547 - Fall 2019 - HW1

August 22, 2019

1. Prove that $\mathcal{T}$ with objects the topological spaces and with hom sets the continuous functions is a category. In particular, show that $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is a set for any pair of topological spaces.

2. Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$ be an object. Define a new category $\mathcal{C}_{X/}$ consisting of pairs (Y, f) where $Y \in \mathcal{C}$ and $f : X \to Y$. Morphisms $(Y, f) \to (Z, g)$ are maps $h : Y \to Z$ in $\mathcal{C}$ such that $g = h \circ f$.

3. Prove that the category $\mathcal{T}_*$ of pointed spaces is equivalent to $\mathcal{T}_{*/}$. Note: there is a unique topology on $*$, the set with one point.

4. How many topologies are there on the set $\{1, 2\}$ with two points? How many are there up to homeomorphism?

5. Construct a forgetful functor $\mathcal{T}_* \to \mathcal{T}$. Prove that it admits a left adjoint and describe it.

6. Prove that homotopy is an equivalence relation on $\mathrm{Hom}_{\mathcal{T}}(X, Y)$.

7. Construct a "quotient" functor $\mathcal{T} \to \mathrm{Ho}(\mathcal{T})$.

8. Do Hatcher, Exercise 0.1.

9. Do Hatcher, Exercise 0.5.

10. Do Hatcher, Exercise 0.16.