Rational Homotopy Theory - Lecture 19

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1. Long exact sequences

Lemma 1.1. *If X is an augmented cdga, then*

(1) $\mathrm{Hom}_*(X, U(n)) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Q}X^n, \mathbb{Q})$,
(2) *right homotopy is an equivalence relation on* $\mathrm{Hom}_*(X, V(n))$*, and*
(3) $[X, V(n)]_* \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n X, \mathbb{Q})$.

Proof. We proved (1) last time.

To prove (2), it is enough to show that $\mathrm{map}_*(X, V(n))$ is a Kan complex. Note that because $V(n)$ has trivial multiplication, so does $\nabla(p,*)\tilde{\otimes}V(n)$ for all $p \geq 0$. In particular, we can add augmented maps $X \to \nabla(p,*)\tilde{\otimes}V(n)$, and hence $\mathrm{map}_*(X, V(n))$ has the structure of a simplicial abelian group. Hence, it's a Kan complex, so that right homotopy is an equivalence relation on $\mathrm{Hom}_*(X, V(n))$. Give more details!

Now, given an augmented map $f : X \to V(n)$, there is an induced map $\mathrm{Q}f : \mathrm{Q}X \to \mathrm{Q}V(n) \cong \mathbb{Q}[n]$. We get an induced map $\pi^n(X) \to \pi^n V(n) = \mathbb{Q}$. The map $\mathrm{Hom}_*(X, V(n)) \to \mathrm{Hom}_{\mathbb{Q}}(\pi^n X, \mathbb{Q})$ is evidently additive. Now, if X has trivial multiplication, then the map is surjective. But, we can replace X by $X/\overline{X}\cdot\overline{X}$ to achieve this. So, the map is always surjective. The Proposition 3.1 from last time shows that the assignment factors through $[X, V(n)]_*$.

Thus, we have to check that the induced map $[X, V(n)]_* \to \mathrm{Hom}_{\mathbb{Q}}(\pi^n X, \mathbb{Q})$ is injective, where we still assume that X has trivial multiplication. Suppose then that $f, g : X \to V(n)$ are pointed maps such that $f_* = g_* : \pi^n X \to \pi^n V(n) = \mathbb{Q}$. Let $H : \overline{X} \to V(n)[-1]$ be a chain homotopy between f and g, with no assumption about multiplicativity. Define $h : X \to \nabla(1,*)\tilde{\otimes}V(n)$ be defined by

$$h(x) = dt_1 \otimes H(x) + 1 \otimes f(x) - t_1 \otimes f(x) + t_1 \otimes g(x)$$

for $x \in \overline{X}$. Then, $\partial_0 h = g$ and $\partial_1 h = f$. Hence, f and g are homotopic maps, as desired. $\square$

One of the key parts of classical homotopy theory is the long exact sequence

$$\cdots \to \pi_2 Y \to \pi_1 F \to \pi_1 X \to \pi_1 Y \to \pi_0 F \to \pi_0 X \to \pi_0 Y$$

associated to a fibration $F \to X \to Y$. This has an exact analogue in rational cdgas.

Proposition 1.2. *Suppose that*

$$\begin{array}{ccc} V & \xrightarrow{j} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle k} \\ W & \xrightarrow{h} & Y \end{array}$$

is a pushout square in cdga_**, and assume that i is a cofibration. Then, there is a long exact sequence*

$$\pi^0 V \to \pi^0 W \oplus \pi^0 X \to \pi^0 Y \to \pi^1 V \to \cdots.$$

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Proof. It is enough to check that the induced square

$$\begin{array}{ccc} QV & \longrightarrow & QX \\ \downarrow & & \downarrow \\ QW & \longrightarrow & QY \end{array}$$

is a pushout square of chain complexes with Qi^n injective for $n > 0$. Consider the augmented cdga $U(n)$ as above. Then, $U(n) \to \mathbb{Q}$ is an acyclic fibration, so it satisfies the right lifting property with respect to the cofibration $V \to W$. In particular, from Lemma 1.1(a), $\mathrm{Hom}_{\mathbb{Q}}(QW^n, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Q}}(QV^n, \mathbb{Q})$ is surjective. Hence, $QV^n \to QW^n$ is injective for $n > 0$.

To prove that the square is a pushout square, note that after taking duals, one obtains a pullback square by Lemma 1.1(1). But, over a field, if the dual of a square with Qi^n injective is a fiber square, then the initial square is a pushout square (exercise). □

Corollary 1.3. *If $X = \mathbb{Q}$ in the proposition, we get a long exact sequence*

$$\pi^0 V \to \pi^0 W \to \pi^0(W/V) \to \pi^1(V) \to \cdots .$$

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