Rational Homotopy Theory - Lecture 21

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1. The adjoint to the de Rham functor

We constructed a functor $\mathrm{A} : \mathrm{sSets}^{\mathrm{op}} \to \mathrm{cdga}_{\mathbb{Q}}^{\geq 0}$ by setting

$$\mathrm{A}^q(X) = \mathrm{Hom}_{\mathrm{sSets}}(X, \nabla(\bullet, q)).$$

We will see now that this has a right adjoint.

Proposition 1.1. *The functor* $\mathrm{A} : \mathrm{sSets}^{\mathrm{op}} \to \mathrm{cdga}_{\mathbb{Q}}^{\geq 0}$ *has a left adjoint given by*

$$\mathrm{F}(X)_p = \mathrm{Hom}_{\mathrm{cdga}}(X, \nabla(p, *))$$

for a cdga X.

Proof. This is a special case of Exercise 1.1 of Lecture 18. □

Hence, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{cdga}}(X, \mathrm{A}(Y)) \cong \mathrm{Hom}_{\mathrm{sSets}^{\mathrm{op}}}(\mathrm{F}(X), Y) \cong \mathrm{Hom}_{\mathrm{sSets}}(Y, \mathrm{F}(X)).$$

Note that this looks slightly strange because of the op decorating sSets.

In any case, we want to check that these functors are derivable. because of the op this is again slightly strange. We must prove for instance that

$$\mathrm{F} : \mathrm{cdga}_{\mathbb{Q}}^{\geq 0} \to \mathrm{sSets}^{\mathrm{op}}$$

preserves cofibrations and acyclic cofibrations. But, on the right hand side we must use the opposite model category structure. This means that if $X \to Y$ is a *cofibration* of cdgas, then we need to check that $\mathrm{F}(X) \to \mathrm{F}(Y)$ is a cofibration in $\mathrm{sSets}^{\mathrm{op}}$, or that $\mathrm{F}(X) \leftarrow \mathrm{F}(Y)$ is a *fibration* in sSets. A similar remark holds for acyclic cofibrations.

To check that this works, let's think about what $\mathrm{F}(X)$ actually is. Recall that we defined the mapping space $\mathrm{map}_{\mathrm{cdga}}(X, Y)$ between two cdgas as a simplicial set with p-simplices given by

$$\mathrm{map}_{\mathrm{cdga}}(X, Y)_p = \mathrm{Hom}_{\mathrm{sSets}}(X, \nabla(p, *) \otimes Y).$$

Corollary 1.2. *The left adjoint* $\mathrm{F}(X)$ *is* $\mathrm{map}_{\mathrm{cdga}}(X, \mathbb{Q})$.

Hence, we can use the **SM7** axiom to check various properties.

Proposition 1.3. *If* $X \to Y$ *is a cofibration of cdgas (resp. acyclic cofibration of cdgas), then* $\mathrm{F}(X) \leftarrow F(Y)$ *is a fibration of simplicial sets (resp. acyclic Kan fibration of simplicial sets).*

Proof. Consider the map $\mathbb{Q} \to 0$ from the unit to the zero cdga. Then, axiom **SM7** says that

$$\mathrm{map}_{\mathrm{cdga}}(Y, \mathbb{Q}) \to \mathrm{map}_{\mathrm{cdga}}(Y, 0) \times_{\mathrm{map}_{\mathrm{cdga}}(X,0)} \mathrm{map}_{\mathrm{cdga}}(X, \mathbb{Q})$$

is a Kan fibration which is acyclic if $X \to Y$ is acyclic. But, $\mathrm{map}_{\mathrm{cdga}}(Y, 0) = * = \mathrm{map}_{\mathrm{cdga}}(X, 0)$, so this just says that $\mathrm{F}(Y) \to \mathrm{F}(X)$ is a Kan fibration which is acyclic if $X \to Y$ is an acyclic cofibration, as desired. □

Corollary 1.4. *The pair*

$$\mathrm{F} : \mathrm{cdga}_{\mathbb{Q}}^{\geq 0} \rightleftarrows \mathrm{sSets}^{\mathrm{op}} : \mathrm{A}$$

is a Quillen pair of adjoint functors.

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We will write $\mathbf{L}\mathrm{F}$ and $\mathbf{R}\mathrm{A}$ for the derived functors.

Exercise 1.5. Show that given a space (i.e., simplicial set) X, one has $\mathbf{R}\mathrm{A}(X) \cong \mathrm{A}(X)$ in the homotopy category $\mathrm{Ho}(\mathrm{cdga}_{\mathbb{Q}}^{\geq 0})$. Hint: see Remark 4.3 of Lecture 12.

Exercise 1.6. Show that the results of this section have natural *pointed* analogues. We will use these without comment.

2. The Sullivan-de Rham theorems

Recall that a group G is **nilpotent** if its lower central series $G = \Gamma_1 G \supseteq \Gamma_2 G \supseteq \Gamma_3 G \supseteq \cdots$ stabilizes with $\Gamma_n G = *$ for some finite n. Here, $\Gamma_n G = [G, \Gamma_{n-1} G]$. A G-module M is **nilpotent** if the series $\Gamma_n M \supseteq \Gamma_{n+1} M \supseteq \cdots$ terminates with 0 for some n. Here, $\Gamma_n M$ is the sub-G-modules of M generated by $gm - m$ for $g \in \Gamma_n M$.

Exercise 2.1. Show that $\Gamma_n G / \Gamma_{n+1} G$ is abelian, and that the action of G on $\Gamma_n M / \Gamma_{n+1} M$ is trivial.

We say that a space (i.e., Kan complex) X is **nilpotent** if

(1) X is connected,
(2) $\pi_1 X$ is nilpotent, and
(3) the action of $\pi_1 X$ on $\pi_n X$ is nilpotent for $n \geq 1$.

If moreover $\pi_n X$ is uniquely divisible for $n \geq 1$, X is **rational**. Finally, if X is nilpotent and $\mathrm{H}_n(X, \mathbb{Q})$ is finite dimensional for $n \geq 1$, then X is a nilpotent space **of finite $\mathbb{Q}$-type**. Let $\mathrm{Ho}(\mathrm{sSets})^{\mathrm{fn}\mathbb{Q}}$ denote the full subcategory of $\mathrm{Ho}(\mathrm{sSets})$ consisting of the nilpotent rational spaces of finite $\mathbb{Q}$ type, and similarly for $\mathrm{Ho}(\mathrm{sSets}_*)^{\mathrm{fN}\mathbb{Q}}$.

Exercise 2.2. Show that for a nilpotent space X, $\pi_n X$ is uniquely divisible for $n \geq 1$ if and only if $\mathrm{H}_n(X, \mathbb{Z})$ is uniquely divisible for $n \geq 1$.

Here are the cdga analogues of these ideas. A *coconnective cofibrant* cdga $X \in \mathrm{cdga}_{\mathbb{Q}}^{\geq 0}$ is of finite $\mathbb{Q}$-type if X^n is finite dimensional for $n \geq 1$. This happens if and only if $\pi^n X$ is finite dimensional (over $\mathbb{Q}$) for all $n \geq 0$. Write $\mathrm{Ho}(\mathrm{cdga}_{\mathbb{Q}}^{\geq 0})^{\mathrm{f}}$ for the full subcategory on the finite $\mathbb{Q}$-type coconnective cdgas, and similarly for $\mathrm{Ho}(\mathrm{cdga}_{/\mathbb{Q}}^{\geq 0})^{\mathrm{f}}$ for the finite $\mathbb{Q}$-type augmented cdgas.

Theorem 2.3 (The Sullivan-de Rham theorem)**.** *The derived functors*

$$\mathbf{L}\mathrm{F} : \mathrm{Ho}(\mathrm{cdga}_{\mathbb{Q}}^{\geq 0}) \rightleftarrows \mathrm{Ho}(\mathrm{sSets}^{\mathrm{op}}) : \mathbf{R}\mathrm{A}$$

restrict to inverse equivalences

$$\mathrm{Ho}(\mathrm{cdga}_{\mathbb{Q}}^{\geq 0})^{\mathrm{f}} \rightleftarrows \mathrm{Ho}(\mathrm{sSets}^{\mathrm{op}})^{\mathrm{fN}\mathbb{Q}}.$$

One can prove a slightly bigger version of this theorem by using pro-nilpotent spaces.

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