MATH 215 – Induction (I)

Proposition 1 *Let a, b be integers and n a natural number. If $a \equiv b \bmod n$, then $a^k \equiv b^k \bmod n$ for every natural number k.*

Notation 2 *Let $P(k)$ be a statement that depends on the integer k.*

Example 3 *Let a, b be integers and n a natural number. Consider the statement $a^k \equiv b^k \bmod n$. We can denote this statement by $P(k)$ where*

$$P(k) : a^k \equiv b^k \bmod n.$$

If we write $P(5)$, then this means the statement

$$a^5 \equiv b^5 \bmod n.$$

Theorem 4 *Let $P(k)$ denote a statement for every integer $k = 0, 1, 2, \ldots$. If the following are true:*

1. *$P(0)$ is true; and*
2. *The truth of $P(\ell - 1)$ implies the truth of $P(\ell)$ for every integer $\ell = 1, 2, 3, \ldots$,*

then $P(k)$ is true for all integers $k = 0, 1, 2, 3 \ldots$.

Remark 5 *Proving a statement $P(k)$ is true for all integer $k = 0, 1, 2, 3, \ldots$ using Theorem 4 is called a* **proof by induction***. Verifying the step $P(0)$ is true is called the* **base case** *and verifying the step that $P(\ell - 1)$ implies $P(\ell)$ is called the* **inductive hypothesis***.*

Theorem 6 *Let n be a non-negative integer and let $P(k)$ denote a statement for every integer $k = n, n + 1, n + 2, \ldots$. If the following are true:*

1. *$P(n)$ is true; and*
2. *The truth of $P(\ell - 1)$ implies the truth of $P(\ell)$ for every integer $\ell = n + 1, n + 2, n + 3, \ldots$,*

then $P(k)$ is true for all integers $k = n, n + 1, n + 2, \ldots$.

Theorem 7 *Let n be a non-negative integer and let $P(k)$ denote a statement for every integer $k = n, n + 1, n + 2, \ldots$. If the following are true:*

1. *$P(n)$ is true; and*
2. *For all integers $\ell > n$, the truth of $P(n), P(n+1), P(n+2), \ldots,$ and $P(\ell-1)$ imply the truth of $P(\ell)$,*

then $P(k)$ is true for all integers $k = n, n + 1, n + 2, \ldots$.

Remark 8 *Using Theorem 7 to prove a result is a proof using* **strong induction**. *Using either Theorems 4 or 6 to prove a result is a proof using* **weak induction**.

Proposition 9 *Let a, b be integers and n a natural number. Using induction, prove that if $a \equiv b \bmod n$, then*

$$a^k \equiv b^k \bmod n$$

for every natural number k.

Challenge 10 *Let k be a natural number. Consider a $2^k \times 2^k$ checkerboard with any single 1×1 square removed. Prove that the checkerboard could be covered using only L-shaped blocks that are made of three 1×1 squares. [Make sure you understand what shaped tiles I mean before starting this problem.]*

Problem 11 *Find the following sums.*

- $1 + 3 + 5 =$
- $1 + 3 + 5 + 7 + 9 =$
- $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 =$

Using your sums above, develop (which means guess and prove) a formula for the sum of the first k odd integers:

$$1 + 3 + 5 + 7 + \cdots + 2k - 1 =$$

Proposition 12 *Prove that $2304 | (7^{2n} - 48n - 1)$ for every natural number n.*

Question 13 *For what natural numbers n (if any) is $4n < 2^n$? Prove it.*

Problem 14 *Consider the sequence $\{x_n\}_{n=1}^{\infty}$ defined recursively by $x_1 = 1$ and $x_{n+1} = \frac{1}{2}x_n + 1$ for $n \geq 1$.*

1. *Show that $x_n \leq 2$ for all $n \geq 1$.*
2. *Show that $x_n \leq x_{n+1}$ for all $n \geq 1$.*
3. *What do the two steps above imply about the sequence?*

Notation 15 *Let n, k be non-negative integers. The* **binomial coefficient** $\left(\begin{array}{c} n \\ k \end{array} \right)$ *is defined by*

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}.$$

Theorem 16 (Binomial Theorem) *Let n be a non-negative integer. Then*

$$(x + y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k y^{n-k}.$$

Theorem 17 (De Moivre's Theorem) *For any positive integer n*

$$(\cos t + i \sin t)^n = \cos(nt) + i \sin(nt)$$