Nikolaus.

On TC and cyclotomic [redacted] spectra: examples.

$$\begin{tikzcd} TC(R) \arrow[r] \arrow[d, phantom, "\simeq" rotate=90] & THH(R)^{h\mathbb{T}} \arrow[r, "can - \varphi_p"] & \left(THH(R)^{t\mathbb{T}}\right)^{\wedge} \\ \mathrm{map}_{CycSp}\left(\mathbb{S}^{triv}, THH(R)\right). & & \end{tikzcd}$$

$$THH(R)^{C_{p^n}} \simeq THH(R)^{hC_{p^n}} \times \dots \times_{THH(R)^{tC_p}} THH(R)$$

R is forget first factor

F is forget last, and shift everything down.

__Ex.__ $R = \mathbb{S}[G]$, $G \simeq \Omega X$, $X \in S_{\geq 1}$.

$$THH(\mathbb{S}[G]) \simeq \Sigma^\infty_+ \mathcal{L}BG$$

$\mathbb{T}$-action corresponds to rotation.

$$\begin{tikzcd} \Sigma^\infty_+ \mathcal{L}BG \arrow[r, "\psi_p"', "\text{p-fold repetition of loops.}"] & \left(\Sigma^\infty \mathcal{L}BG\right)^{hC_p} \arrow[d, "can"] \\ & \left(\Sigma^\infty \mathcal{L}BG\right)^{tC_p} \end{tikzcd}$$

$\mathbb{T}$-equivariant

$$\begin{tikzcd} \Sigma^\infty_+ \mathcal{L}BG \arrow[r] \arrow[d] & \left(\Sigma^\infty \mathcal{L}BG\right)^{hC_p} \arrow[d] \\ \left(\Sigma^\infty_+ \mathcal{L}BG\right)^{hC_q} \arrow[r] & \left(\left(\Sigma^\infty_+ \mathcal{L}BG\right)^{hC_p}\right)^{hC_q} \end{tikzcd}$$

__Def.__ ~~X is~~ A cyclotomic spectrum w/ Frobenius lifts is a cyclotomic spectrum $(X, (\varphi_p))$ together with lifts

$$\begin{tikzcd} & X^{hC_p} \arrow[d] \\ X \arrow[ur, dashed, "\text{equivariant } \psi_p"] \arrow[r, "\varphi_p"'] & X^{tC_p} \end{tikzcd}$$

which are compatible in the sense of commutativity as above.

<u>Exs</u>. (1) $\Sigma^\infty_+ \mathcal{L}BG$.

(2) $X \in Sp$, X^{triv}, via

$$X \longrightarrow X^{hC_p} \xrightarrow{can} X^{tC_p}.$$

Everything now will p-complete.

Assume X has Frobenius lifts, X bounded below.

$$\begin{tikzcd}
TC(X) \arrow[r] \arrow[d] & \Sigma X_{h\mathbb{T}} \arrow[r] \arrow[d, "N"] & 0 \arrow[d] \\
X^{h\mathbb{T}} \arrow[r, "\varphi_p - id"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{h\mathbb{T}} \arrow[r, "can"] \arrow[d] & X^{t\mathbb{T}} \\
X \arrow[r, "\tilde{\varphi}_p - id"] & X &
\end{tikzcd}$$

Norm for compact Lie is shift by adjoint.

$\tilde{\varphi}_p : X \xrightarrow{\varphi_p} X^{hC_p} \longrightarrow X$

<u>Lemma</u>. Lower left square is a pullback. (again, after p-completion).

So,

$$\begin{tikzcd}
TC(X) \arrow[r] \arrow[d] & \Sigma X_{h\mathbb{T}} \arrow[d] \\
X \arrow[r] & X
\end{tikzcd}$$

is a pullback.

Example. X^{triv}.

$$\begin{tikzcd} TC(X^{triv}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \otimes \Sigma \mathbb{S}_{hT} \arrow[d] \\ X \arrow[r, "0", "\tilde{\varphi}_p \simeq id"'] & X \end{tikzcd}$$

$\Rightarrow \quad TC(X^{triv}) \simeq X \oplus X \otimes \Sigma \mathbb{CP}^{\infty}_{-1}$.

tensor

cells in degrees $-1, 1, 3, \ldots$

Example. $\Sigma^\infty_+ \mathcal{L}BG$.

$$\begin{tikzcd} TC(\Sigma^\infty_+ \mathcal{L}BG) \arrow[r] \arrow[d] & \Sigma(\Sigma^\infty_+ \mathcal{L}BG_{hT}) \arrow[d, "N"] \\ \Sigma^\infty_+ \mathcal{L}BG \arrow[r, "id - \varphi_p"] \arrow[d] & \Sigma^\infty_+ \mathcal{L}BG \arrow[d] \\ \Sigma^\infty_+ BG \arrow[r, "0"] & \Sigma^\infty_+ BG \end{tikzcd}$$

(BCCGHM)

Prop. Bottom square is a pullback if BG[redacted] is nilpotent.

Cor. BG nilpotent,

$$TC(\mathbb{S}[G]) \simeq \Sigma^\infty_+ BG \oplus \mathrm{fib}(\Sigma\, \Sigma^\infty_+ \mathcal{L}BG_{hT} \to \Sigma^\infty_+ BG).$$

Prop. For R $\mathbb{E}_1$, there are natural maps

$$\Omega^\infty R \longrightarrow \Omega^\infty THH(R)^{C_{p^n}}$$

which are multiplicative and comp. with the R maps $\Omega^\infty THH(R)^{C_{p^n}} \to \Omega^\infty THH(R)^{C_{p^{n-1}}}$.

Proof.

$$\begin{tikzcd}
R \arrow[rr, bend left] \arrow[dr, dashed] & & \\
 & THH(R)^{C_p} \arrow[r] \arrow[d] & THH(R) \arrow[d] \\
 & THH(R)^{hC_p} \arrow[r] & THH(R)^{tC_p}
\end{tikzcd}$$

Exists after Ω^∞. This follows from the

Fact. There is a unique lax sym. monoidal transformation

$$\begin{tikzcd}
\Omega^\infty X \arrow[r] \arrow[dr, dashed, "\Omega^\infty \Delta_p"'] & \Omega^\infty\left((X^{\otimes p})^{hC_p}\right) \arrow[d] \\
 & \Omega^\infty\left((X^{\otimes p})^{tC_p}\right).
\end{tikzcd}$$

Can use multiplicativity of the Tate diagonal.

Thm. The identity functor $Sp \to Sp$ is initial among lax sym monoidal $\boxed{\text{exact}}$ endofunctors.

This gives Δ_p since $(X^{\otimes p})^{tC_p}$ is sym monoidal.

Proof. Use Day convolution.

Same with $\Omega^\infty (X^{\otimes p})^{t}$

Thm. $\Omega^\infty : Sp \to S_*$ is initial among lax symmetric monoidal functors.

Q. Why doesn't this work before delooping?

Because $(X^{\otimes p})^{hC_p}$ lax monoidal, but not exact.

Thanks Lars!

Putting these together, we get the Fact.

Rem. Wrong for chain complexes, unless you say $H\mathbb{Z}$-linear.

__Segal conjecture.__ R bounded below.

__Q.__ When is $\mathrm{THH}(R) \longrightarrow \mathrm{THH}(R)^{tC_p}$ p-completion?

$\uparrow$

Always p-completion.

Or, in degs $* \geq n$.

__Consequence.__ If so, then $TC^-_*(R) \underset{\varphi}{\cong} TP_*(R)$ for $* \gg n$.

Known cases.

(i) $\mathrm{THH}(\mathbb{F}_p)$. True for $* \geq 0$.

(ii) $\mathrm{THH}(\mathbb{Z})$. True for $* \geq 0$.

(iii) $X/\mathbb{F}_p$ smooth. $\mathrm{THH}(X)$. True for $* \geq \dim X$.

(iv) $\mathrm{THH}(\mathbb{S}) \longrightarrow \mathrm{THH}(\mathbb{S})^{tC_p}$. True for all $*$.

$$\begin{array}{ccc} \mathrm{THH}(\mathbb{S}) & \longrightarrow & \mathrm{THH}(\mathbb{S})^{tC_p} \\ \wr\| & & \wr\| \\ \mathbb{S} & \longrightarrow & \mathbb{S}^{tC_p} \end{array}$$

Segal conjecture for C_p.

(v) $\mathrm{THH}(MU)$. True for all $*$.
Same for BP. (Rognes–Nielsen).

(vi) $\Sigma^\infty_+ \Omega BG$. True for all $*$ if X 1-connected.
Carlsson finite type.
N–S. X 1-connected.

Def. R a ring spectrum.

$$K_{\mathrm{Aut}}(R) := K(\mathrm{Fun}(S^1, \mathrm{Perf}_R)).$$

This has an S^1-action.

There are lifts of Frobenius.

Composition of morphisms

$$\begin{tikzcd} K_{\mathrm{Aut}}(R) \arrow[r, "\varphi_p"] \arrow[dr, "\varphi_p"'] & K_{\mathrm{Aut}}(R)^{hC_p} \arrow[d] \\ & K_{\mathrm{Aut}}(R)^{tC_p}. \end{tikzcd}$$

Cyclotomic w/ Frobenius lifts.

$$\boxed{\begin{array}{l} \textbf{Thm. } \text{There is a map of cyclotomic spectra} \\ \qquad K_{\mathrm{Aut}}(R) \to THH(R). \\ \text{Call this the } \underline{\text{trace}}. \\ \text{Apply } TC. \\ \qquad TC(K_{\mathrm{Aut}}(R)) \to TC(R). \\ \qquad\qquad \uparrow \\ \text{Get } K(R). \text{ Takes } P \mapsto (P, \mathrm{id}). \end{array}}$$

$$K_{\mathrm{End}}(R) = K(\mathrm{Fun}(B\mathbb{N}, \mathrm{Perf}_R)).$$

$$\begin{tikzcd} K(R) \arrow[r, "0"] & K_{\mathrm{End}}(R) \arrow[r] & K_{\mathrm{cyc}}(R). \\ & K_{\mathrm{Aut}}(R) \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$\pi_0 K_{\mathrm{cyc}}(R) \cong W_0(R) \subseteq W(R) \cong 1 + R[[t]].$$