply the Chain Rule:

$$\frac{d}{dx}\left(e^{g(x)}\right) = \left[f\big(g(x)\big)\right]' = f'\big(g(x)\big)g'(x) = e^{g(x)}g'(x) \quad [\text{since } f'(x) = e^x]$$

A special case is $(e^{kx+b})' = ke^{kx+b}$, where k and b are constants.

$$\frac{d}{dx}\left(e^{g(x)}\right) = g'(x)e^{g(x)}, \qquad \frac{d}{dx}\left(e^{kx+b}\right) = ke^{kx+b} \quad (k, b \text{ constants}) \qquad \boxed{3}$$

EXAMPLE 6 Differentiate:

$$\textbf{(a)}\ f(x) = e^{9x-5} \quad \text{and} \quad \textbf{(b)}\ f(x) = e^{\cos x}$$

Solution Apply Eq. (3):

$$\textbf{(a)}\ \frac{d}{dx}e^{9x-5} = 9e^{9x-5} \quad \text{and} \quad \textbf{(b)}\ \frac{d}{dx}\left(e^{\cos x}\right) = -(\sin x)e^{\cos x}$$

EXAMPLE 7 **Graph Sketching Involving e^x** Sketch the graph of $f(x) = xe^x$ on the interval $[-4, 2]$.

Solution As usual, the first step is to solve for the critical points:

$$f'(x) = \frac{d}{dx}xe^x = xe^x + e^x = (x+1)e^x = 0$$

Since $e^x > 0$ for all x, the unique critical point is $x = -1$ and

$$f'(x) = \begin{cases} < 0 & \text{for } x < -1 \\ > 0 & \text{for } x > -1 \end{cases}$$

Thus, $f'(x)$ changes sign from $-$ to $+$ at $x = -1$ and $f(-1)$ is a local minimum. For the second derivative, we have

$$f''(x) = (x+1) \cdot (e^x)' + e^x \cdot (x+1)' = (x+1)e^x + e^x = (x+2)e^x$$

$$f''(x) = \begin{cases} < 0 & \text{for } x < -2 \\ > 0 & \text{for } x > -2 \end{cases}$$

Thus, $x = -2$ is a point of inflection, where the graph changes from concave down to concave up at $x = -2$. Figure 5 shows the graph with its local minimum and point of inflection.

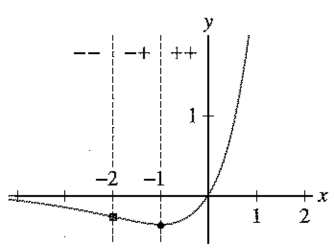


FIGURE 5 Graph of $f(x) = xe^x$. The sign combinations $--$, $-+$, $++$ indicate the signs of f' and f''.

Integrals Involving e^x

The formula $(e^x)' = e^x$ says that the function $f(x) = e^x$ is its own derivative. But this means $f(x) = e^x$ is also *its own antiderivative*. In other words,

$$\boxed{\int e^x\, dx = e^x + C}$$

More generally, for any constants b and k with $k \neq 0$,

$$\int e^{kx+b}\, dx = \frac{1}{k} e^{kx+b} + C$$

We verify this formula using substitution, or by noting that $\dfrac{d}{dx}\left(\dfrac{1}{k} e^{kx+b}\right) = e^{kx+b}$.

EXAMPLE 8 Evaluate:

(a) $\displaystyle\int e^{7x-5}\, dx$ **(b)** $\displaystyle\int xe^{2x^2}\, dx$ **(c)** $\displaystyle\int \frac{e^t}{1+2e^t+e^{2t}}\, dt$

Solution

(a) $\displaystyle\int e^{7x-5}\, dx = \frac{1}{7}e^{7x-5} + C.$

(b) Use the substitution $u = 2x^2$, $du = 4x\, dx$:

$$\int xe^{2x^2}\, dx = \frac{1}{4}\int e^u\, du = \frac{1}{4}e^u + C = \frac{1}{4}e^{2x^2} + C$$

(c) We have $1 + 2e^t + e^{2t} = (1 + e^t)^2$. The substitution $u = e^t$, $du = e^t\, dt$ gives

$$\int \frac{e^t}{1+2e^t+e^{2t}}\, dt = \int \frac{du}{(1+u)^2} = -(1+u)^{-1} + C = -(1+e^t)^{-1} + C$$

CONCEPTUAL INSIGHT What precisely do we mean by b^x? We have taken for granted that b^x is meaningful for all real numbers x, but we never specified how b^x is defined when x is irrational. If n is a whole number, then b^n is simply the product $b \cdot b \cdots b$ (n times), and for any rational number $x = m/n$,

$$b^x = b^{m/n} = \left(b^{1/n}\right)^m = \left(\sqrt[n]{b}\right)^m$$

When x is irrational, this definition does not apply and b^x cannot be defined directly in terms of roots and powers of b. However, it makes sense to view $b^{m/n}$ as an approximation to b^x when m/n is a rational number close to x. For example, $3^{\sqrt{2}}$ should be approximately equal to $3^{1.4142} \approx 4.729$ because 1.4142 is a good rational approximation to $\sqrt{2}$. Formally, then, we may define b^x as a limit over rational numbers m/n approaching x:

$$b^x = \lim_{m/n \to x} b^{m/n}$$

It can be shown that this limit exists and that the function $f(x) = b^x$ thus defined is not only continuous but also differentiable.

7.1 SUMMARY

- $f(x) = b^x$ is the *exponential function* with base b (where $b > 0$ and $b \neq 1$).
- $f(x) = b^x$ is increasing if $b > 1$ and decreasing if $b < 1$.
- The derivative of b^x is proportional to b^x:

$$\frac{d}{dx} b^x = m(b)b^x$$

where $m(b) = \displaystyle\lim_{h \to 0} \frac{b^h - 1}{h}$.

- There is a unique number $e \approx 2.718$ with the property $m(e) = 1$, so that

$$\frac{d}{dx} e^x = e^x$$

- By the Chain Rule:

$$\frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}, \qquad \frac{d}{dx} e^{kx+b} = ke^{kx+b} \quad (k, b \text{ constants})$$

- $\int e^x \, dx = e^x + C.$
- $\int e^{kx+b} \, dx = \frac{1}{k} e^{kx+b} + C \quad (k, b \text{ constants with } k \neq 0).$

7.1 EXERCISES

Preliminary Questions

1. Which of the following equations is incorrect?
(a) $3^2 \cdot 3^5 = 3^7$ (b) $(\sqrt{5})^{4/3} = 5^{2/3}$
(c) $3^2 \cdot 2^3 = 1$ (d) $(2^{-2})^{-2} = 16$

2. What are the domain and range of $\ln x$? When is $\ln x$ negative?

3. To which of the following does the Power Rule apply?
(a) $f(x) = x^2$ (b) $f(x) = 2^e$ (c) $f(x) = x^e$
(d) $f(x) = e^x$ (e) $f(x) = x^x$
(f) $f(x) = x^{-4/5}$

4. For which values of b does b^x have a negative derivative?

5. For which values of b is the graph of $y = b^x$ concave up?

6. Which point lies on the graph of $y = b^x$ for all b?

7. Which of the following statements is not true?
(a) $(e^x)' = e^x$
(b) $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$
(c) The tangent line to $y = e^x$ at $x = 0$ has slope e.
(d) The tangent line to $y = e^x$ at $x = 0$ has slope 1.

Exercises

1. Rewrite as a whole number (without using a calculator):
(a) 7^0 (b) $10^2(2^{-2} + 5^{-2})$
(c) $\frac{(4^3)^5}{(4^5)^3}$ (d) $27^{4/3}$
(e) $8^{-1/3} \cdot 8^{5/3}$ (f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2}$

2. Compute $(16^{-1/16})^4$.

In Exercises 3–10, solve for the unknown variable.

3. $9^{2x} = 9^8$
4. $e^{t^2} = e^{4t-3}$
5. $3^x = \left(\frac{1}{3}\right)^{x+1}$
6. $(\sqrt{5})^x = 125$
7. $4^{-x} = 2^{x+1}$
8. $b^4 = 10^{12}$
9. $k^{3/2} = 27$
10. $(b^2)^{x+1} = b^{-6}$

In Exercises 11–14, determine the limit.

11. $\lim_{x \to \infty} 4^x$
12. $\lim_{x \to \infty} 4^{-x}$
13. $\lim_{x \to \infty} \left(\frac{1}{4}\right)^{-x}$
14. $\lim_{x \to \infty} e^{x-x^2}$

In Exercises 15–18, find the equation of the tangent line at the point indicated.

15. $y = 4e^x, \quad x_0 = 0$
16. $y = e^{4x}, \quad x_0 = 0$
17. $y = e^{x+2}, \quad x_0 = -1$
18. $y = e^{x^2}, \quad x_0 = 1$

In Exercises 19–40, find the derivative.

19. $f(x) = 7e^{2x} + 3e^{4x}$
20. $f(x) = e^{-5x}$
21. $f(x) = e^{\pi x}$
22. $f(x) = e^3$
23. $f(x) = e^{-4x+9}$
24. $f(x) = 4e^{-x} + 7e^{-2x}$
25. $f(x) = \frac{e^{x^2}}{x}$
26. $f(x) = x^2 e^{2x}$
27. $f(x) = (1 + e^x)^4$
28. $f(x) = (2e^{3x} + 2e^{-2x})^4$
29. $f(x) = e^{x^2+2x-3}$
30. $f(x) = e^{1/x}$
31. $f(x) = e^{\sin x}$
32. $f(x) = e^{(x^2+2x+3)^2}$
33. $f(\theta) = \sin(e^\theta)$
34. $f(t) = e^{\sqrt{t}}$
35. $f(t) = \frac{1}{1 - e^{-3t}}$
36. $f(t) = \cos(te^{-2t})$

37. $f(x) = \dfrac{e^x}{3x+1}$

38. $f(x) = \tan(e^{5-6x})$

39. $f(x) = \dfrac{e^{x+1}+x}{2e^x - 1}$

40. $f(x) = e^{e^x}$

In Exercises 41–46, calculate the derivative indicated.

41. $f''(x);\quad f(x) = e^{4x-3}$

42. $f'''(x);\quad f(x) = e^{12-3x}$

43. $\dfrac{d^2y}{dt};\quad y = e^t \sin t$

44. $\dfrac{d^2y}{dt};\quad y = e^{-2t}\sin 3t$

45. $\dfrac{d^2}{dt}e^{t-t^2}$

46. $\dfrac{d^3}{d\theta}\cos(e^\theta)$

In Exercises 47–52, find the critical points and determine whether they are local minima, maxima, or neither.

47. $f(x) = e^x - x$

48. $f(x) = x + e^{-x}$

49. $f(x) = \dfrac{e^x}{x}$ for $x > 0$

50. $f(x) = x^2e^x$

51. $g(t) = \dfrac{e^t}{t^2+1}$

52. $g(t) = (t^3 - 2t)e^t$

In Exercises 53–58, find the critical points and points of inflection. Then sketch the graph.

53. $y = xe^{-x}$

54. $y = e^{-x} + e^x$

55. $y = e^{-x}\cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

56. $y = e^{-x^2}$

57. $y = e^x - x$

58. $y = x^2e^{-x}$

59. Find $a > 0$ such that the tangent line to the graph of $f(x) = x^2e^{-x}$ at $x = a$ passes through the origin (Figure 6).

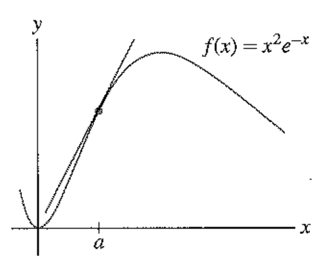


FIGURE 6

60. Use Newton's Method to find the two solutions of $e^x = 5x$ to three decimal places (Figure 7).

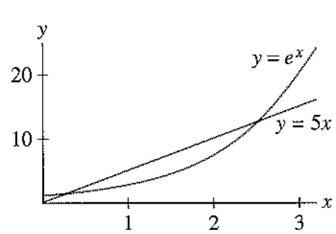


FIGURE 7 Graphs of e^x and $5x$.

61. Compute the linearization of $f(x) = e^{-2x}\sin x$ at $a = 0$.

62. Compute the linearization of $f(x) = xe^{6-3x}$ at $a = 2$.

63. Find the linearization of $f(x) = e^x$ at $a = 0$ and use it to estimate $e^{-0.1}$.

64. Use the linear approximation to estimate $f(1.03) - f(1)$ where $y = x^{1/3}e^{x-1}$.

65. A 2005 study by the Fisheries Research Services in Aberdeen Scotland showed that the average length of the species Clupea Harengus (Atlantic herring) as a function of age t (in years) can be modeled by $L(t) = 32(1 - e^{-0.37t})$ cm for $0 \le t \le 13$.

(a) How fast is the average length changing at age $t = 6$ yrs?

(b) At what age is the average length changing at a rate of 5 cm/yr?

(c) Calculate $L = \lim_{t\to\infty} L(t)$.

66. According to a 1999 study by Starkey and Scarnecchia, the average weight (kg) at age t (years) of channel catfish in the Lower Yellowstone River can be modeled by

$$W(t) = \left(3.46293 - 3.32173e^{-0.03456t}\right)^{3.4026}$$

Find the rate at which weight is changing at age $t = 10$.

67. The functions in Exercises 65 and 66 are examples of the **Von Bertalanffy growth function**

$$M(t) = \left(a + (b-a)e^{kmt}\right)^{1/m}$$

introduced in the 1930's by Austrian-born biologist Karl Ludwig Von Bertalanffy. Calculate $M'(0)$ in terms of the constants a, b, k, and m.

68. Find an approximation to $m(4)$ using the limit definition and estimate the slope of the tangent line to $y = 4^x$ at $x = 0$ and $x = 2$.

In Exercises 69–86, evaluate the integral.

69. $\displaystyle\int (e^x + 2)\,dx$

70. $\displaystyle\int e^{4x}\,dx$

71. $\displaystyle\int_0^1 e^{-3x}\,dx$

72. $\displaystyle\int_2^6 e^{-x/2}\,dx$

73. $\displaystyle\int_0^3 e^{1-6t}\,dt$

74. $\displaystyle\int_2^3 e^{4t-3}\,dt$

75. $\displaystyle\int (e^{4x} + 1)\,dx$

76. $\displaystyle\int (e^x + e^{-x})\,dx$

77. $\displaystyle\int_0^1 xe^{-x^2/2}\,dx$

78. $\displaystyle\int_0^2 ye^{3y^2}\,dy$

79. $\displaystyle\int e^t\sqrt{e^t+1}\,dt$

80. $\displaystyle\int (e^{-x} - 4x)\,dx$

81. $\displaystyle\int \frac{e^{2x} - e^{4x}}{e^x}\,dx$

82. $\displaystyle\int e^x\cos(e^x)\,dx$

83. $\displaystyle\int \frac{e^x}{\sqrt{e^x+1}}\,dx$

84. $\displaystyle\int e^x(e^{2x}+1)^3\,dx$

85. $\displaystyle\int \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$

86. $\displaystyle\int x^{-2/3}e^{x^{1/3}}\,dx$

87. Find the area between $y = e^x$ and $y = e^{2x}$ over [0, 1].

88. Find the area between $y = e^x$ and $y = e^{-x}$ over [0, 2].

89. Find the area bounded by $y = e^2$, $y = e^x$, and $x = 0$.

90. Find the volume obtained by revolving $y = e^x$ about the x-axis for $0 \le x \le 1$.

91. Wind engineers have found that wind speed v (in m/s) at a given location follows a **Rayleigh distribution** of the type

$$W(v) = \frac{1}{32} v e^{-v^2/64}$$

This means that the probability that v lies between a and b is equal to the shaded area in Figure 8.

(a) Show that the probability that $v \in [0, b]$ is $1 - e^{-b^2/64}$.

(b) Calculate the probability that $v \in [2, 5]$.

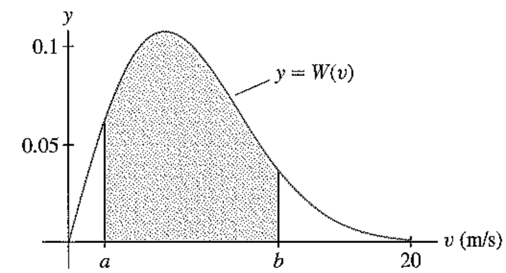


FIGURE 8 The shaded area is the probability that v lies between a and b.

92. The function $f(x) = e^x$ satisfies $f'(x) = f(x)$. Show that if $g(x)$ is another function satisfying $g'(x) = g(x)$, then $g(x) = Ce^x$ for some constant C. *Hint:* Compute the derivative of $g(x)e^{-x}$.

Further Insights and Challenges

93. Prove that $f(x) = e^x$ is not a polynomial function. *Hint:* Differentiation lowers the degree of a polynomial by 1.

94. Recall the following property of integrals: If $f(t) \ge g(t)$ for all $t \ge 0$, then for all $x \ge 0$,

$$\int_0^x f(t)\,dt \ge \int_0^x g(t)\,dt \qquad \boxed{4}$$

The inequality $e^t \ge 1$ holds for $t \ge 0$ because $e > 1$. Use (4) to prove that $e^x \ge 1 + x$ for $x \ge 0$. Then prove, by successive integration, the following inequalities (for $x \ge 0$):

$$e^x \ge 1 + x + \frac{1}{2}x^2, \qquad e^x \ge 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

95. Generalize Exercise 94; that is, use induction (if you are familiar with this method of proof) to prove that for all $n \ge 0$,

$$e^x \ge 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n \quad (x \ge 0)$$

96. Use Exercise 94 to show that $\dfrac{e^x}{x^2} \ge \dfrac{x}{6}$ and conclude that $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = \infty$. Then use Exercise 95 to prove more generally that $\lim\limits_{x\to\infty} \dfrac{e^x}{x^n} = \infty$ for all n.

97. Calculate the first three derivatives of $f(x) = xe^x$. Then guess the formula for $f^{(n)}(x)$ (use induction to prove it if you are familiar with this method of proof).

98. Consider the equation $e^x = \lambda x$, where λ is a constant.

(a) For which λ does it have a unique solution? For intuition, draw a graph of $y = e^x$ and the line $y = \lambda x$.

(b) For which λ does it have at least one solution?

99. Prove in two ways that the numbers $m(a)$ satisfy

$$m(ab) = m(a) + m(b)$$

(a) First method: Use the limit definition of m_b and

$$\frac{(ab)^h - 1}{h} = b^h\left(\frac{a^h - 1}{h}\right) + \frac{b^h - 1}{h}$$

(b) Second method: Apply the Product Rule to $a^x b^x = (ab)^x$.

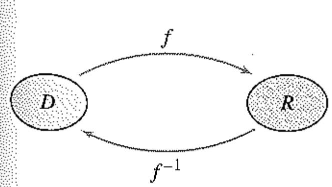


FIGURE 1 A function and its inverse.

7.2 Inverse Functions

In the next section, we will define logarithmic functions as inverses of exponential functions. But first, we review inverse functions and compute their derivatives.

The inverse of $f(x)$, denoted $f^{-1}(x)$, is the function that *reverses* the effect of $f(x)$ (Figure 1). For example, the inverse of $f(x) = x^3$ is the cube root function $f^{-1}(x) = x^{1/3}$. Given a table of function values for $f(x)$, we obtain a table for $f^{-1}(x)$ by interchanging the x and y columns: