548 - Spring 2020 - HW3

Due 19 February 2020

1. Let $\mathcal{C}$ be a model category. Prove that cofibrations are closed under cobase change. In other words, prove that if $X \xrightarrow{i} Y$ is a cofibration and $X \xrightarrow{f} Z$ is any map, then the pushout $Z \xrightarrow{j} Z \cup_X Y$ is a cofibration.

2. Based on the work in class with $\mathrm{Ch}_{\geqslant 0}(A)$, propose a model category structure on $\mathrm{Ch}^{\geqslant 0}$, the category of non-negatively graded *cochain* complexes of left A-modules.

3. Prove that if $\mathcal{C}$ is a model category (with respect to W, C, F), then for any object X of $\mathcal{C}$, $\mathcal{C}_{X/}$ admits a model category structure where a map

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & \xrightarrow{f} & Z \end{array}$$

in $\mathcal{C}_{X/}$ is a weak equivalence, cofibration, or cofibration if and only if f is a weak equivalence, cofibration, or fibration in $\mathcal{C}$.

4. Let $\mathcal{C}$ be a model category. Prove that $f : X \to Y$ maps to an isomorphism in $\mathrm{Ho}(\mathcal{C})$ if and only if $f \in W$.

5. Prove using only what we've done with model categories that if $f : M \to N$ is a morphism of two left A-modules and if $P_* \to M$ and $Q_* \to N$ are projective resolutions, then there exists a morphism $\tilde{f} : P_* \to Q_*$ making

$$\begin{array}{ccc} P_* & \xrightarrow{\tilde{f}} & Q_* \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

commute.

6. Let A be an associative ring and let M and N be two left A-modules. View M and N as chain complexes concentrated in degree 0. Compute $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Ch}_{\geqslant 0}(A))}(M, N)$.

7. Let M and N be left A-modules. Denote, for $n \geqslant 0$, by $N[n]$ the chain complex with N in degree n and zeros elsewhere. Compute $\mathrm{Hom}_{\mathrm{Ch}_{\geqslant 0}(A)}(M, N[n])$.

8. In the situation of Problem 7, compute $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Ch}_{\geqslant 0}(A))}(M, N[n])$.