Rational Homotopy Theory - Lecture 15

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1. Dold-Kan

Let $K^\bullet$ be a cosimplicial abelian group. We let C^*K be the cochain complex $0 \to K^0 \to K^1 \to K^2 \to \cdots$, where $d : K^{n-1} \to K^n$ is $\sum_{i=0}^n \epsilon_i$. This is sometimes called the **unnormalized cochain complex** of K. The **normalized cochain complex** of K is a subcomplex N^*K of C^*K given by $\mathrm{N}^pK = \cap_{i=0}^p \ker(\epsilon_i)$, with differential $\mathrm{N}^pK \to \mathrm{N}^{p+1}K$ given by ϵ_{p+1}. It is not difficult to show that the natural map $\mathrm{N}^*K \to \mathrm{C}^*K$ is a natural quasi-isomorphism for every K.

There is a dual picture for chain complexes. If X is a simplicial abelian group, then $\mathrm{C}_*(X) \cong \mathrm{N}_*(X) \oplus \mathrm{D}_*(X)$, where $\mathrm{D}_*(X)$ is the subcomplex generated by degenerate simplices. Again, $\mathrm{D}_*(X)$ is contractible. In the special case when $K = \mathrm{Hom}(X, \mathbb{Z})$, it follows that the normalized cochains are precisely those homomorphisms $X_n \to \mathbb{Z}$ that vanish on the degenerate simplices.

A key fact is that taking normalized chains gives an equivalence of categories $\mathrm{N}^* : \mathrm{cAb} \to \mathrm{Ch}_{\mathbb{Z}}^{\geq 0}$. Moreover, this is lax symmetric monoidal, so it preserves monoids. We used this fact last time. There are dual forms

2. Rational differential forms

Last time we discussed the simplicial cdga $\nabla(\bullet, *)$. This algebra supports a notion of integration as follows. There is a natural map $\nabla(p, q) \to \mathrm{A}^q_{\mathrm{dR}}(\Delta^p_{\mathrm{top}})$ given by viewing a rational differential q-form on Δ^p as a de Rham form. If $p = q$, a form $\omega \in \nabla(p, p)$ can be written as $w(t)dt_1 \wedge \cdots dt_p$, and we can define

$$\int \omega = \int_{\Delta^p_{\mathrm{top}}} w(t)\, \mathrm{d}t_1 \cdots \mathrm{d}t_p,$$

which is a rational number as $w(t)$ is a polynomial in the t_i with rational coefficients. Hence, $\int : \nabla(p, p) \to \mathbb{Q}$, which is evidently a group homorphism. Here, the t_i are the coordinates on Δ^p_{top} embedded in the standard way in $\mathbb{R}^{p+1}$.

Consider

$$\partial = \sum_{i=0}^p (-1)^i \partial_i : \nabla(p, q) \to \nabla(p-1, q),$$

the boundary in the chain complex $\mathrm{C}_*\nabla(\bullet, q)$ associated to the simplicial vector space $\nabla(\bullet, q)$. Since $\nabla(\bullet, *)$ is a simplicial chain complex, it follows that $\mathrm{C}_*\nabla(\bullet, *)$ is a cochain complex in chain complexes. That is, $\partial d = d\partial$, where d is the algebraic de Rham differential.

Theorem 2.1 (Stokes' theorem)**.** *For $\omega \in \nabla(p, p-1)$,*

$$\int d\omega = \int \partial w.$$

Proof. This theorem is a special case of Stokes' theorem for a simplex. □

Now comes a key definition in this course. Let X be a simplicial set, and let $\mathrm{N}^*X = \mathrm{N}^* \mathrm{Hom}(\mathbb{Q}[X], \mathbb{Q})$ be the normalized cochain complex. We mentioned last time that $\mathrm{H}^*(\mathrm{N}^* \mathrm{Sing}(X)) \cong \mathrm{H}^*(X, \mathbb{Q})$.

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Definition 2.2. Let $\mathrm{A}^*(X) = \mathrm{Hom}_{\mathrm{sSets}}(X, \nabla(\bullet, *))$, which is naturally a cdga with differential induced by $d : \nabla(\bullet, q) \to \nabla(\bullet, q+1)$.

There is a canonical unit map $\eta : \mathbb{Q} \to \mathrm{A}^*(X)$ given by $\eta(\alpha)(x) = \eta(\alpha)$ for all $x \in X_0$, where we abuse notation and use η for the unit of $\Omega^0_{\mathbb{A}^\bullet/\mathbb{Q}}$.

3. Thom's problem

Recall that part of the story of rational homotopy theory began with Thom's problem to find cdga models for the cohomology rings of spaces. This is impossible in general with integer coefficients, but is solved by $\mathrm{A}^*(X)$ with rational coefficients, as we shall see.

We start by defining a natural map $\rho : \mathrm{A}^*(X) \to \mathrm{N}^*X = \mathrm{N}^*\,\mathrm{Hom}(\mathbb{Q}[X], \mathbb{Q})$. Given $\omega \in \mathrm{A}^q(X) = \mathrm{Hom}(X, \nabla(\bullet, q))$ and $x \in X_q$, we let

$$\langle \rho\omega, x \rangle = \int \omega(x).$$

This is natural in X. We have to check several things, which will take a little while.

(1) We have to check that ρ lands in the normalized cochains.
(2) We have to check that ρ is a map of cochain complexes.
(3) We have to show that ρ is a quasi-isomorphism.
(4) We have to show that ρ is compatible with multiplication in a certain sense.

Lemma 3.1. *Given $\omega \in \mathrm{A}^q(X)$, $\rho\omega$ is a normalized cochain.*

Proof. Let $x = s_i y$ be a degenerate q-simplex. It is enough to show that $(\rho\omega)(x) = 0$. But, $\omega(x) = \omega(s_i y) = s_i\omega(y) = 0$, since $\omega(y) \in \nabla(q-1, q) = 0$. □

Lemma 3.2. *The map ρ is a map of cochain complexes.*

Proof. Let $\omega \in \mathrm{A}^q(X)$, and view this by adjunction as a map $\mathbb{Q}[X] \to \nabla(\bullet, q)$ of simplicial $\mathbb{Q}$-modules. Then, for $x \in \mathbb{Q}[X_{q+1}]$, we have

$$\begin{aligned} \langle \rho d\omega, x \rangle &= \int (d\omega)x \\ &= \int \partial\omega x \\ &= \int \omega\partial x \\ &= \langle \rho\omega, \partial x \rangle \\ &= \langle d\rho\omega, x \rangle. \end{aligned}$$

This is what we wanted to show. □

Showing the other two properties is quite a bit harder and will have to wait. In the meantime consider the unit map $\eta : \mathbb{Q} \to \mathrm{N}^0 X$ which sends $(\eta\alpha)(x) = \alpha$. We claim that ρ commutes with the unit maps η. Let $\omega \in \mathrm{A}^0(X) = \mathrm{Hom}(X, \nabla(\bullet, 0))$, and let $x \in X_0$. Then,

$$\langle \rho\omega, x \rangle = \omega(x) \in \mathbb{Q} = \nabla(0, 0).$$

It follows that the $\rho\eta_{\mathrm{A}^*X} = \eta_{\mathrm{N}^*X}$.

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