MATH 215 – Fall 2017 – Axioms for the Integers (AI)

We will assume the existence of a set $\mathbb{Z} = \{0, 1, -1, 2, -2, \cdots\}$, whose elements are called integers, along with a well-defined binary operation $+$ on $\mathbb{Z}$ (called addition), a second well-defined binary operation $\cdot$ on $\mathbb{Z}$ (called multiplication), and a relation $<$ on $\mathbb{Z}$ (called less than), and we will assume that the following statements involving $\mathbb{Z}$, $+$, $\cdot$, and $<$ are true:

A1. For all a, b, c in $\mathbb{Z}$, $(a + b) + c = a + (b + c)$.
A2. There exists an integer 0 in $\mathbb{Z}$ such that $a + 0 = 0 + a = a$ for every integer a.
A3. For every a in $\mathbb{Z}$, there exists a unique integer $-a$ in $\mathbb{Z}$ such that $a + (-a) = (-a) + a = 0$.
A4. For all a, b in $\mathbb{Z}$, $a + b = b + a$.
M1. For all a, b, c in $\mathbb{Z}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
M2. There exists an integer 1 in $\mathbb{Z}$ such that $a \cdot 1 = 1 \cdot a = a$ for all a in $\mathbb{Z}$.
M4. For all a, b in $\mathbb{Z}$, $a \cdot b = b \cdot a$.
D1. For all a, b, c in $\mathbb{Z}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
NT1. $1 \neq 0$.
O1. For all a in $\mathbb{Z}$, exactly one of the following statements is true: $0 < a$, $a = 0$, $0 < -a$.
O2. For all a, b in $\mathbb{Z}$, if $0 < a$ and $0 < b$, then $0 < a + b$.
O3. For all a, b in $\mathbb{Z}$, if $0 < a$ and $0 < b$, then $0 < a \cdot b$.
O4. For all a, b in $\mathbb{Z}$, $a < b$ if and only if $0 < b + (-a)$.
WOP. A mystery axiom to be introduced later.

Notation 1 *We will use the common notation ab to denote $a \cdot b$.*

Notation 2 *We will also use the notation $a > b$ (greater than) to denote $b < a$ (less than).*

Proposition 3 *The element 0 is unique. In other words, if $0'$ is another element such that $a + 0' = 0' + a = a$ for all a in $\mathbb{Z}$, then $0' = 0$.*

Proposition 4 *For every a in $\mathbb{Z}$, $a \cdot 0 = 0$.*

Proposition 5 *Let a, b be integers. If $ab = 0$, then $a = 0$ or $b = 0$.*

Proposition 6 *The element 0 in $\mathbb{Z}$ has no multiplicative inverse. In other words, there is no integer a such that $a \cdot 0 = 1$.*

Proposition 7 *The element 1 is unique. In other words, if $1'$ is an integer such that $a \cdot 1' = 1' \cdot a = a$ for all a in $\mathbb{Z}$, then $1' = 1$.*

Proposition 8 *For all a, b, c in $\mathbb{Z}$, if $a + b = a + c$, then $b = c$.*

Proposition 9 *For every a in $\mathbb{Z}$, $-(-a) = a$.*

Proposition 10 *For all integers a and b, $(-a)b = -(ab)$.*

Proposition 11 *For all integers a and b, $(-a)(-b) = ab$.*

Proposition 12 $(-1)(-1) = (1)(1) = 1$.

Proposition 13 $0 < 1$.