Rational Homotopy Theory - Lecture 3

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1. Some classical results

Theorem 1.1 (Whitehead)**.** *Suppose that X and Y are based, connected CW complexes. If $f : X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.*

The idea is to construct a homotopy inverse g to f inductively on the n-skeleta of Y.

Theorem 1.2 (Hurewicz)**.** *If X is $(n-1)$-connected for some $n \geq 2$, then $\mathrm{H}_i(X, \mathbb{Z}) = 0$ for $1 \leq i \leq n-1$ and $\pi_n(X) \cong \mathrm{H}_n(X, \mathbb{Z})$. Similarly, if (X, A) is $(n-1)$-connected for $n \geq 2$ and A is simply connected and non-empty, then $\mathrm{H}_i(X, A, \mathbb{Z}) = 0$ for $1 \leq i \leq n-1$ and $\pi_n(X, A) \cong \mathrm{H}_n(X, A, \mathbb{Z})$.*

Corollary 1.3. *If $f : X \to Y$ is a map between simply-connected CW complexes such that $\mathrm{H}_*(X, \mathbb{Z}) \cong \mathrm{H}_*(Y, \mathbb{Z})$, then f is a homotopy equivalence.*

Proof. One can assume that f includes X as a subcomplex of Y. Then, the hypotheses imply that $\mathrm{H}_*(Y, X, \mathbb{Z}) = 0$, and hence all the relative homotopy groups of the pair vanish. □

Remark 1.4. The corollary is false if X or Y is not simply connected. For example, let Z be the homology 3-sphere constructed by Poincaré. It is an orientable 3-manifold with $\mathrm{H}_n(Z, \mathbb{Z}) \cong \mathbb{Z}$ if $n = 0, 3$ and $\mathrm{H}_n(Z, \mathbb{Z}) = 0$ otherwise. Thus, it has the same homology as the 3-sphere. In fact, $Z = \mathrm{SO}(3)/\mathrm{A}_5$, where the alternating group embeds in $\mathrm{SO}(3)$ as the group of orientation-preserving isometries that preserve the regular icosahedron. The fundamental group of Z is an extension of A_5 by $\mathbb{Z}/2$, isomorphic to the binary icosahedral group. If we let X be the complement of a point in Z, then it follows that the integral homology of X is that of a point, while the fundamental group of X is the binary icosahedral group again, and hence non-zero. In particular, X is not contractible, even weakly.

A map $f : X \to Y$ is a **homology equivalence** if the induced map $\mathrm{H}_*(f) : \mathrm{H}_*(X, \mathbb{Z}) \to \mathrm{H}_*(Y, \mathbb{Z})$ is an isomorphism.

There are a couple consequences of these facts. First is that *every* topological space is weakly homotopy equivalent to a CW complex. The second is that on 1-connected CW complexes, the three classes

(1) homotopy equivalences,
(2) weak homotopy equivalences, and
(3) homology equivalences

all agree.

2. Localization of categories

In the definition below, we need the functor category between two categories. If M and N are categories, then $\mathrm{Fun}(M, N)$ is the category whose objects are functors from M to N and whose morphisms are natural transformations. This expresses the fact that there is a 2-categorical enhancement of the category of (small) categories.

Definition 2.1. Let M be a category and W a class of morphisms in M. The **localization** of M by W, if it exists, is a category $M[W^{-1}]$ with a functor $L : M \to M[W^{-1}]$ such that

(1) $L(w)$ is an isomorphism for every $w \in W$,

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(2) every functor $F : M \to N$ having the property that $F(w)$ is an isomorphism for all $w \in W$ factors uniquely through L in the sense that there is a functor $G : M[W^{-1}] \to N$ and a natural isomorphism of functors $G \circ L \simeq F$, and
(3) for any category N, the functor $\mathrm{Fun}(M[W^{-1}], N) \to \mathrm{Fun}(M, N)$ induced by composition with $L : M \to M[W^{-1}]$ is fully faithful.

The localization of M by W, if it exists, is unique up to categorical equivalence.

Let Ho(Spaces) be the localization of the category of topological spaces at the weak local equivalences, and let Ho(CW) be the localization of the category of CW complexes (spaces which admit at least one CW structure) at the weak homotopy equivalences. The facts from the previous section imply that the the natural map $\mathrm{Ho(CW)} \to \mathrm{Ho(Spaces)}$ is an equivalence of categories. We will call Ho(CW) **the homotopy category of spaces**.

We can also consider $\mathrm{Ho}(\tau_{\geq 2}\mathrm{CW})$, the homotopy category obtained by localizing the category of simply-connected CW complexes at the weak homotopy equivalences. The Hurewicz theorem implies that $\mathrm{Ho}(\tau_{\geq 2}\mathrm{CW})$ is equivalent to the localization of $\tau_{\geq 2}\mathrm{CW}$ at the homology equivalences.

3. Rational homotopy theory: the outline

The fundamental category of study in rational homotopy theory is $\tau_{\geq 2}\mathrm{Spaces}$, the full subcategory of Spaces consisting of the 1-connected spaces. A **rational homotopy equivalence** is a map $f : X \to Y$ of 1-connected spaces such that

$$f_* : \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_n(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for all $n \geq 2$. We will be interested in $\mathrm{Ho}_{\mathbb{Q}}(\tau_{\geq 2}\mathrm{Spaces}) \simeq \mathrm{Ho}_{\mathbb{Q}}(\tau_{\geq 2}\mathrm{CW})$.

Rational homotopy theory is the study of spaces up to rational homotopy equivalences, or in other words of the category $\mathrm{Ho}_{\mathbb{Q}}(\tau_{\geq 2}\mathrm{Spaces})$. A **rational space** (or a **$\mathbb{Q}$-local space**) is a 1-connected space X such that $\pi_n(X)$ is a rational vector space for $n \geq 2$.

The goals of rational homotopy theory are

(1) to find a closest rational approximation $X \to \mathrm{L}_{\mathbb{Q}}X$ to a given 1-connected space X,
(2) to compute the rational homotopy type of a 1-connected space via algebraic invariants.

This is entirely similar to rationalization of chain complexes of abelian groups. In that case, given a chain complex $C_\bullet$ of abelian groups, there is a new chain complex $\mathrm{L}_{\mathbb{Q}}C_\bullet$ together with a map $C_\bullet \to \mathrm{L}_{\mathbb{Q}}C_\bullet$ such that $\mathrm{L}_{\mathbb{Q}}C_\bullet$ is a rational space and the map $\mathrm{H}_*(C_\bullet) \otimes \mathbb{Q} \to \mathrm{H}_*(\mathrm{L}_{\mathbb{Q}}C_\bullet)$ is an isomorphism. Of course, in this case the functor $\mathrm{L}_{\mathbb{Q}}$ is simply obtained by tensoring with $\mathbb{Q}$. We search therefore for a non-abelian analogue of the localization process.

Motivated by the observation that there is an adjunction $\otimes_{\mathbb{Z}} : \mathrm{Mod}_{\mathbb{Z}} \rightleftarrows \mathrm{Mod}_{\mathbb{Q}} : U$ and the localization $M \to \mathrm{L}_{\mathbb{Q}}M$ of an abelian group is the unit map of the adjunction, we hope to find an adjunction

$$\mathrm{L}_{\mathbb{Q}} : \mathrm{Ho}(\tau_{\geq 2}\mathrm{Spaces}) \rightleftarrows \mathrm{Ho}_{\mathbb{Q}}(\tau_{\geq 2}\mathrm{Spaces}) : U$$

such that the right adjoint U is fully faithful and the unit $X \to U\mathrm{L}_{\mathbb{Q}}X$ of the adjunction is precisely the localization morphism.

4. Rational homotopy theory: the main results

The first structure results are due to Quillen. Let $\mathrm{DGL}_{\mathbb{Q}}$ be the category of rational dg Lie algebras, and let $\tau_{\geq 1}\mathrm{DGL}_{\mathbb{Q}}$ be the category of dg Lie algebras $L_\bullet$ such that $\mathrm{H}_n(L) = 0$ for $n \leq 0$. Recall that a dg Lie algebra is a chain complex $L_\bullet$ equipped with a map of chain complexes $[,] : L_\bullet \otimes_{\mathbb{Q}} L_\bullet \to L_\bullet$ satisfying

(1) $[x, y] + (-1)^{pq}[y, x] = 0$ for $x \in L_p$ and $y \in L_q$,
(2) $(-1)^{pr}[x, [y, z]] + (-1)^{qp}[y, [z, x]] + (-1)^{rq}[z, [x, y]]$ for $z \in L_r$, and
(3) $d[x, y] = [dx, y] + (-1)^p[x, dy]$.

Note that the last axiom follows from the fact that the bracket map is a map of chain complexes. We let $\mathrm{Ho}(\tau_{\geq 1}\mathrm{DGL}_{\mathbb{Q}})$ be the localization at the quasi-isomorphisms, i.e., morphisms $f : L_\bullet \to M_\bullet$ of rational dg Lie algebras such that $\mathrm{H}_*(f) : \mathrm{H}_*(L) \to \mathrm{H}_*(M)$ is an isomorphism.

Similarly, let $\mathrm{CDGC}_{\mathbb{Q}}$ be the category of rational cocommutative dg coalgebras, and let $\tau_{\geq 2}\mathrm{CDGC}_{\mathbb{Q}}$ be the category of cocommutative dg coalgebras C such that $\mathrm{H}_n(C) = 0$ for $n \leq 1$. Let $\mathrm{Ho}(\tau_{\geq 2}\mathrm{CDGC}_{\mathbb{Q}})$ be the localization at the quasi-isomorphisms. Recall that a cocommutative coalgebra object in $\mathrm{Ch}_{\mathbb{Q}}$, the symmetric monoidal category of chain complexes over $\mathbb{Q}$ is just a commutative algebra object in the opposite symmetric monoidal category $\mathrm{Ch}_{\mathbb{Q}}^{\mathrm{op}}$. The reader can work out the axioms from this fact.

Theorem 4.1 (Quillen [8])**.** *There are equivalences of categories*

$$\mathrm{Ho}_{\mathbb{Q}}(\tau_{\geq 2}\mathrm{Spaces}) \simeq \mathrm{Ho}(\tau_{\geq 1}\mathrm{DGL}_{\mathbb{Q}}) \simeq \mathrm{Ho}(\tau_{\geq 2}\mathrm{CDGC}_{\mathbb{Q}}).$$

Lurie has proven moreover that these categories are equivalent to the homotopy category of 1-connected formal moduli problems over $\mathbb{Q}$.

Quillen's theorem is nicely augmented by Sullivan's, which gives computational strength to rational homotopy theory. Let $\tau^{\geq 2}\mathrm{CDGA}_{\mathbb{Q}}$ be the category of commutative cochain dg algebras $A^\bullet$ over $\mathbb{Q}$ such that $\mathrm{H}^n(A) = 0$ for $n < 0$ and $n = 1$, and $\mathrm{H}^0(A) = \mathbb{Q}$. Let $\mathrm{Ho}(\tau^{\geq 2}\mathrm{CDGA}_{\mathbb{Q}})$ be the localization at the quasi-isomorphisms.

Theorem 4.2 (Sullivan, Bousfield-Gugenheim [3])**.** *There is an equivalence*

$$\mathrm{Ho}^{\mathrm{ft}}_{\mathbb{Q}}(\tau_{\geq 2}\mathrm{Spaces}) \simeq \mathrm{Ho}^{\mathrm{ft}}(\tau^{\geq 2}\mathrm{CDGA}_{\mathbb{Q}}),$$

where $\mathrm{Ho}^{\mathrm{ft}}_{\mathbb{Q}}(\tau_{\geq 2}\mathrm{Spaces}) \subseteq \mathrm{Ho}_{\mathbb{Q}}(\tau_{\geq 2}\mathrm{Spaces})$ *is the full subcategory of homotopy types of* 1*-connected spaces* X *such that* $\mathrm{H}_n(X, \mathbb{Q})$ *is finite dimensional for all* n*, and* $\mathrm{Ho}^{\mathrm{ft}}(\tau^{\geq 2}\mathrm{CDGA}_{\mathbb{Q}})$ *is the full subcategory of* $\mathrm{Ho}(\tau^{\geq 2}\mathrm{CDGA}_{\mathbb{Q}})$ *of commutative dgas* $A^\bullet$ *over* $\mathbb{Q}$ *such that* $\mathrm{H}^*(A)$ *is finite dimensional for all* n*.*

The equivalence in Sullivan's theory is a generalization of the functor that assigns to a manifold M the de Rham complex $\mathrm{A}^\bullet_{\mathrm{dR}}(M)$. The other equivalences are somewhat less transparent at the moment.

Using the diagonal map $X \to X \times X$, one obtains a map $\mathrm{C}_*(X, \mathbb{Q}) \to \mathrm{C}_*(X \times X, \mathbb{Q})$. The Eilenberg-Zilber theorem gives a canonical quasi-isomorphism $\mathrm{C}_*(X \times X, \mathbb{Q}) \to \mathrm{C}_*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{C}_*(X, \mathbb{Q})$, and hence we get a map $\mathrm{C}_*(X, \mathbb{Q}) \to \mathrm{C}_*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{C}_*(X, \mathbb{Q})$. It turns out that this is a comultiplication on $\mathrm{C}_*(X, \mathbb{Q})$, which makes it into a cocommutative differential graded $\mathbb{Q}$-coalgebra.

Given a 1-connected space X, the loopspace ΩX is a connected space which has a homotopy-associative multiplication given by composition of loops. Thus, the homotopy groups

$$\pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

gain a product structure. In fact, this makes $\pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$ a graded Lie algebra. An enrichment of this gives the equivalence in Quillen's theorem.

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