Rational Homotopy Theory - Lecture 17

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1. The model category on rational cdgas

Throughout this section, $\mathrm{Ch} = \mathrm{Ch}_{\mathbb{Q}}^{\geq 0}$ denotes the category of non-negatively graded *rational* cochain complexes, and $\mathrm{cdga} = \mathrm{cdga}_{\mathbb{Q}}^{\geq 0}$ is the category of commutative algebra objects in $\mathrm{Ch}_{\mathbb{Q}}^{\geq 0}$. Recall that we introduced two model category structures on Ch. One is the injective model category structure, in which the weak equivalences are the quasi-isomorphisms and the cofibrations are the positive-degreewise monomorphisms. The other has weak equivalences the quasi-isomorphisms and fibrations the degree-wise surjections. In this section we view Ch as equipped with the second model category structure, which we will call the **big** model category structure.

Theorem 1.1. *There is a (big) model category structure on* cdga *with weak equivalences the quasi-isomorphisms and fibrations the degree-wise surjections making the adjunction*

$$\mathrm{Sym} : \mathrm{Ch} \rightleftarrows \mathrm{cdga} : U$$

a Quillen adjunction, where Sym *is the free commutative algebra functor and* U *is the forgetful functor.*

Proof. Since U preserves fibrations and acyclic fibrations, it is enough to prove that the claimed classes of morphisms are part of a model category structure on cdga. The proof is basically the same as the proof of the big model category structure on Ch, which we explained in detail in Lectures 7 and 8. The main differences are that we use different elementary cofibrations. This time, we let $S(n)$ (for $n \geq 0$) be the free commutative $\mathbb{Q}$-algebra on an element x of degree n, with $d(x) = 0$. The algebra $D(n)$ (for $n \geq 1$) is the free commutative algebra on elements x, y of degree $n-1$ and n, respectively, with $d(x) = y$. Besides this change in definitions, the proof remains exactly the same as for Ch. To convince you, let me do a single example, the verification of the first part of **M4**. So, let $f : X \to Y$ be an arbitrary map in cdga. We want to factor f as an acyclic cofibration followed by a fibration. We define Y_f as

$$Y_f = X \otimes \bigotimes_{y \in Y} D(|y| + 1).$$

Then, there are natural maps $X \to Y_f \to Y$, and $Y_f \to Y$ is surjective, so it is a fibration. The map $X \to Y_f$ is a quasi-isomorphism and it is easily seen to be a cofibration. □

Exercise 1.2. Prove that $S(n)$ and $D(n)$ are indeed cofibrant, and that $S(n-1) \to D(n)$ is a cofibration.

Of course, the same model category structure exists on $\mathrm{Ch}_{\mathbb{Z}}^{\geq 0}$, so we can wonder about the existence of a compatible model category structure on $\mathrm{cdga}_{\mathbb{Z}}^{\geq 0}$.

Proposition 1.3. *Consider the adjunction*

$$\mathrm{Sym} : \mathrm{Ch}_{\mathbb{Z}}^{\geq 0} \rightleftarrows \mathrm{cdga}_{\mathbb{Z}}^{\mathrm{cdga}} : U.$$

There is no model category structure $\mathrm{cdga}_{\mathbb{Z}}^{\geq 0}$ *where the weak equivalences are the quasi-isomorphisms and where this adjunction is a Quillen adjunction where* $\mathrm{Ch}_{\mathbb{Z}}^{\geq 0}$ *is given either the big or the injective model category structure.*

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Proof. Consider the cdga $D(2)$ as in the proof of the theorem. What is its homology over $\mathbb{Z}$? As an algebra, $D(2)$ is $\mathbb{Z}[x,y]/(2x^2)$, where $|x|=1$, $|y|=2$, and $d(x)=y$. Now, $d(x^m y^n) = mx^{m-1}y^n$, which is zero if $m \geq 4$ and m is even. Hence, the monomials $mx^{m-1}y^n$ for $m \geq 0$, m even, and $n \geq 1$ represent non-zero homology classes. It follows that Sym does not preserve acyclic cofibrations. □

2. Comma categories

Let $x \in C$. A **comma category** is a category of objects over or under x. Specifically, the under-category $C_{x/}$ is the category of objects of C equipped with a map from x, while the over-category $C_{/x}$ is the category of objects of C equipped with a map to x.

Proposition 2.1. *Suppose that M is a model category and $X \in M$. Then, $M_{/X}$ and $M_{X/}$ are model categories where a morphism in one is in W, C, F if the underlying morphism in M is in W, C, F, respectively.*

Proof. Exercise. □

Example 2.2. The model category of pointed simplicial sets is $\mathrm{sSets}_{*/}$, but we will usually write this as sSets_*.

Example 2.3. The model category of augmented cdgas is $(\mathrm{cdga}_{/\mathbb{Q}} = \mathrm{cdga}_{\mathbb{Q}}^{\geq 0})_{/\mathbb{Q}}$.

3. Commutative dgas as a (co)simplicial model category

We will not need the full structure of the (co)simplicial model category, but we will need the mapping spaces and the action objects. So, let R be a cdga, and let X be a simplicial set. How should we define $R \otimes X$? Well, we simply define it as

$$R \otimes X = R \otimes \mathrm{A}^*(X) = R \otimes \mathrm{A}^*_{\nabla}(X).$$

Why am I throwing around (co)simplicial? Note that $R \otimes (-)$ is contravariant in maps of simplicial sets with this definition. Hence, the slightly different convention. This doesn't cause any issues.

The mapping space $\mathrm{map}_{\mathrm{cdga}}(R,S)$ is the simplicial set with p-simplices

$$\mathrm{map}_{\mathrm{cdga}}(R,S)_p = \mathrm{Hom}_{\mathrm{cdga}}(R, \nabla(p,*) \otimes S) = \mathrm{Hom}_{\mathrm{cdga}_{\nabla(p,*)}}(\nabla(p,*) \otimes R, \nabla(p,*) \otimes S).$$

Exercise 3.1. Define composition $\mathrm{map}_{\mathrm{cdga}}(S,T) \times \mathrm{map}_{\mathrm{cdga}}(R,S) \to \mathrm{map}_{\mathrm{cdga}}(R,T)$.

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