Rational Homotopy Theory - Lecture 8

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1. Finishing the proof

Proposition 1.1. *The class of morphisms W, C, F described in the previous example give a model category structure on $\mathrm{Ch}_A^{\geq 0}$.*

End of proof. We discussed in more detail the fact that $S(n)$ is cofibrant, which we mentioned in passing last time. Consider the standard test diagram

$$\begin{array}{ccc} 0 & \longrightarrow & E \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle p} \\ S(n) & \xrightarrow{\mathrm{id}_X} & B, \end{array}$$

where p is an acyclic fibration. Giving $S(n) \to B$ amounts to choosing a cycle $z \in \mathrm{Z}^n(B)$. Since p is a fibration, we can lift z to an element $w \in E^n$. Unfortunately, there is no reason we must have $d(w) = 0$. However, there is anoether class $w' \in \mathrm{Z}^n(B)$ such that $p^*(\overline{w'}) = \overline{z}$ in $\mathrm{H}^n(B)$ since p is a quasi-isomorphism. In particular, $p(w') - z = d(b)$ for some $b \in B^{n-1}$. Lift b to $c \in E^{n-1}$, and let $w = w' - d(c)$. Then, $d(z) = 0$, and we also have that $p(w) = p(w') - p(d(c)) = p(w') - d(b) = z$, as desired. This proves that $S(n)$ is cofibrant. The complexes $D(n)$ are also cofibrant. This is left as an exercise. It is an easy corollary of what we have just said and the fact that $S(n) \to D(n)$ is cofibrant.

For the second part of **M4**, let

$$L_f(1) = X \oplus (\bigoplus_{z \in Z} D(|z|+1)) \oplus \bigoplus_{z \in \mathrm{Z}^*(Z)} S(|z|).$$

There is a natural factorization $X \to L_f(1) \to Z$, the first map is a cofibration, and the second map is a fibration. We iterate this inductively to kill the kernel of $H^*(L_f(1)) \to \mathrm{H}^*(Y)$ to get what we want. For example, to construct $L_f(2)$, take the coproduct

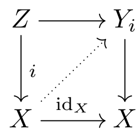
$$\begin{array}{ccc} \bigoplus_z S(|z|) & \longrightarrow & L_f(1) \\ \downarrow & & \downarrow \\ \bigoplus_z D(|z|) & \longrightarrow & L_f(2), \end{array}$$

where we set $D(0) = 0$. Inductively construct the $L_f(n)$ in this way. The colimit has the desired properties. Note we see why it's nice to have all colimits. In any case, there is a map from $L_f(2)$ to Z. Induct, and let $L_f = \mathrm{colim}_n L_f(n)$.

Having constructed the factorizations, consider **M3**. The only thing left to prove is that acyclic cofibrations $i : Z \to X$ have the LLP with respect to all fibrations. Suppose that i is a cofibration and a weak equivalence, and consider the factorization : $Z \to Y_i \to X$, wth $Z \to Y_i$ an acyclic cofibration and $Y_i \to X$ a fibration. However, by **M1**, $Y_i \to X$ is acyclic. The map $Z \to Y_i$ is again an acyclic cofibration, and $Y_i \to X$ is a fibration. But, it is easy

Date: 4 February 2016.

to see that we have a diagram



which has a dotted filling. Hence, i is a retract of $Z \to Y_i$, and hence satisfies the LLP for all fibrations because $Z \to Y_i$ does (by the same argument we used before to prove that cofibrations are closed under retracts). This completes the proof. □

Definition 1.2. A model category M has an initial object $\emptyset$ and a final object $*$, since it is closed under colimits and limits. An object X of M is **fibrant** if $X \to *$ is a fibration, and X is **cofibrant** if $\emptyset \to X$ is a cofibration. Given an object X of M, an acyclic fibration $QX \to X$ such that QX is cofibrant is called a **cofibrant replacement**. Similarly, if $X \to RX$ is an acyclic fibration with RX fibrant, then RX is called a **fibrant replacement** of X. These replacements always exist, by applying **M4** to $\emptyset \to X$ or $X \to *$.

Example 1.3. Consider the degree-wise surjective model category structure of the proposition. Let M be an arbitrary (right) A-module, viewed as a chain complex in degree 0. Let us ask if M is cofibrant. So, given any map $M \to \mathrm{H}^0(B)$, we need to find a lift to $M \to \mathrm{H}^0(E)$ making the usual diagram commute. However, since there are no degree 0 boundaries, $\mathrm{H}^0(E) \cong \mathrm{Z}^0(E) \cong \mathrm{Z}^0(B) \cong \mathrm{H}^0(B)$. So, M is cofibrant!

Question 1.4. When is $M(n)$, the module M placed in degree $n \geq 1$ cofibrant?

Remark 1.5. In the degree-wise surjective model category structure on $\mathrm{Ch}_A^{\geq 0}$, every object is fibrant.

Exercise 1.6. Show that if a complex X is cofibrant in $\mathrm{Ch}_A^{\geq 0}$, then X^n is projective for $n \geq 1$.

Example 1.7. In $\mathrm{Ch}_{\geq 0}(A)$, let M be a right A-module (viewed as a chain complex concentrated in degree zero). A projective resolution $P_\bullet \to M$ is an example of a cofibrant replacement. Indeed, such a resolution is cearly an acyclic fibration. Moreover, the map $0 \to P_\bullet$ is a cofibration, since the cokernel is projective in each degree. This time, an object M placed in degree 0 is cofibrant if and only if it is projective.

Exercise 1.8. Think about fibrant and cofibrant replacements in all of the model categories we've discussed.

Definition 1.9. A model category M is **pointed** if the natural map $\emptyset \to *$ is an isomorphism. Examples of pointed model categories include $\mathrm{Ch}_A^{\geq 0}$, which is pointed by the 0 object, and sSets_*, the category of *pointed* simplicial sets.

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