Rational Homotopy Theory - Lecture 11

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1. Simplicial sets

Topological spaces are horrible, so for the purposes of proving our main theorems in rational homotopy theory, we will use a homotopically equivalent category, the category of simplicial sets. We will endow this category with a model category structure such that the corresponding homotopy category is equivalent to the homotopy category of CW complexes.

Let Δ be the category of finite non-empty ordered sets. We call Δ the **simplex category** for reasons that will become clear below. A **cosimplicial object** in a category C is a functor $\Delta \to C$. A **simplicial object** in C is a functor $\Delta^{\mathrm{op}} \to C$, i.e., a presheaf on Δ with coefficients in C. The category of cosimiplicial objects in C is $\mathrm{Fun}(\Delta, C)$, while the category of simplicial objects in C is $\mathrm{Fun}(\Delta^{\mathrm{op}}, C)$. We are in particular interested in

$$\mathrm{sSets} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sets}),$$

the category of **simplicial sets**.

In practice, it is often convenient to take a slightly different approach to simplicial objects. Recall that a full subcategory $C \subseteq D$ is skeletal if every object of D is isomorphic to an object in D and if two isomorphic objects of C are equal. For all practical purposes, C and D are the same. Indeed, the inclusion $C \subseteq D$ is a categorical equivalence.

Let $[n]$ denote the ordered set $0 < 1 < \cdots < n$. The full subcategory of Δ on the non-empty partially ordered sets $[n]$ for $n \geq 0$ is a skeleton for Δ. Hence, we often view a simplicial set as being a functor on this full subcategory. Given a simplicial set X, we set $X_n = X([n])$. This set is called the set of **n-simplices of X**.

In what follows, we follow [5, Section 8.1] closely. There are two special classes of morphisms in Δ. The first are the **face maps** $\epsilon_i : [n-1] \to [n]$, which is the unique injective map that misses i for $0 \leq i \leq n$. The second class consists of the **degeneracy maps** $\eta_i : [n+1] \to [n]$ which is the unique surjective map with two elements mapping to i, for $0 \leq i \leq n$. These satisfy the following easy-to-check conditions:

$$\begin{aligned}
\epsilon_j \epsilon_i &= \epsilon_i \epsilon_{j-1} \text{ if } i < j, \\
\eta_j \eta_i &= \eta_i \eta_{j+1} \text{ if } \leq j, \\
\eta_j \epsilon_i &= \begin{cases} \epsilon_i \eta_{j-1} & \text{if } i < j, \\ \text{identity} & \text{if } i = j, j+1, \\ \epsilon_{i-1} \eta_j & \text{if } i > j+1. \end{cases}
\end{aligned}$$

Moreover, the morphisms in Δ are generated by these morphisms, and these are the only conditions.

It follows that a simplicial set consists of a set X_n for each $n \geq 0$ together with **face maps** $\partial_i : X_n \to X_{n-1}$ for $0 \leq i \leq n$ and **degeneracy maps** $\sigma_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$

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satisfying the identities

$$\partial_i\partial_j = \partial_{j-1}\partial_i \text{ if } i < j,$$
$$\sigma_i\sigma_j = \sigma_{j+1}\sigma_i \text{ if } \leq j,$$
$$\partial_i\sigma_j = \begin{cases} \sigma_{j-1}\partial_i & \text{if } i < j, \\ \text{identity} & \text{if } i = j, j+1, \\ \sigma_j\partial_{i-1} & \text{if } i > j+1. \end{cases}$$

This is quite a handful, and usually one doesn't specificy a simplicial object by hand, but rather by first constructing a cosimplicial object, which has **coface maps** ∂^i and **codegeneracy maps** σ^i and mapping out.

Example 1.1. Let Δ^n be the simplicial set represented by $[n]$. Thus,

$$\Delta^n_m = \Delta^n([m]) = \mathrm{Hom}_\Delta([m],[n]).$$

Note here that Δ^n is a simplicial set by construction! We will see that Δ^n behaves much like a topological simplex.

Exercise 1.2. A simplex $s \in X_n$ is **degenerate** if $s = \sigma_j(t)$ for some $t \in X_{n-1}$. In particular, $n \geq 1$ if s is degenerate. Show that if $m > n$, then every simplex in Δ^n_m is degenerate.

We can view $[n] = 0 < 1 < \cdots < n$ as a category with $n+1$ objects and a unique arrow $i \to j$ if $i \leq j$. In this way, we get a functor $\Delta \to \mathrm{Cat}$ from Δ to the category of small categories. That is, we get a cosimplicial category. The **nerve** of a small category C is $N_m(C) = \mathrm{Fun}([n], C)$. By construction, this is a simplicial set.

An important special case is when C is a category consisting of a single object $*$, and where $\mathrm{Hom}_C(*,*) = G$, some discrete group G. In this case, $NC = BG$ is called the **classifying space** of the group G.

Exercise 1.3. Show that $B_nG \cong G^n$. Compute in terms of these coordinates the face and degeneracy maps $G^n \to G^{n-1}$ and $G^n \to G^{n+1}$. This is one of the most important examples in the whole business.

A more topological examples is as follows. Let Δ^n_{top} be the topological n-simplex. That is,

$$\Delta^n_{\mathrm{top}} = \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} : \; x_i \geq 0 \text{ for } 0 \leq i \leq n \text{ and } \sum x_i = 1\}.$$

Exercise 1.4. Show that $[n] \mapsto \Delta^n_{\mathrm{top}}$ defines a cosimplicial space, i.e., a cosimplicial object in topological spaces.

Now, given any topological space X, we can construct a simplicial set called the singular simplicial set of X, written $\mathrm{Sing}(X)$. The n-simplices are

$$\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, X).$$

This defines a simplicial set because $\Delta^\bullet_{\mathrm{top}}$ is a cosimplicial space.

Weibel says how to make a smaller simplicial set for an abstract simplicial complex. Recall that an **(abstract) simplicial complex** on a set W is a subset $K \subseteq P(W)$ of the power set of W with the property that if $\sigma \in K$ and if $\tau \subseteq \sigma$, then $\tau \in K$. Fix an ordering on W, so we obtain an ordered simplicial complex. Let $S_n(K)$ be the set of all ordered tuples $(v_0,\ldots,v_n)$ such that the underlying set is in K. Prove that $S_n(K)$ is naturally a simplicial set with the obvious deletion and insertion face and degeneracy maps.

Exercise 1.5. Let W be a set with $n+1$ elements, and let $K = P(W)$. Prove that $S(K) \cong \Delta^n$ as simplicial sets. On the other hand, $\mathrm{Sing}(\Delta^n_{\mathrm{top}})$ is gigantic. Why?

References

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