Hesselholt.

THH and arithmetic: de Rham cohomology over the sphere.

THH de Rham complex over the sphere.

$S^1$-action gives an operator to raise degree by 1.

Bryan etc: don't first take homotopy groups.
Build something directly.

$$TP(X) \simeq THH(X)^{tT}.$$

Natural notion of de Rham cohomology of $X/\mathbb{S}$.

Morava...

$X/\mathbb{F}_p$ smooth.

Calculated Postnikov filtration of, e.g., $TR^n(\mathcal{O}_X, p)$.

$\vdots$

4 $W_n\Omega^4_X \oplus W_n\Omega^2_X \cdot x \oplus W_n\Omega^0_X \cdot x^2$ weight 2.

3 $W_n\Omega^3_X \oplus W_n\Omega^1_X \cdot x$

2 $W_n\Omega^2_X \oplus W_n\Omega^0_X \cdot x$ weight 1

divided Bott element

1 $W_n\Omega^1_X$

0 $W_n\Omega^0_X$ weight 0

$Fr_p^*$ geometric Frobenius.

$Fr_p^* = p^j \phi_p$  $j =:$ weight

internal Frobenius. Like the weight in motivic cohomology.

$W_n\Omega^i_X$ has weight $i$.

$x$ has weight 1.

So, e.g., $W_n\Omega^0_X \cdot x$ has weight 1,

$W_n\Omega^2_X \cdot x$ has weight 3.

Want: a different theory when the weights are not mixed in each degree.

<u>Thm</u> (Bhatt, Bhatt-Morrow-Scholze). New results, to appear.

If $f: A \to B$ is faithfully flat,

$$THH(A) \xrightarrow{\sim} \lim_{\Delta} THH(B^{\otimes_A \bullet}).$$

That is, THH satisfies faithful flat descent.

$A/\mathbb{F}_p$ smooth

$$\underset{\text{faithfully flat}}{A \longrightarrow} A^{perf} \cong \operatorname{colim}_{\phi} A$$

$$THH(A) \simeq \lim_{\Delta} THH(A^{perf, \otimes_A \bullet}).$$

$$Fil^j THH(A) \simeq \lim_{\Delta} \tau_{\geq 2j} THH(A^{perf, \otimes \bullet}).$$ BMS filtration.

Not by equivariant or cyclotomic spectra.

If it were, Frobenius would act by weight 0.

$$TC(X) \to TC^-(X) \underset{can}{\overset{\phi_p}{\rightrightarrows}} TP(X)$$

$$\mathbb{Z}_p(j) \xrightarrow{\text{"dlog"}} \underset{\text{Nygaard filtration}}{Fil^j R\Gamma_{crys}} \underset{incl}{\overset{\frac{\phi_p}{p^j}}{\rightrightarrows}} R\Gamma_{crys}(X/\mathbb{Z}_p)$$

$j^{\text{th}}$ graded piece for BMS filtration.

Just make some definition.

$$TP(A) \simeq \lim_{\Delta} TP(A^{perf, \otimes_A \bullet}).$$

Get finite generation results for TP from those for crystalline cohomology for some prop. $/X$.

**Thm** (H.). $X \xrightarrow{f} \operatorname{Spec} \mathbb{F}_p$ is sm. prop.

$\iota : \mathbb{Z}_p \hookrightarrow \mathbb{C}$. Then,

$$\zeta(X,s) = \frac{\det_\infty\left(\frac{1}{2\pi}(s \cdot id - \Theta)\right) \mid TP_{odd}(X) \otimes_{\mathbb{Z}_p,\iota} \mathbb{C}}{\det_\infty\left(\frac{1}{2\pi}(s \cdot id - \Theta)\right) \mid TP_{ev}(X) \otimes_{\mathbb{Z}_p,\iota} \mathbb{C}}.$$

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Deninger: "$\Theta = \frac{d}{dt} Fr_t^* \big|_{t=1}$"

Frobenius flow.

$Fr_p^* = p^\Theta$.

$$\zeta(X,s) = \zeta\left(X, s + \frac{2\pi i}{\log p}\right).$$

$$0 \to \frac{2\pi i}{\log p}\mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$$

$\mathbb{Z}$ worth of indeterminacy.

Want periodicity in the cohomology theory.

Different from $Z(X,t)$, rational, for varieties in char. p.

Can prove everything but the Riemann hyp. part of the Weil conjectures.

Really want: Hodge $*$, conjugate operator.

No complex conjugation.

Need something like TP, but algebraic.

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$C = \widehat{\overline{\mathbb{Q}_p}}$. complete alg. closed field.

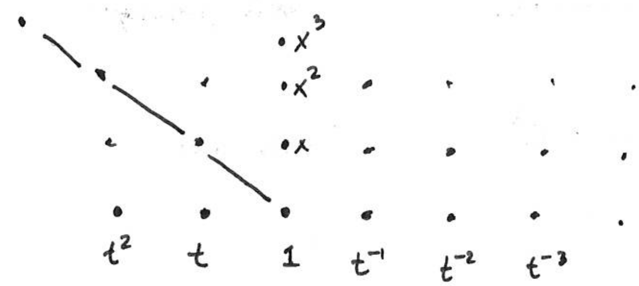
$\mathcal{O}_C$ p-adic unit disk.

$$THH_*(\mathcal{O}_C, \mathbb{Z}_p) = \pi_*(THH(\mathcal{O}_C)_p^\wedge) \simeq \mathcal{O}_C[x], \quad |x| = 2.$$

Notes on Copenhagen webpage.

What about TP?

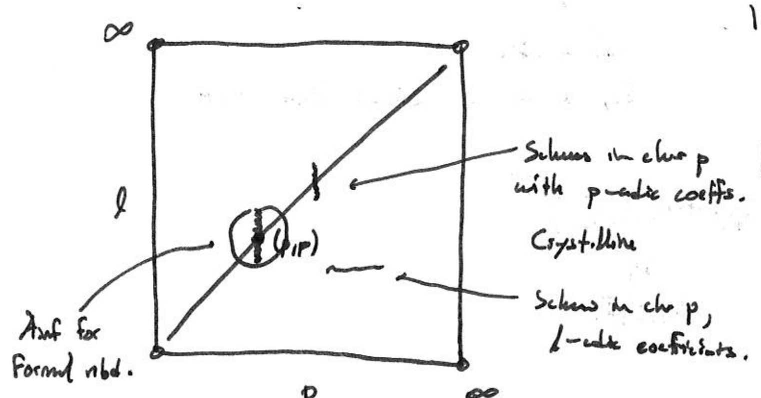
Tate spectral sequence.




$$TP_0(\mathcal{O}_C, \mathbb{Z}_p) \simeq \lim TR^n_0(\mathcal{O}_C, p, \mathbb{Z}_p)$$
$$\simeq \lim_{n,F} W_n(\mathcal{O}_C) \simeq \lim_F W(\mathcal{O}_C)$$
$$\cong \lim_F W(\mathcal{O}_C/p\mathcal{O}_C)$$
$$\cong W(\lim_F \mathcal{O}_C/p\mathcal{O}_C)$$
$$\cong W(\mathcal{O}_{C^\flat})$$
$$\cong A_{inf} \text{ of Fontaine.}$$

$\mathcal{O}_{C^\flat}$ tilt.

This = Frobenius, but not the identity.




diagonal is de Rham cohomology

"$\operatorname{Spec} \mathbb{Z} \times_{\operatorname{Spec} \mathbb{F}_1} \operatorname{Spec} \mathbb{Z}$" ??

Sheaf on perfectoid spaces.

Watch Scholze's plenary talk at Clay in Oxford. Useful explanation.

"$\operatorname{Spec} \mathbb{Z}_p \times_{\operatorname{Spec} \mathbb{F}_1} \operatorname{Spec} \mathbb{Z}_p$" exists as a diamond, but not as a scheme.

"$\operatorname{Spec} \mathbb{Z}_p \times_{\operatorname{Spec} \mathbb{F}_p} \operatorname{Spec} \mathcal{O}_C$" exists as a scheme, Spec Ainf.

$\mathcal{O}_{C^\flat}$ is a valuation ring with residue field $\bar{\mathbb{F}}_p$, fraction field alg. closed field of char. p.

$$\begin{tikzcd} & W(k) \arrow[r] & W(k)[\tfrac{1}{p}] & \text{"kills } 1\otimes p\text{"} \\ A_{\inf} \arrow[ur, bend left, "\text{"exp"}"] \arrow[r, "\text{"de Rham"}"] \arrow[dr, "\text{"étale cohomology with } \mathbb{Z}/p\text{-coefficients."}"'] & \mathcal{O}_C \arrow[r] & C & \text{"kills } p\otimes 1 - 1\otimes p\text{"} \\ & \mathcal{O}_C^b \arrow[r] & C^b & \text{"kills } p\otimes 1\text{"} \end{tikzcd}$$

Edge map in Tate s.s.

The elmt $x$.

$$\begin{array}{ccc} K_*(\mathcal{O}_C, \mathbb{Z}_p) & \xrightarrow{\text{trc}} & THH_*(\mathcal{O}_C, \mathbb{Z}_p) \\ \text{SI}\wr\ \text{Suslin} & & \wr\text{I} \\ \mathbb{Z}_p[\beta] & & \mathcal{O}_C[x] \end{array}$$

$$\beta \longmapsto (\zeta_p - 1)\cdot x$$

not a unit.

So, $x$ is a divided Bott elmt.

$$\begin{array}{ccc} \pi_*\left(K(\mathcal{O}_C)^\wedge_p\right)^{tT} & \longrightarrow & \pi_*\left(THH(\mathcal{O}_C)^\wedge_p\right)^{tT} \\ \wr\text{I} & & \wr\text{I} \\ \mathbb{Z}_p[[q-1]][x^{\pm 1}] & & A_{\inf}[x^{\pm 1}] \end{array}$$

Same Tate s.s.

Want $x$. It is the K-theoretic Thom class of $-\mathcal{O}(1)/\mathbb{P}^\infty(\mathbb{C})$.

$$\begin{array}{ccc} K_*(\mathcal{O}_C, \mathbb{Z}_p) & \longrightarrow & TP_*(\mathcal{O}_C, \mathbb{Z}_p) \\ \wr\text{I} & & \wr\text{I} \\ \mathbb{Z}_p[\beta] & & A_{\inf}[x^{\pm 1}] \\ \beta & \longmapsto & \mu\cdot x \end{array}$$

$\mu = 0$ is the an infinitesimal thickening of de Rham diagonal.

$$\mu = \zeta\cdot\zeta^{1/p}\cdot\zeta^{1/p^2}\cdot\zeta^{1/p^3}\cdots$$

$TP(\mathcal{O}_C, \mathbb{Z}_p)$ is complex oriented.

$$x \underset{TP}{+} y = x + y + \mu \cdot xy.$$

Exercise. Use that $A_{inf}$ is an integral domain.

Thus, TP interpolates between additive (crystalline realization) and multiplicative formal groups (or étale realization).

BMS filtration over $\mathcal{O}_C$.

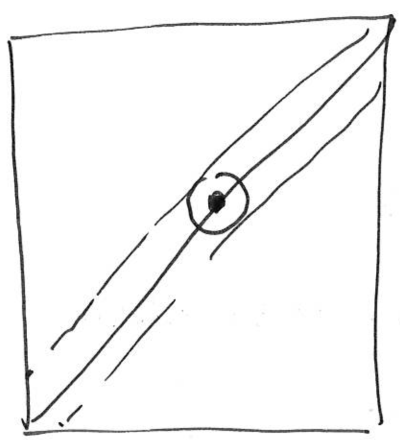
$X/\mathcal{O}_C$ smooth.

$$TC(X) \longrightarrow TC^-(X) \underset{can}{\overset{\phi_p}{\rightrightarrows}} TP(X) \qquad \text{complete at } p.$$

$$\mathbb{Z}_p(j) \longrightarrow \underbrace{\mathrm{Fil}^j A\Omega_X}_{\text{Nygaard filtration}} \underset{\text{inclusion}}{\overset{\phi_p / \xi^j}{\rightrightarrows}} A\Omega_X \qquad j\text{th graded piece for BMS.}$$

Replaces syntomic cohomology for $j \geq p$.

Syntomic coho w/o denominators.



Complexes seem not to depend on p.