Yeakel.

The Dundas-McCarthy Theorem III: analyticity.

Thm (DGM). If $f : R \to S$ map of connective $\mathbb{E}_1$-rings with nilpotent kernel, then

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & TC(R) \arrow[d] \\ K(S) \arrow[r] & TC(S) \end{tikzcd}$$

is cartesian.

Thm (Goodwillie, analytic continuation). If $F, G : S_* \to Sp$ are p-analytic with a n.t. $F \to G$ inducing $D_1F \simeq D_1G$, then for $(p+1)$-connective $Y \to X$,

Aaron's talk.

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & G(Y) \arrow[d] \\ F(X) \arrow[r] & G(X) \end{tikzcd}$$

is cartesian.

sketch of proof. Reduce to $G \simeq *$, note that D_1 commutes with homotopy fibers. Need to show: [redacted] if

(i) F is p-analytic,

(ii) $D_1F \simeq *$,

then $F(Y) \simeq F(X)$ for $Y \to X$ $(p+1)$-connected.

Recall: $E_1(p-q, p+1)$:

$$\begin{tikzcd} \cdot \arrow[r, "k_0\text{-conn}"] \arrow[d, "k_1\text{-conn}"'] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \quad \overset{F}{\curvearrowright} \quad \begin{tikzcd} \cdot \arrow[dr, "(k_0+k_1-(p-q))\text{-conn}"] & & \\ & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ & \cdot \arrow[r] & \cdot \end{tikzcd}$$

$k_0, k_1 \geq p+1$

$Y \to X$ k-connected, $k > p$:

$$\cancel{F(Y) \to P_1F(Y) \quad P_0F(X)}$$

$$\begin{tikzcd} & D_1F(Y) \simeq * \arrow[d] \\ F(Y) \arrow[r] & P_1F(Y) \arrow[d, "\wr"] \\ & P_0F(Y) \end{tikzcd}$$

WTS: $F(Y) \simeq P_1F(Y)$.

Recall: $P_1 F(Y) \simeq \mathrm{colim}\left(F(Y) \to T_1 F(Y) \to T_1^2 F(Y) \to \cdots \right)$

By def.

$$\begin{tikzcd} F(Y) \arrow[r] & T_1 F(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F(C_X Y) \arrow[d] \\ & F(C_X Y) \arrow[r] & F(\Sigma_X Y) \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, "k\text{-conn}"] \arrow[d, "k\text{-conn}"'] & C_X Y \simeq X \arrow[d] \\ X \simeq C_X Y \arrow[r] & \Sigma_X Y \end{tikzcd}$$

$\Rightarrow$ $F(Y) \to T_1 F(Y)$ is $(2k - (p-q))$-conn.

Similar for $T_1 F(Y) \to T_2 F(Y) \to \cdots$ etc.

Exercise.

So, $T_1 F(Y) \to T_1^2 F(Y)$ is $2k-(p-q)$-connective.

WTS this works for all q, or that F satisfies $E_1(p-q, p+1)$ $\forall q$.

Goodwillie: F satisfies $E_n(pn-q, p+1)$ $\forall n, \forall q$.

Get that $F(Y) \simeq F(X)$.

<u>TC is analytic.</u> Specifically, $TC(A \otimes [-])^\wedge_p : S_+ \to Sp$ is (-1)-analytic.

Extension to integral TC is in DGM.

Step 1: reduce to THH.

Step 2: show it for THH.

Do this for Nikolaus-Scholze.

Prop [McCarthy]. If $\mathrm{THH}(A\oplus M[\cdot])$ satisfies $E_n(-n) = E_n(-n,-1)$ $\forall n\geq 1$, then $\mathrm{TC}(A\oplus M[\cdot])^\wedge_p$ is (-1)-analytic.

proof. $E_n(-n)$ boils down to

0 $(\sum k_i + n)$-conn

$h\ell \longrightarrow$

1 $\searrow$ $(n+1)$-cube

Under the hypothesis,

$$\mathrm{THH}(A\oplus M[\cdot])_{hC}$$

is $E_n(-n)$ for all finite $C\subset S^1$. Use fundamental diagram

$$\mathrm{THH}(A\oplus M[\cdot])_{hC_{p^k}} \longrightarrow \mathrm{THH}(A\oplus M[\cdot])^{C_{p^k}} \xrightarrow{R} \mathrm{THH}(A\oplus M[\cdot])^{C_{p^{k-1}}}.$$

By induction, get that fixed points are $E_n(-n)$ for all C_{p^k}.

This implies that $\mathrm{TR}(A\oplus M[\cdot]) \simeq \lim_R \mathrm{THH}(A\oplus M[\cdot])^{C_{p^k}}$ is $E_n(-n)$ $\forall n\geq 1$. Then,

$$\mathrm{TC}(A\oplus M[\cdot];p) \simeq \mathrm{eq}\left(\mathrm{TR} \overset{1}{\underset{F}{\rightrightarrows}} \mathrm{TR}\right).$$

So, $\mathrm{TC}(A\oplus M[\cdot])$ is $E_n(-n+1)$. What matters is the $\boxed{\text{slope}}$ of $-n, -n+1$, which is still -1.

Finally, p-completion doesn't change anything. And by Hesselholt–Madsen, $\mathrm{TC}(R)^\wedge_p \simeq \mathrm{TC}(R;p)^\wedge_p$. Done.

Step 2.

Exercise. $F_\bullet$ a simplicial functor for $F_\bullet : S_+ \longrightarrow Sp^{cn}$ such that (i) F_v satisfies $E_n(c,-1)$, for all $v\geq 0$,
(ii) $F_v \simeq *$ for $0\leq v\leq q-1$ for some q,

then $|F_\bullet|$ satisfies $E_n(c-q,-1)$.

Recall from Hesselholt, explained in Aaron's talk,

$$THH(A \oplus M[\cdot]) \simeq \left| \bigvee_a T_{a,\bullet}(A^{\cdot}, M[\cdot]) \right|$$

$$T_{a,v}^{(M,X)} \simeq \bigvee_{0 \le j_1 \le \cdots \le j_a \le v} A^{\otimes(v+1-a)} \otimes M^{\otimes a} \otimes (\Sigma^\infty_+ X)^{\otimes a}.$$

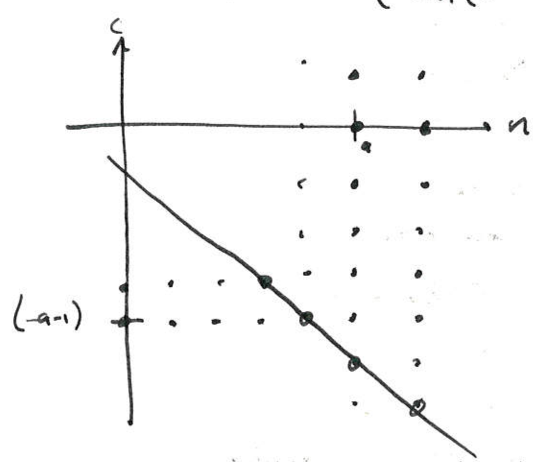
Note: $T_{a,v} \simeq *$ for $0 \le v \le a-2$.

[Goodwillie]. $(\Sigma^\infty_+ X)^{\otimes a}$ is a-excisive. Get $E_n(c,-1)$ $\forall c$ when $n \ge a$. In fact, it is $E_n(0,-1)$ for $n < a$. Since smashing and $\vee$ are linear, $T_{a,v}(M,X)$ is

$E_n(c,-1)$ $\forall c$, $n \ge a$,

$E_n(0,-1)$ $n < a$.

Hence, $|T_{a,\bullet}|$ is $\begin{cases} E_n(c-(a-1),-1) & n \ge a \\ E_n(0-(a-1),-1) & n < a. \end{cases}$



$h = -1$

Best line. Maybe

Thus, $THH(A \oplus M[J])$ is $E_n(-n)$.

$K(A \oplus M[\cdot])$ is analytic.

Need 2 things.

a. Dual higher Blakers–Massey theorem [Ellis–Steiner].
If Y is strongly cartesian $(n+1)$-cube of spaces with the final maps k_i-connected, then Y is $(\Sigma k_i + n)$-cocartesian.

b. An $(n+k)$-cocartesian $(n+1)$-cube of spectra is k-cartesian.

[Dundas–McCarthy]. $K(A \oplus M[\cdot]) \simeq K(A; M[\Sigma \cdot])$.

Exc. If F is p-analytic, $F \circ \Sigma$ is $(p+1)$-analytic.

How to show: $K(A; M[\cdot])$ is 0-analytic.
Just show that $\tilde{K}(A; M[\cdot])$ is 0-analytic. fiber of $K(A,M[\cdot]) \to K(A)$.

Recall:

$$\tilde{K}(A,M[\cdot]) \simeq \lim_{l \to \infty} \Omega^l \Big| \underbrace{\bigvee_{P \in S^{(l)}_\cdot P} \mathrm{Hom}_{S^{(l)}_\cdot \mathrm{Mod}}(\bar{P}, \bar{P} \wedge M[\cdot])}_{G} \Big| .$$

Happens to be connective mapping spectra.

Let X be str. cocart $(n+1)$-cube with k_i-conn initial maps, then $G[X]$ is $(n + \Sigma k_i)$-cocart.

Use that G is linear.
Preserves strongly cocartesian.

Actually, $G[X]$ is $(n + \Sigma k_i + l)$-conn.
But, we Ω^l, so get
$(n + \Sigma k_i)$-conn.

Thus, $\tilde{K}(A, M[X])$ is $(n + \Sigma k_i)$-cocartesian.
So, by (b) above, $\tilde{K}(A, M[\cdot])$ is Σk_i-cocartesian.
Thus, $\tilde{K}(A, M[\cdot])$ is $E_n(0,-1)$ and so 0-analytic.