308G - Midterm

9 May 2014

Name:
Student ID #:

This is a closed-book, closed-notes exam. Calculators are not allowed.

Show all work.

If you need more room, write on the back, and make a note on the front.

POINTS:

1.

2.

3.

4.

TOTAL:

1.a. (10 points). Determine the rank of the matrix

$$A = \begin{pmatrix} -8 & 8 & 0 & 0 \\ 1 & -1 & -1 & -2 \\ 1 & 1 & -1 & 4 \end{pmatrix}.$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -1 & -1 & -2 \\ -8 & 8 & 0 & 0 \end{bmatrix}$$

$R_2 - R_1$
$R_3 + 8R_1$

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -2 & 0 & -6 \\ 0 & 16 & -8 & 32 \end{bmatrix}$$

$R_3 + 8R_2$

$$\begin{bmatrix} \boxed{1} & 1 & -1 & 4 \\ 0 & \boxed{-2} & 0 & -6 \\ 0 & 0 & \boxed{-8} & -16 \end{bmatrix}$$

3 non-zero pivots

$\Rightarrow$ rank$(A) = 3$.

The next three problems involve the matrix A from Question 1.a.

1.b. (5 points). What is the rank of A^T? 3: $\mathrm{rank}(A^T) = \mathrm{rank}(A)$.

1.c. (5 points). Is there a solution to the the equation

$$A\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}?$$

Yes. The range of A is all of $\mathbb{R}^3$, since $\mathrm{rank}(A) = 3$.

1.d. (5 points). Suppose that $\mathbf{b} \in \mathbf{R}^3$ is a vector such that $A\mathbf{x} = \mathbf{b}$ has a solution. Is this solution unique?

No. There is an independent variable.

2.a. (20 points). Find the inverse of the matrix

$$A = \begin{pmatrix} 0 & -2 & 2 \\ 3 & 2 & -3 \\ 1 & -1 & 1 \end{pmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 0 & -2 & 2 & 1 & 0 & 0 \\ 3 & 2 & -3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 2 & 1 & 0 & 0 \\ 3 & 2 & -3 & 0 & 1 & 0 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} (-\tfrac{1}{2})R_2 \\ R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -\tfrac{1}{2} & 0 & 0 \\ 0 & 5 & -6 & 0 & 1 & -3 \end{bmatrix}$$

$$\Big\downarrow R_3 - 5R_2$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -\tfrac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & \tfrac{5}{2} & 1 & -3 \end{bmatrix}$$

$$\Big\downarrow (-1)R_3$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -\tfrac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -\tfrac{5}{2} & -1 & 3 \end{bmatrix} \xrightarrow[R_1 - R_3]{R_2 + R_3} \begin{bmatrix} 1 & -1 & 0 & \tfrac{5}{2} & 1 & -2 \\ 0 & 1 & 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & -\tfrac{5}{2} & -1 & 3 \end{bmatrix}$$

$$\Big\uparrow R_1 + R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & -\tfrac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & -\tfrac{5}{2} & -1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\tfrac{1}{2} & 0 & 1 \\ -3 & -1 & 3 \\ -\tfrac{5}{2} & -1 & 3 \end{bmatrix}$$

2.b. (5 points). Find the unique solution to $A\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\vec{x} = A^{-1}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & 1 \\ -3 & -1 & 3 \\ -\frac{5}{2} & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

3. (20 points). Find a basis for the null space of the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 3 \\ -19 & 1 & 0 & 1 & -1 \end{pmatrix}.$$

We see that x_1 and x_3 are dependent, while $x_2, x_4,$ and x_5 are independent. So, we fill in

$$x_3 + x_4 + 3x_5 = 0$$
$$-19x_1 + x_2 + x_4 - x_5 = 0.$$

$$\begin{bmatrix} ? \\ 1 \\ ? \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} ? \\ 0 \\ ? \\ 0 \\ 1 \end{bmatrix},$$

subject to the constraint that $A\vec{x} = \vec{0}$. We get that

$$\left\{ \begin{bmatrix} \frac{1}{19} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{19} \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{19} \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $N(A)$.

5.a. (20 points). Describe geometrically the column space of the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -1 & -2 & -1 \\ 2 & -1 & -5 \\ -3 & -8 & 0 \end{pmatrix}.$$

We find which $\bar{b}$ lead to a consistent $A\vec{x} = \bar{b}$.

$$\left[\begin{array}{cccc} 1 & 0 & -2 & b_1 \\ -1 & -2 & -1 & b_2 \\ 2 & -1 & -5 & b_3 \\ -3 & -8 & 0 & b_4 \end{array}\right]$$

$R_2 + R_1$
$R_3 - 2R_1$
$R_4 + 3R_1$

$$\left[\begin{array}{cccc} 1 & 0 & -2 & b_1 \\ 0 & -2 & -3 & b_1+b_2 \\ 0 & -1 & -1 & b_3-2b_1 \\ 0 & -8 & -6 & b_4+3b_1 \end{array}\right]$$

$R_2 - 2R_3$
$R_4 - 8R_3$

$$\left[\begin{array}{cccc} 1 & 0 & -2 & b_1 \\ 0 & 0 & -1 & b_1+b_2-2b_3+4b_1 \\ 0 & -1 & -1 & b_3-2b_1 \\ 0 & 0 & 2 & b_4+3b_1-8b_3+16b_1 \end{array}\right] \xrightarrow{R_4 + 2R_2} \left[\begin{array}{cccc} \boxed{1} & 0 & -2 & b_1 \\ 0 & 0 & \boxed{-1} & 5b_1+b_2-2b_3 \\ 0 & \boxed{-1} & -1 & b_3-2b_1 \\ 0 & 0 & 0 & (19b_1-8b_3+b_4)+2(5b_1+b_2-2 \end{array}\right]$$

Hence, to get a consistent system, we need

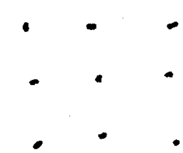
$$0 = (19b_1 - 8b_3 + b_4) + 2(5b_1 + b_2 - 2b_3)$$
$$= 29b_1 + 2b_2 - 12b_3 + b_4.$$

That is, $R(A)$ is the hyperplane in $\mathbb{R}^4$ of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ satisfying

$$29b_1 + 2b_2 - 12b_3 + b_4 = 0.$$

5.b. (10 points) Circle the dimension of the null space of A: 0, 1, 2, 3, 4, 5.

Use rank + nullity = 3.



$n^2 (n-1)^2$