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On topological cyclic homology and cyclotomic spectra: cyclotomic spectra and TC.

- $\mathbb{T}$-action on $THH(R)$.
- $\varphi_p : THH(R) \longrightarrow THH(R)^{tC_p}$ $\mathbb{T}$-equivariant
- $\varphi : THH(R)^{h\mathbb{T}} \xrightarrow{\prod \varphi_p^{h\mathbb{T}}} \prod_p \left(THH(R)^{tC_p}\right)^{h\mathbb{T}}$.

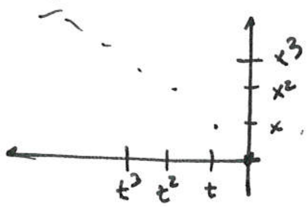
<u>Lemma</u>. The canonical map
$$TP(R) \simeq THH(R)^{t\mathbb{T}} \longrightarrow \prod_p \left(THH(R)^{tC_p}\right)^{h\mathbb{T}}$$
is profinite completion for R connective.

So, we can view φ as
$$THH(R)^{h\mathbb{T}} \simeq TC^-(R) \longrightarrow TP(R)^\wedge .$$
And, we get, when R is connective,
$$TC(R) \longrightarrow TC^-(R) \underset{\varphi}{\overset{can}{\rightrightarrows}} TP(R)^\wedge .$$

<u>Thm</u> (N-Scholze). When R is connective, this is equivalent to Goodwillie's integral TC.

<u>Ex</u>. $R = \mathbb{F}_p$. $THH_*(\mathbb{F}_p) \simeq \mathbb{F}_p[x]$.
$$TC^-_*(\mathbb{F}_p) \simeq \mathbb{Z}_p[x,t]/(xt=p)$$
$$|x|=2, \quad |t|=-2.$$

spectral seq.



No differentials. $H^*(B\mathbb{T}, \pi_* THH(\mathbb{F}_p)) \Rightarrow TC^-_*(\mathbb{F}_p)$

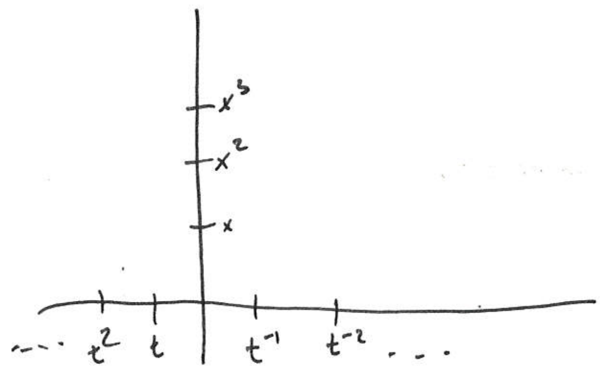
Just have to check $xt=p$.
Can check in $HC^-_*(\mathbb{F}_p)$
since the degree is so low.

Good exercise.

$$TP_*(\mathbb{F}_p) \cong \mathbb{Z}_p[t^{\pm 1}]$$
$$\simeq TC^-(\mathbb{F}_p)[t^{-1}].$$

Use Tate s.s.

Euler class.



Since $TP_*(\mathbb{F}_p)$ is p-complete, we get

$$TC(\mathbb{F}_p) \longrightarrow TC^-(\mathbb{F}_p) \rightrightarrows TP(\mathbb{F}_p).$$

What are the maps?

$$\mathbb{Z}_p[x,t]/(xt=p) \xrightarrow{can} \mathbb{Z}_p[t^{\pm 1}]$$

$$can(x) = pt^{-1}$$
$$can(t) = t$$

$$\phi(x) = t^{-1}$$
$$\phi(t) = pt$$

: Claim. Maybe some units depending on choices.

$(\mathrm{THH}(\mathbb{F}_p)^{tC_p})^{h\mathbb{T}}$
$\simeq$ LEMMA

$$\begin{tikzcd}
\mathrm{THH}(\mathbb{F}_p)^{h\mathbb{T}} \arrow[r, "\phi"] \arrow[d] & \mathrm{THH}(\mathbb{F}_p)^{t\mathbb{T}} \arrow[r] \arrow[d] & \mathbb{F}_p^{t\mathbb{T}} \arrow[d] \\
\mathrm{THH}(\mathbb{F}_p) \arrow[r, "\phi_p"] & \mathrm{THH}(\mathbb{F}_p)^{tC_p} \arrow[r] & \mathbb{F}_p^{tC_p}
\end{tikzcd}$$

connective.

$\Longrightarrow$

by saying that $\phi(t)$ is <u>not</u> a unit.

Since $TC(\mathbb{F}_p)$ is the fiber of

$$TC^-(\mathbb{F}_p) \xrightarrow{can-\phi} TP(\mathbb{F}_p),$$

get a les.

$$\begin{tikzcd} TC^-_{2i}(\mathbb{F}_p) \arrow[r, "can-\phi"] & TP_{2i}(\mathbb{F}_p) \\ \mathbb{Z}_p \arrow[u, "\cong"] & \mathbb{Z}_p \arrow[u, "\cong"] \end{tikzcd}$$

iso in all degrees except zero.

in degree 0, it is 0.

So, $TC_0(\mathbb{F}_p) \cong \mathbb{Z}_p \cong TC_{-1}(\mathbb{F}_p)$. 0 otherwise.

$$TC_*(\mathbb{F}_p) \cong \mathbb{Z}_p[\varepsilon]/(\varepsilon^2).$$

Def. A cyclotomic spectrum is an $X \in Sp^{B\mathbb{T}}$ together with $\mathbb{T}$-equivariant maps

$$\phi_p : X \longrightarrow X^{tC_p}$$

for each prime p.

CyclSp presentable, not compactly gen.

Exs. · $THH(R)$. Uses the Tate diagonals.

· $X \in Sp$. Equip w/ trivial action.

$$X \xrightarrow[\mathbb{T}\text{-equivariant}]{} X^{hC_p} \longrightarrow X^{tC_p}.$$

Call this X^{triv}.

Interesting, subtle X^{hC_p}, X^{tC_p} have non-trivial $\mathbb{T}$-actions.

· $THH(\mathbb{S}) \simeq \mathbb{S}^{triv}$ as cyclotomic spectra.

Prop (Blumberg-Mandell, Nikolaus-Scholze). $TC : CyclSp \longrightarrow Sp$ is corepresentable by $\mathbb{S}^{triv}$; so, in particular,

$$TC(R) \simeq \underline{Map}_{CyclSp}(\mathbb{S}^{triv}, THH(R)).$$

Rem. CyclSp is stable because it is a pullback.

$$\begin{tikzcd}
\mathrm{CyclSp} \arrow[r] \arrow[d] & \prod_p (\mathrm{Sp}^{B\mathbb{T}})^{\Delta^1} \arrow[d, "\text{source, target}"] \\
\mathrm{Sp}^{B\mathbb{T}} \arrow[r, "(id, ()^{tC_p})"'] & \prod_p \mathrm{Sp}^{B\mathbb{T}} \times \mathrm{Sp}^{B\mathbb{T}}
\end{tikzcd}$$

proof of prop. Mapping space in pullback is pullback of mapping spaces.

$$\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d] & X \arrow[d] \\
\mathbb{S}^{tC_p} \arrow[r] \arrow[ur, Rightarrow] & X^{tC_p}
\end{tikzcd} \qquad X^{h\mathbb{T}} \qquad X^{h\mathbb{T}} \rightrightarrows (X^{tC_p})^{h\mathbb{T}}$$

How to recover fixed points.

What happens for other compact abelian Lie group.

Def. $X \in \mathrm{CyclSp}$. (X, ϕ_p).

$$X^{C_{p^n}} := \boxed{X^{hC_{p^n}} \underset{(X^{tC_p})^{hC_{p^n}/C_p}}{\times} X^{hC_{p^n}/C_p} \underset{\phi_p^{hC_{p^n}/C_p}}{\times} \cdots \underset{X^{tC_p}}{\times} X}$$

Just uses action and ϕ_p.

Need Tate orbit lemma and isotropy separation.

Cyclotomic structure leads back to a genuine $\mathcal{F}^{fin}_{S^1}$-spectrum.

Prop. If X is bounded below, then $X^{C_{p^n}}$ are fixed points of a genuine fin-S^1-spectrum.

"Proof".

$$\begin{tikzcd}
\prod \mathrm{Sp}^{\mathcal{F}in_p} \arrow[r, shift left, "\text{forget}"] \arrow[d, shift right, "\phi^{C_p}"'] & \mathrm{Sp}^{B\mathbb{T}} \arrow[l, shift left, "\substack{B = \text{Borel,} \\ \text{right adjoint.}}"] \\
\prod \mathrm{Sp}^{\mathcal{F}in_p} \arrow[u, shift right, "\substack{R \\ \text{right adj.}}"']
\end{tikzcd}$$

$(RX)^{C_{p^n}}$

$\wr\wr$

$X^{C_{p^{n-1}}}$

Lemma. X bounded below with $\mathbb{T}$-spectrum.

$$\Phi^{C_p}(BX) \simeq B(X^{tC_p}).$$

Equivalent to the Tate orbit lemma.

$$\mathbb{X} = BX \times_{RBX^{tC_p}} RBX \times_{R^2BX^{tC_p}} \cdots$$

$$BX \xrightarrow{\text{unit}} RBX^{tC_p}$$

Check that fixed points are the fixed points above.

Thm (N-S). The canonical forgetful functor

$$\left\{\begin{matrix}\text{genuine} \\ \text{cyclotomic} \\ \text{spectra}\end{matrix}\right\} \longrightarrow \mathrm{CycSp}$$

is an $\simeq$ on bounded belows.

The right adjoint is $R(X) = \mathbb{X}$.

"Identify $THH(\mathbb{F}_p)$".

$$\mathrm{Sp} \xrightarrow{\text{triv}} \mathrm{CycSp}$$

Left adjoint functor.

Right adjoint: TC.

$$\mathrm{Sp} \underset{\text{triv}}{\overset{TC}{\rightleftarrows}} \mathrm{CycSp}.$$

Thm is

$$\mathbb{Z}_p \longrightarrow TC(\mathbb{F}_p)$$

from connective cover. This is adjoint to

$$\mathbb{Z}_p^{\text{triv}} \longrightarrow THH(\mathbb{F}_p).$$

Construction. X connective.

$$X \xrightarrow{\varphi_p} \tau_{\geq 0} X^{tC_p} =: F_p(X).$$

$$F_p(X) \xrightarrow{F_p(\varphi_p)} F_p^2(X).$$

New cyclotomic spectra.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & F_p X \arrow[d] \\
\mathrm{sh}_p X \arrow[r] & F_p^2 X
\end{tikzcd}$$

$\mathrm{sh}_p X \simeq F_p(X)$

$X \longrightarrow \mathrm{sh}_p X$ map of cyclotomic spectra.

Thm (Hesselholt, N-S).

$$\begin{tikzcd}
H\mathbb{Z}^{triv} \arrow[r] \arrow[d] & THH(\mathbb{F}_p) \arrow[d, "\wr"'] \\
\mathrm{sh}_p(H\mathbb{Z}^{triv}) \arrow[r, "\sim"] & \mathrm{sh}_p(THH(\mathbb{F}_p))
\end{tikzcd}$$

Cor. $THH(\mathbb{F}_p) \simeq \tau_{\geq 0} H\mathbb{Z}^{tC_p}$ as $\mathbb{E}_\infty$-rings with $\mathbb{T}$-action.

Proof still uses Bökstedt.

Easy to then read off TC.

Fundamental Question. How to understand Bökstedt periodicity.

Q. What about $\widetilde{KEnd}(\mathbb{F}_p)$?
As a cyclotomic spectrum.
I guess it probably doesn't help.