F18OS - Perfectoid rings

August 15, 2018

The goal of this seminar is to begin working with perfectoid rings and spaces. We will do this by understanding first the short paper of Bhatt–Iyengar–Ma and then Bhatt's proof of the (derived) direct summand conjecture.

To start, we describe briefly the theorem of [BIM18]

Definition 1. Let R be a p-adically complete ring. We say that R is **perfectoid** if (1) $p = u\pi^p$ for some unit u and some element π, (2) the Frobenius map $R/p \to R/p$ is surjective, and (3) the kernel of the natural map $W(R^\flat) \to R$ is principal, where $R^\flat$ is the limit-perfection of R/p.

Remark 2. Equivalently, a ring R is perfectoid if and only if there is a perfect $\mathbb{F}_p$-algebra S and an isomorphism $W(S)/(\xi) \cong R$ where $\xi = a_0 + a_1 p + a_2 p^2 + \cdots$ is some element such that a_1 is a unit in S and S is a_0-adically complete. See [BIM18, Lemma 3.6] for a proof.

Theorem 3 (Bhatt–Iyengar–Ma [BIM18])**.** *Let R be a p-adically complete Noetherian local ring. Then, R is regular if and only if it admits a faithfully flat map to a perfectoid ring.*

Warning 4. There are many different notions of perfectoid rings in the literature. The ones appearing here might be called the integral perfectoid rings. Other flavors include various notions over perfectoid fields.

Now, we recall the direct summand conjecture.

Theorem 5 (André [And16], Bhatt [Bha18])**.** *Suppose that R is a regular noetherian ring and $R \hookrightarrow S$ is a finite R-algebra. Then, the structure map is split as an R-module map.*

These two results have rather different proofs. The first is primarily restricted to the world of commutative algebra, despite its use of perfectoid rings. The proof of the direct summand conjecture relies on almost mathematics and perfectoid spaces. We aim to understand these and to bear in mind Wenliang Zhang's question on annihilators of local cohomology modules.

Question 6 (Zhang)**.** Suppose that R is a regular local ring and $I \subseteq R$ an ideal. Fix some integer n. Is it the case that $\mathrm{ann}_R \mathrm{H}^n_I(R) \neq 0$ if and only if $\mathrm{H}^n_I(R) = 0$?

We will start with the proof of Theorem 3 before moving to the more technically challenging Theorem 5. For the latter, we follow the presentation in Bhatt's notes [Bha17, Chapter 11].

Week 1. Perfectoid rings. See [BMS16, Section 3] and [BMS18, Section 4]. In particular, we will need the construction of the map $W(R^\flat) \to R$ and the description of the distinguished elements. Roughly 3.1-3.19 of [BMS16]. Fontaine's article [Fon94] is also helpful.

Week 2. Cover [BIM18, Section 3]. In some sense, this will be a recapitulation of Week 1, but there are good examples and everyone will need time to get familiar with these.

Week 3. Cover [BIM18, Section 2]. This gives the main homological criterion used to prove Theorem 3.

Week 4. Finish the proof of Theorem 3 by covering [BIM18, Section 4].

Week 5. We sit down together in the seminar room and actually try to solve Question 6.

Week 6. Almost mathematics I. See [Bha17, Section 4].

Week 7. Almost mathematics II. See [Bha17, Section 4].

Week 8. Adic spaces I. See [Bha17, Sections 7 and 8].

Week 9. Adic spaces II. See [Bha17, Section 7 and 8].

Week 10. Perfectoid spaces I. See [Bha17, Section 9].

Week 11. Perfectoid spaces II. See [Bha17, Section 9].

Week 12. Almost purity. Probably we just need to know what the idea of this theorem is and some outline of the proof. See [Bha17, Section 10].

Week 13. Overview of the proof (p.125 of [Bha17]) of DSC and the case of extensions unramified in characteristic zero (Theorem 11.1.1 of [Bha17]).

Week 14. The proof. Survey Sections 11.2 and 11.3 of [Bha17] and then give the proof as in Section 11.4.

References

[And16] Y. André, *La conjecture du facteur direct*, ArXiv e-prints (2016), available at http://arxiv.org/abs/1609.00345.

[Bha18] B. Bhatt, *On the direct summand conjecture and its derived variant*, Invent. Math. **212** (2018), no. 2, 297–317.

[Bha17] ———, *Lecture notes for a class on perfectoid spaces* (2017), available at http://www-personal.umich.edu/~bhattb/teaching/mat679w17/lectures.pdf.

[BIM18] B. Bhatt, S. Iyengar, and L. Ma, *Regular rings and perfect(oid) algebras*, ArXiv e-prints (2018), available at http://arxiv.org/abs/1803.03229.

[BMS16] B. Bhatt, M. Morrow, and P. Scholze, *Integral p-adic Hodge theory*, ArXiv e-prints (2016), available at http://arxiv.org/abs/1602.03148.

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[Fon94] J.-M. Fontaine, *Le corps des périodes p-adiques*, Astérisque **223** (1994), 59–111. With an appendix by Pierre Colmez; Périodes p-adiques (Bures-sur-Yvette, 1988).