Extremal cuts:
From sparse random graphs to spin-glasses

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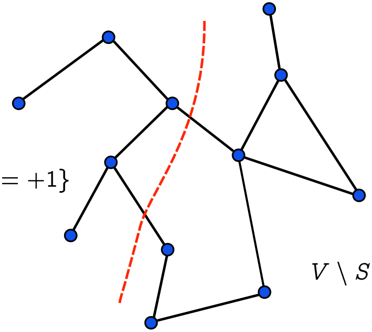
July 20, 2016

Cuts in graph $G = (V, E)$



$S = \{i : \sigma_i = +1\}$

$V \setminus S = \{i : \sigma_i = -1\}$

$$\mathrm{cut}_G(S) \equiv \Big| \{(i,j) \in E : \; i \in S, \, j \in V \setminus S\} \Big|$$

$$2\mathrm{cut}_G(S) - |E| = - \sum_{(i,j)\in E} \sigma_i \sigma_j$$

AF-Ising on G

Extremal cuts

Minimum bisection

$$\mathsf{mcut}(G) = \min\left\{\mathrm{cut}_G(S)\ :\ S \subseteq V\,,\ |S| = |V|/2\right\}$$

Maximum bisection

$$\mathsf{MCUT}(G) = \max\left\{\mathrm{cut}_G(S)\ :\ S \subseteq V\,,\ |S| = |V|/2\right\}$$

Maximum Cut

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Complexity: hard to solve (worst case)

$$\left.\begin{array}{r}\mathsf{mcut}\\ \mathsf{MCUT}\\ \mathsf{MaxCut}\end{array}\right\} \qquad \text{are NP-hard}$$

(hard to approximate to within $1 + o(1)$ factor:
SDP [Goemans,Williamson '95] gives 0.878
hardness [Trevisan,Sorkin,Sudan,Williamson '00] best be 0.941)

Typical complexity (average graph)?

Random graph models

Erdős-Renyi Random graph $G = (V, E) \sim \mathcal{G}(n, p)$

- $|V| = n$ vertices
- Edges independently present with probability p

  (or uniform among graphs of $m = [\binom{n}{2} p]$ edges)
- Average degree $\gamma = np$

Random regular graph $G = (V, E) \sim \mathcal{G}^{\text{reg}}(n, \gamma)$

- $|V| = n$ vertices
- Uniformly random among graphs with $\deg(i) = \gamma \quad \forall i \in V$.

$p = \gamma/n$, $\gamma = O(1)$, sparse graphs

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Extremal cuts of random graphs: long history

- Bollobas '88 ($\mathcal{G}^{\text{reg}}$, mcut, concentration bound)
- Alon '97 ($\mathcal{G}^{\text{reg}}$, mcut, algorithmic bound)
- Coppersmith, Gamarnik, Hajiaghayi, Sorkin '04 ($\mathcal{G}$, MaxCut transition $\gamma = 0.5$)
- Díaz,Serna,Wormald, '07 ($\mathcal{G}^{\text{reg}}$, MCUT, algorithmic bound)
- Daudé,Martínez,Rasendrahasina,Ravelomanana '12 (MaxCut, scaling-window)
- Gamarnik,Li '14 ($\mathcal{G}$, MaxCut, improved concentration)
- ...

Typical result (sparse case):

If $G \sim \mathcal{G}(n, \gamma/n)$ then, with high probability

$$\frac{n\gamma}{4} + C_1 n\sqrt{\gamma} \leq \text{MCUT}(G) \leq \frac{n\gamma}{4} + C_2 n\sqrt{\gamma}$$

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A classical argument (Bollobas '88)

$|V| = n$, $|E| = n\gamma/2$. Fix $S \subseteq V$, $|S| = n/2$

- Each edge is cut with probability 1/2

$$\mathbb{E}[\text{cut}(S)] = \frac{|E|}{2} = \frac{n\gamma}{4} .$$

(also factor 0.5 approximation of MaxCut)

- Azuma-Höeffding argument

$$\mathbb{P}\Big\{|\text{cut}(S) - \mathbb{E}\text{cut}(S)| \geq \Delta\Big\} \leq 2\exp\Big(-\frac{\Delta^2}{4n\gamma}\Big)$$

(simpler: Binomial($\frac{n^2}{4}, \frac{\gamma}{n}$) in $\mathcal{G}(n, \gamma/n)$)

- Union bound

$$\mathbb{P}\Big\{\max_{S,|S|=n/2}\Big|\text{cut}(S) - \frac{n\gamma}{4}\Big| \geq \delta n\sqrt{\gamma}\Big\} \leq 2^{n+1}\, e^{-n\delta^2/4} .$$

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So, w.h.p.

$$\frac{n\gamma}{4} - C_1 n\sqrt{\gamma} \leq \mathsf{mcut}(G) \leq \mathsf{MCUT}(G) \leq \frac{n\gamma}{4} + C_2 n\sqrt{\gamma}$$

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Extremal cuts and Ising spins

Ising Hamiltonian: $$\mathcal{H}_G(\sigma) = \frac{1}{\sqrt{\gamma}} \sum_{(i,j)\in E} \sigma_i \sigma_j \ : \{-1,+1\}^n \mapsto \mathbb{R}\,.$$

$$\mathsf{MaxCut}(G) = \max_{\sigma} \sum_{(i,j)\in E} \left(\frac{1-\sigma_i\sigma_j}{2}\right)$$
$$= \frac{1}{2}|E| - \frac{1}{2}\min_{\sigma} \sum_{(i,j)\in E} \sigma_i\sigma_j = \frac{n\gamma}{4} - \sqrt{\frac{\gamma}{4}}\min_{\sigma}\{\mathcal{H}_G(\sigma)\}$$

$(S = \{i : \sigma_i = +1\}, \quad |E| = \frac{n\gamma}{2})$.

$$|S| = \frac{n}{2} \quad \Longleftrightarrow \quad \Omega_n = \left\{\sigma \in \{-1,+1\}^n : \sum_{i=1}^{n} \sigma_i = 0\right\}$$

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Statistical physics: Ising measures

Ising measures (on balanced configurations):

$$\mathcal{H}_G(\sigma) = \frac{1}{\sqrt{\gamma}} \sum_{(i,j)\in E} \sigma_i \sigma_j \qquad (\sigma \in \Omega_n)\,,$$

$$\mu_{n,\beta}(\sigma) \equiv \frac{1}{Z_n(\beta)}\, e^{\beta \mathcal{H}_G(\sigma)}\,, \qquad Z_n(\beta) \equiv \sum_{\tilde\sigma \in \Omega_n} e^{\beta \mathcal{H}_G(\tilde\sigma)}\,.$$

(ferromagnetic iff $\beta \geq 0$).

Ground state energy:

$$\lim_{\beta\to+\infty} \frac{1}{\beta} \log Z_n(\beta) = \max_{\sigma\in\Omega_n} \{\mathcal{H}_G(\sigma)\} \qquad \Longrightarrow \qquad \mathsf{mcut}(G)\,,$$

$$\lim_{\beta\to-\infty} \frac{1}{\beta} \log Z_n(\beta) = \min_{\sigma\in\Omega_n} \{\mathcal{H}_G(\sigma)\} \qquad \Longrightarrow \qquad \mathsf{MCUT}(G)\,.$$

Insights from statistical physics

Conjectures

- Fu,Anderson, '86
- Mézard,Parisi, '01
- [Zdéborova,Boettcher '10] For $G_n \sim \mathcal{G}^{\text{reg}}(n,\gamma)$, w.h.p.

$$\mathsf{MCUT}(G_n) = \mathsf{MaxCut}(G_n) + o(n) = |E_n| - \mathsf{mcut}(G_n) + o(n)\,.$$

Theorem (Bayati,Gamarnik,Tetali, '09)

For $G_n \sim \mathcal{G}(n,\gamma/n)$, or $G_n \in \mathcal{G}^{\text{reg}}(n,\gamma)$ w.h.p.

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Connection to spin glasses

Theorem (D.-Montanari-Sen '15)

For $G_n \sim \mathcal{G}^{\rm reg}(n,\gamma)$, $G_n \sim \mathcal{G}(n,\gamma/n)$ *w.h.p.*

$$\frac{1}{n}\mathsf{mcut}(G_n) = \frac{\gamma}{4} - \mathsf{P}_*\sqrt{\frac{\gamma}{4}} - o_\gamma(\sqrt{\gamma}) + o_n(1)\,,$$

$$\frac{1}{n}\mathsf{MCUT}(G_n) = \frac{\gamma}{4} + \mathsf{P}_*\sqrt{\frac{\gamma}{4}} + o_\gamma(\sqrt{\gamma}) + o_n(1)\,,$$

$$\frac{1}{n}\mathsf{MaxCut}(G_n) = \frac{\gamma}{4} + \mathsf{P}_*\sqrt{\frac{\gamma}{4}} + o_\gamma(\sqrt{\gamma}) + o_n(1)\,.$$

What is P_*?

GOE random matrix:

$$J \in \mathbb{R}^{n\times n}\,, \quad J = J^{\mathsf{T}}\,, \quad (J_{ij})_{i<j} \sim \mathrm{N}(0, n^{-1}), \quad J_{ii} \sim \mathrm{N}(0, 2n^{-1})\,.$$

Sherrington-Kirkpatrick spin-glass model

$$\mathcal{H}_{\rm SK}(\sigma) \equiv \frac{1}{2}\,\sigma^{\mathsf{T}} J\sigma\,.$$

Finally

$$\mathsf{P}_* \equiv \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[\max_{\sigma\in\{-1,+1\}^n}\{\mathcal{H}_{\rm SK}(\sigma)\}\Big]\,.$$

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- The limit exists (consequence of [Guerra,Toninelli '02])
- Given by *'Parisi's formula'* ([Talagrand '06])
- Clarifies why 'standard' combinatorial methods unsuccessful.
- Set of near-extremal cuts predicted to have ∞-RSB structure.

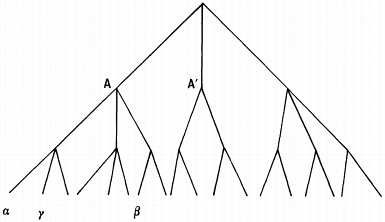

Fig. 1. — The tree of the states. The different states α, β, γ... are the extremities of the branches of the tree. The distance between two states is a monotonic function of the number of steps one has to climb along the tree to find a common ancestor.

Proof strategy

1. Proof for mcut, MCUT with $G \sim \mathcal{G}(n, \gamma/n)$:

   Concentration, interpolation,'smooth max'

2. Extend to $G' \sim \mathcal{G}^{\text{reg}}(n, \gamma)$: Couple G' and $G \sim \mathcal{G}(n, \gamma'/n)$
   (Tricky: need $\gamma - \gamma' \gg \sqrt{\gamma}$ so $|E_G \Delta E_{G'}| \gg n\sqrt{\gamma}$)

3. Prove that $\text{MaxCut}(G) - \text{MCUT}(G) = o(n\sqrt{\gamma})$
   (Tricky part: $\frac{n}{2} - |S| \leq n\gamma^{-\delta}$, $\delta < \frac{1}{4}$)

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mcut for $\mathcal{G}(n, \gamma/n)$:

Recall (our claim)

$$\mathsf{mcut}(G) = \frac{n\gamma}{4} - \sqrt{\frac{\gamma}{4}} \max_{\sigma \in \Omega_n} \{\mathcal{H}_G(\sigma)\} \,,$$

By concentration, suffices to show

$$\frac{1}{n}\mathbb{E}\Big[\max_{\sigma \in \Omega_n} \{\mathcal{H}_G(\sigma)\}\Big] = \mathsf{P}_* + o_\gamma(1) + o_n(1)\,.$$

We know

$$\frac{1}{n}\mathbb{E}\Big[\max_{\sigma \in \{-1,+1\}^n} \{\mathcal{H}_{\mathrm{SK}}(\sigma)\}\Big] = \mathsf{P}_* + o_n(1)\,.$$

Idea: Interpolation ('smart path'; Lindeberg method)

Interpolation: 'smart path'

Step 1: For $t \in [0,1]$, $G(t) = (V, E(t)) \sim \mathcal{G}(n, \gamma(1-t)/n)$

$$\mathcal{H}_t(\sigma;\gamma) = \frac{1}{\sqrt{\gamma}} \sum_{(i,j)\in E(t)} \sigma_i\sigma_j + \frac{\sqrt{t}}{2} \sum_{i,j=1}^{n} J_{ij}\sigma_i\sigma_j$$

$\mathcal{H}_0 = \mathcal{H}_G \qquad \mathcal{H}_1 = \mathcal{H}_{\mathrm{SK}} \qquad\qquad (\sigma \in \Omega_n).$

$G(t)$ multi-graph of i.i.d. Poisson$(\gamma(1-t)/n)$ multi-edges.

Step 2: From $\max_{\sigma\in\Omega_n}\{\mathcal{H}_{\mathrm{SK}}(\sigma)\}$ to $\max_{\sigma\in\{-1,+1\}^n}\{\mathcal{H}_{\mathrm{SK}}(\sigma)\}$.

Step 1: 'smooth max'

Free energy density (balanced configurations):

$$\phi_n(\beta; t, \gamma) \equiv \frac{1}{n}\mathbb{E}\Big[\log\big\{\sum_{\sigma\in\Omega_n} e^{\beta\mathcal{H}_t(\sigma;\gamma)}\big\}\Big]$$

$$\Big|\frac{1}{\beta}\phi_n(\beta; t, \gamma) - \frac{1}{n}\mathbb{E}\big[\max_{\sigma\in\Omega_n}\{\mathcal{H}_t(\sigma;\gamma)\}\big]\Big| \le \frac{\log 2}{\beta}\,.$$

$\Longrightarrow$ just find $\beta(\gamma) \to \infty$ with

$$\sup_{n,t}\frac{1}{\beta}\Big|\frac{\partial\phi_n}{\partial t}(\beta; t, \gamma)\Big| \to 0\,.$$

Step 1: Poisson & Gaussian calculus

Lemma

$$\left|\frac{\partial \phi_n}{\partial t}(\beta; t, \gamma)\right| \le C\left[\frac{\beta^3}{\gamma^{1/2}} + \frac{\beta^4}{\gamma}\right].$$

Proof:
$\langle\cdot\rangle_{n,t}$ denote expectations for i.i.d. $\sigma^{(j)} \in \Omega_n$ under $\mu_{n,t,\beta,J}(\sigma) \equiv Z_{n,t,J}^{-1}\, e^{\beta\mathcal{H}_t(\sigma;\gamma)}$.

$$\text{Multi-replica overlaps} \qquad Q_\ell \equiv \frac{1}{n}\sum_{i=1}^{n}\Big(\prod_{j=1}^{\ell}\sigma_i^{(j)}\Big),$$

so $Q_1 = 0$ and $|Q_\ell| \le 1$. Let $\kappa \equiv \beta/\sqrt{\gamma}$.

$$\frac{\partial \phi_n(\beta; t, \gamma)}{\partial t} = I + II,$$

$$II = \frac{\beta^2}{4}(1 - \mathbb{E}[\langle Q_2^2\rangle_{n,t}]) \qquad \text{by **Gaussian (SK)** calculus}$$

$$I = \frac{\gamma}{2}\Big(-\log\cosh\kappa + \sum_{\ell\ge1}\frac{(-\tanh\kappa)^\ell}{\ell}\mathbb{E}[\langle Q_\ell^2\rangle_{n,t}]\Big) \qquad \text{by **Poisson (ER)** calculus}$$

Step 1: Poisson & Gaussian calculus

Lemma

$$\left|\frac{\partial \phi_n}{\partial t}(\beta; t, \gamma)\right| \le C\left[\frac{\beta^3}{\gamma^{1/2}} + \frac{\beta^4}{\gamma}\right].$$

Proof:

$\langle\cdot\rangle_{n,t}$ denote expectations for i.i.d. $\sigma^{(j)} \in \Omega_n$ under $\mu_{n,t,\beta,J}(\sigma) \equiv Z_{n,t,J}^{-1}\, e^{\beta\mathcal{H}_t(\sigma;\gamma)}$.

Multi-replica overlaps $$Q_\ell \equiv \frac{1}{n}\sum_{i=1}^{n}\Big(\prod_{j=1}^{\ell}\sigma_i^{(j)}\Big),$$

so $Q_1 = 0$ and $|Q_\ell| \le 1$. Let $\kappa \equiv \beta/\sqrt{\gamma}$.

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Universality [Subhabrata Sen, '16]

Not Ising related, for all models when $\gamma \gg 1$ (& ER).
Key: replace 'smart path' with Lindeberg method.

Applications:

- Max q-cut in sparse random graphs

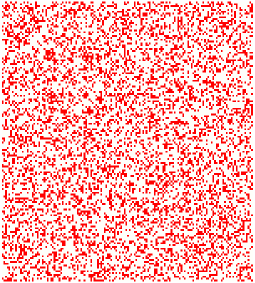
  $\Longrightarrow$ Potts spin-glass.

- Max-SAT for $m = \gamma n/2$ random linear Eqn. over $\{0, 1\}$
  (in UNSAT phase, k variables per Eqn.)

  $\Longrightarrow$ k-spin models.

Finding mcut(G) for specific $G \sim \mathcal{G}(n, \gamma/n)$

$$\text{mcut}(G) \quad \Longleftrightarrow \quad \max \quad \langle \sigma, A_G \sigma \rangle \,,$$
$$\text{for} \quad \sum_{i=1}^{n} \sigma_i = 0 \,, \quad \sigma_i \in \{-1, 1\} \,.$$



$$(A_G)_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Regularization (Lagrangian)

$$\max \quad \langle \sigma, A_G \sigma \rangle - \lambda \Big(\sum_{i=1}^{n} \sigma_i \Big)^2 ,$$
$$\text{for} \quad \sigma_i \in \{-1, 1\} .$$

$G \sim \mathcal{G}(n, \gamma/n) \Rightarrow$ 'sufficient' regularization $\lambda = \gamma/n$:

$$\mathsf{OPT}(G) = \max \quad \langle \sigma, (A_G - \mathbb{E}\{A_G\})\sigma \rangle ,$$
$$\text{for} \quad \sigma_i \in \{-1, 1\} .$$

Recall for such G w.h.p.

$$\mathsf{OPT}(G) = 2n\mathsf{P}_* \sqrt{\gamma} + n\, o_\gamma(\sqrt{\gamma}) \approx 1.5264\, n \sqrt{\gamma}$$

Spectral relaxation: does poorly!

$$\mathrm{OPT}(G) = \max \ \langle \sigma, (A_G - \mathbb{E}\{A_G\})\sigma \rangle \,, \quad \text{for} \quad \sigma_i \in \{+1, -1\} \quad .$$

$$\lambda_{\max}(A_G - \mathbb{E}A_G)\, n = \max \ \langle \sigma, (A_G - \mathbb{E}\{A_G\})\sigma \rangle \,, \quad \text{for} \quad \sout{\sigma_i \in \{+1, -1\}} \ \ \|\sigma\|_2^2 = n$$

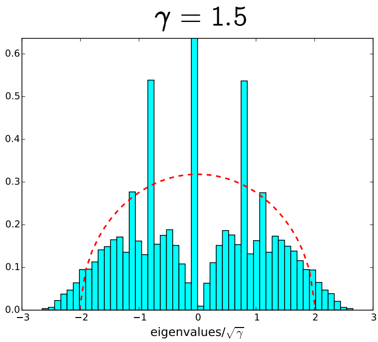
Theorem (Krivelevich, Sudakov 2003+Vu 2005)

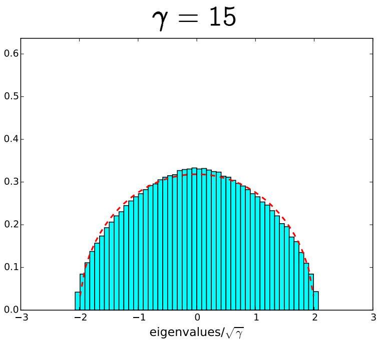
If $G \sim \mathcal{G}(n, \gamma/n)$, then, with high probability,

$$\lambda_{\max}(A_G - \mathbb{E}A_G) = \begin{cases} 2\sqrt{\gamma}\,(1 + o(1)) & \text{if } \gamma \gg (\log n)^4, \\ \sqrt{\log n/(\log\log n)}(1 + o(1)) & \text{if } \gamma = O(1). \end{cases}$$

Compare with $\mathrm{OPT}(G)/n \approx 1.5264\,\sqrt{\gamma}$

Illustration: $n = 10,000$





Semidefinite Programming - does well!

$$\begin{aligned} \mathsf{OPT}(G) = \max \quad & \langle (A_G - \mathbb{E}\{A_G\}), \sigma\sigma^{\mathsf{T}} \rangle \,, \\ \text{for} \quad & \sigma_i \in \{+1, -1\} \,. \end{aligned}$$

$$\begin{aligned} \mathsf{SDP}(G) = \max \quad & \langle (A_G - \mathbb{E}\{A_G\}), X \rangle \,, \\ \text{for} \quad & X_{ii} = 1 \,, \quad X \succeq 0 \,. \end{aligned}$$

Theorem (Montanari-Sen '15)

If $G \sim \mathcal{G}(n; \gamma/n)$, $\gamma = O(1)$, *w.h.p.*

$$\frac{1}{n} \mathsf{SDP}(G) = 2\sqrt{\gamma} + o(\sqrt{\gamma})$$

- $\mathsf{SDP}(G)$ behaves much better than the principal eigenvalue

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Proof outline

'Interpolate' in rank:

$$\mathsf{OPT}_k(G) := \max \quad \langle (A_G - \mathbb{E}\{A_G\}), X \rangle \,, \\ \text{for} \quad X_{ii} = 1\,,\; X \succeq 0\,,\; \text{rank}(X) \le k$$

$\mathsf{OPT} = \mathsf{OPT}_1 \nearrow \mathsf{OPT}_n = \mathsf{SDP}$.

Equivalently:

$$\mathsf{OPT}_k(G) = \max \quad \langle \big(A_G - \mathbb{E}\{A_G\}\big), \boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathsf{T}} \rangle \,, \\ \text{for} \quad \boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \dots, \boldsymbol{\sigma}_n)^{\mathsf{T}} \in \mathbb{R}^{n\times k}\,, \\ \boldsymbol{\sigma}_i \in \mathbb{R}^k\,, \quad \|\boldsymbol{\sigma}_i\|_2 = 1\,.$$

$$\mathsf{OPT}_k(G) = \max \quad \sum_{i,j=1}^{n} \big(A_G - \mathbb{E}\{A_G\}\big)_{i,j} \langle \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j \rangle \,, \\ \text{for} \quad \boldsymbol{\sigma}_i \in \mathbb{R}^k\,, \quad \|\boldsymbol{\sigma}_i\|_2 = 1\,.$$

$\mathbb{S}^{k-1}$-valued spin model

Proof outline

- Higher-rank Grothendieck inequality

$$\Big(1-\frac{c}{k}\Big)\mathsf{SDP}(G)\le\mathsf{OPT}_k(G)\le\mathsf{SDP}(G)\,.$$

- Interpolation for $\mathsf{OPT}_k(G)$ (via asymptotic free energy)

$$(A_G-\mathbb{E}\{A_G\})\longleftrightarrow J\sim\mathrm{GOE}_n$$

- Analyze SDP for $J\sim\mathrm{GOE}_n$.

Conclusion

- Extremal cuts in random graphs

  $\Longrightarrow$ balanced Ising spins $\sigma_i \in \{-1,+1\}$

- SDP $\Rightarrow$ Vector spins $\boldsymbol{\sigma}_i \in \mathbb{S}^{k-1}$, $k \le n$.

- Universal connection to spin-glasses as $\gamma \to \infty$.

- But little is known for γ fixed!

  Open Problem [Zdéborova,Boettcher '10]:

  For $G_n \sim \mathcal{G}^{\text{reg}}(n,\gamma)$, fixed $\gamma \geq 3$, w.h.p.

$$\mathsf{MCUT}(G_n) = \mathsf{MaxCut}(G_n) + o(n) = |E_n| - \mathsf{mcut}(G_n) + o(n)\,.$$