Hardy-Littlewood-Sobolev/Burkholder-Gundy, Doob

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$f \in C_0^\infty(\mathbb{R}^d)$.

$$\|f\|_{2d/(d-2)} \leq C\|\nabla f\|_2, \quad d \geq 3. \quad \text{(classical Sobolev)}$$

More general: $d \geq 2$, $1 \leq p < d$,

$$\|f\|_{pd/(d-p)} \leq C\|\nabla f\|_p, \quad \text{(also classical)}$$

Best constants found Aubin and Talenti in mid 70's. Many other proofs over the years

Without care for best constants, general follows from "isoperimetric" case $p = 1$.

$$\|f\|_{d/(d-1)} \leq C\|\nabla f\|_1$$

applied to f^β for $\beta = \frac{(d-1)p}{d-p}$. Set $q = p/(p-1)$.

$$\|f^\beta\|_{d/(d-1)} \leq \beta C \int_{\mathbb{R}^d} |f(x)|^{\beta-1} |\nabla f(x)|\, dx \leq \beta C \|f^{\beta-1}\|_q \|\nabla f\|_p,$$

Sobolev, log-sobolev and heat kernel decay ($d \geq 3$)

$$\|f\|_{2d/(d-2)} \leq C\|\nabla f\|_2 = \left[C\left(\int_{\mathbb{R}^d} -f\Delta f\right)^{1/2} = C\sqrt{\mathcal{E}(f)}\right] \qquad \text{(S)}$$

$$\int_{\mathbb{R}^d} f^2 \log f \, dx \leq \epsilon \int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \beta(\epsilon)\|f\|_2^2 + \|f\|_2^2 \log \|f\|_2, \quad f > 0, \qquad \text{(Log-S)}$$

$\beta(\epsilon) = C + \frac{d}{4} \log \frac{1}{\epsilon}$.

$$p_t(x,y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}} \leq \frac{C}{t^{d/2}} \qquad \text{(Heat(d))}$$

S $\Rightarrow$ Log-S $\Rightarrow$ Heat(d) $\Rightarrow$ S

S $\Rightarrow$ Log-S: Take $\|f\|_2 = 1$. $p = \frac{2d}{d-2}$. Jensen gives

$$\int_{\mathbb{R}^d} f^2 \log f = \frac{1}{p-2} \int_{\mathbb{R}^d} \log(f^{p-2}) f^2 dx \leq \frac{1}{p-2} \log \int_{\mathbb{R}^d} f^p dx = \frac{p}{2(p-2)} \log \|f\|_p^2$$

use $\log(a) \leq \epsilon a + \log \frac{1}{\epsilon}$ and apply (S)

The Hardy-Littlewood-Sobolev (HLS) inequality–(HL-1932, S-1938)

$$\mathcal{I}_\alpha(f)(x) = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{d/2}\Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy.$$

Let $0 < \alpha < d$, $1 < p < \frac{d}{\alpha}$ and set $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ or $q = \frac{pd}{d-p\alpha}$.

$$\|\mathcal{I}_\alpha(f)\|_{\frac{pd}{d-p\alpha}} = \|\mathcal{I}_\alpha(f)\|_q \leq C_{p,d,\alpha}\|f\|_p. \tag{HLS}$$

If $d > 2$, $p = 2$, $\alpha = 1$,

$$\|f\|_{2d/d-2} \leq c_d\|\nabla f|\|_2 \tag{S}$$

The relation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ comes from scaling.

Indeed: With $\tau_\delta f(x) = f(\delta x)$ one sees that $\tau_{\delta^{-1}}\mathcal{I}_\alpha\tau_\delta = \delta^{-\alpha}\mathcal{I}_\alpha$, $\delta > 0$.

$$\begin{aligned}
\delta^{-\alpha}\|\mathcal{I}_\alpha f\|_q &= \|\tau_{\delta^{-1}}\mathcal{I}_\alpha\tau_\delta f\|_q = \delta^{d/q}\|\mathcal{I}_\alpha(\tau_\delta f)\|_q \\
&\leq \delta^{d/q}C\|\tau_\delta f\|_p = C\delta^{d/q}\delta^{-d/p}\|f\|_p \\
&\Rightarrow \alpha + d/q - d/p = 0
\end{aligned}$$

$$\mathcal{I}_\alpha(f)(x) = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{d/2} \Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy.$$

Our Aim: Write $\mathcal{I}_\alpha(f)$ as a martingale transform and obtain (HLS) from the Burkholder-Gundy and Doob inequalities.

Why in the world would we want to do this?

Motivation: Find a new proof of Eliot Lieb's (1983) sharp (HLS) on $\mathbb{R}^d$ and extend it to other geometric settings?

Not completely crazy

Martingale techniques have been extremely successful in obtaining sharp inequalities for various singular integral operators and Fourier multipliers, including Riesz transforms, the Beurling-Ahlfors operator on $\mathbb{R}^d$, Lie groups and manifolds under curvature assumptions. **Large literature on this subject.**

Burkholder 1984

B_t d–dimensional BM. Set

$$X_t = \int_0^t H_s \cdot dB_s$$

and M_s an $d \times d$ predictable matrix with

$$\sup_{s>0} \|M(s,\omega)\|_\infty \leq 1.$$

Set

$$M * X_t = \int_0^t M_s H_s \cdot dB_s.$$

For any $1 < p < \infty$,

$$\|M * X\|_p \leq (p^* - 1)\|X\|_p,$$

$$p^* = \max\{p, \frac{p}{p-1}\}.$$

The inequality is sharp but never attained unless $p = 2$.

Theorem (E. Lieb: Sharp constants on Hardy-Littlewood-Sobolev and related inequalities (1983))

- *There is "an" extremal function h for HLS for all p, q as above.*
- *When $p = 2$, or $q = 2$ or $p = \frac{q}{q-1}$ (conjugate exponent of q) he identifies the extremal explicitly and computes the constant explicitly. That is he computes the constant explicitly in three cases.*

(i) $\|\mathcal{I}_\alpha(f)\|_{2d/d-2\alpha} \leq B_1 \|f\|_2$. ($\alpha = 1$, *Aubin (1976), Talenti (1976). S. G. Bobkov & M. Ledoux (2010) from Brunn–Minkowski.*)

(ii) $\|\mathcal{I}_\alpha(f)\|_2 \leq B_2 \|f\|_{2d/d+2\alpha}$

(iii) $\|\mathcal{I}_\alpha(f)\|_{2d/d-\alpha} \leq B_3 \|f\|_{2d/d+\alpha}$

For the above cases the, extremal is (up to multiplication by constants and dilations) unique and has form

$$h(x) = (1 + |x|^2)^{-\mu}$$

Very large literature on sharp Sobolev!

R. Frank and E. Lieb–2010, 2012: Sharp version for Heisenberg group (rearrangement free)

Proof of HLS: L. Hedberg (1972).

Fix $\delta > 0$.

$$\int_{\{|x-y|\le\delta\}} \frac{|f(y)|}{|x-y|^{d-\alpha}}dy = \sum_{k=0}^{\infty}\int_{\{\delta 2^{-k-1}<|x-y|\le\delta 2^{-k}\}} \frac{|f(y)|}{|x-y|^{d-\alpha}}dy$$

$$\le C\sum_{k=0}^{\infty}(\delta 2^{-k})^{\alpha}\frac{1}{(\delta 2^{-k})^{d}}\int_{\{|x-y|\le\delta 2^{-k}\}}|f(y)|dy$$

$$\le \le C\delta^{\alpha}M(f)(x)\sum_{k=0}^{\infty}2^{-k\alpha} = C_{\alpha}\delta^{\alpha}Mf(x),$$

Mf Hardy-Littlewood maximal function: $Mf(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy$.
Apply Hölder ($p > 1$)

$$\int_{\{|x-y|>\delta\}} \frac{|f(y)|}{|x-y|^{d-\alpha}}dy \le C_d\|f\|_p\left(\int_{\delta}^{\infty}\frac{r^{d-1}}{r^{p'(d-\alpha)}}\right)^{1/p'} \le C\|f\|_p\delta^{\alpha-d/p}$$

$$|\mathcal{I}_{\alpha}(f)(x)| \le C_{\alpha,d}[\delta^{\alpha}Mf(x) + \|f\|_p\delta^{\alpha-d/p}]$$ **(a Peter-Paul inequality)**

Minimizing in δ,

$$|\mathcal{I}_{\alpha}(f)(x)| \le CMf(x)^{1-p\alpha/d}\|f\|_p^{p\alpha/d}$$

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d} \Rightarrow \frac{p}{q} = 1 - \frac{p\alpha}{d}, \quad \& \qquad \frac{p\alpha}{d} = 1 - \frac{p}{q} = \frac{q-p}{q}$$

$$\Rightarrow \; |\mathcal{I}_\alpha(f)(x)| \leq CMf(x)^{1-p\alpha/d}\|f\|_p^{p\alpha/d} = CMf(x)^{p/q}\|f\|_p^{\frac{q-p}{q}}$$

$$\begin{aligned} \Rightarrow \int_{\mathbb{R}^d} |\mathcal{I}_\alpha(f)(x)|^q dx &\leq C\left(\int_{\mathbb{R}^d} Mf(x)^p dx\right)\|f\|_p^{(q-p)} \\ &\leq C\left(\|f\|_p^p\right)\|f\|_p^{(q-p)} = \|f\|_p^q. \end{aligned}$$

Used fact:

$$\|Mf\|_p \leq C\|f\|_p, \quad 1 < p < \infty.$$

Remark

- *The "classical" proof (E. Stein "Singular Integrals . . ." page 120) uses the "off diagonal" version of the Marcinkiewicz interpolation and not the Hardy-Littlewood Max Function M.*

$$T_t f(x) = \int_{\mathbb{R}^d} p_t(x-y) f(y) dy, \quad p_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}$$

By Fubini and doing the integral in t, one finds that

$$\mathcal{I}_\alpha(f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} T_t f(x)\, dt,$$

Taking Fourier transform:

$$\begin{aligned} \widehat{\mathcal{I}_\alpha(f)}(\xi) &= \left(\frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t|\xi|^2}\, dt \right) \widehat{f}(\xi) \\ &= \left(\frac{1}{\Gamma(\alpha/2)} (|\xi|^2)^{-\alpha/2} \int_0^\infty t^{\alpha/2-1} e^{-t}\, dt \right) \widehat{f}(\xi) \\ &= (|\xi|^2)^{-\alpha/2} \widehat{f}(\xi). \end{aligned}$$

Thus (Riesz potentials)

$$\mathcal{I}_\alpha(f)(x) = (-\Delta)^{-\alpha/2} f(x).$$

(i) Semigroup is symmetric Markovian, contraction on all L^p, $1 \le p \le \infty$.

(ii) Semigroup is ultracontractive: $\|T_t f\|_\infty \le \frac{C}{t^{d/2}}\|f\|_1 \;\; \forall t > 0$. In fact, Jensen's shows for any $1 \le p < \infty$,

$$\|T_t f\|_\infty \le \frac{C}{t^{d/2p}}\|f\|_p, \;\; \forall t > 0$$

(ii) If $f^*(x) = \sup_{t>0}|T_t f(x)|$, then $\|f^*\|_p \le c_p\|f\|_p, \;\; 1 < p \le \infty$.
Follows from: p_t is an approximation to the identity, radial, decreasing.

Theorem (E. Stein 1961 "On the maximal ergodic theorem.")

$\|f^*\|_p \le c_p\|f\|_p$, $1 < p < \infty$ *holds for any symmetric Markovian semigroup.*

Remark (Carl Herz)

Stein follows from Doob and holds with $c_p = \frac{p}{p-1}$*. (I learned the proof from Doob and Burkholder about 30 years ago. Other proofs.)*

$$\begin{aligned}|\mathcal{I}_\alpha(f)(x)| &\leq C_1\left(\int_0^\delta t^{\alpha/2-1}\,dt\right) f^*(x)\\ &\quad+\frac{\|f\|_p}{\Gamma(\alpha/2)(4\pi)^{d/(2p)}}\int_\delta^\infty t^{[\frac{\alpha}{2}-1]-\frac{d}{2p}}\,dt\\ &\leq C_{d,\alpha}\left(\delta^{\alpha/2}f^*(x)+\delta^{[\frac{\alpha}{2}-\frac{d}{2p}]}\|f\|_p\right).\end{aligned}$$

(Used the fact that $\alpha-\frac{d}{p}=-\frac{d}{q}<0$.) Minimize by picking

$$\delta=\left(\frac{\|f\|_p}{f^*}\right)^{2p/d}$$

$$|\mathcal{I}_\alpha(f)(x)|\leq C_{d,\alpha}\,(f^*)^{1-\alpha p/d}\,\|f\|_p^{\alpha p/d}\leq C_{d,\alpha}\,(f^*)^{p/q}\,\|f\|_p^{\alpha p/d}.$$

and as before

$$\|\mathcal{I}_\alpha(f)\|_q\leq C\|f\|_p \Leftrightarrow$$

$$\|(-\Delta)^{-\alpha/2}f(x)\|_q\leq C\|f\|_p.$$

Same as "Hedberg 2^{nd} time" gives: $(S, \mathcal{S}, \mu)$ measure space, T_t a symmetric Markovian semigroup with dimension n in the sense of Varopoulos, i.e., T_t is

$$T_t f(x) = \int_S p_t(x, y) f(y) \mu(dy),$$

there exists $C > 0$ so that for all $t > 0, x, y \in S$, $p_t(x, y) \leq C t^{-\frac{n}{2}}$. ($n$ does not have to be an integer, many examples exist.) Define

$$\mathcal{I}_\alpha(f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} T_t f(x)\, dt,$$

Let $-A$ be the (self-adjoint) infinitesimal generator of $(T_t, t \geq 0)$.

$$\mathcal{I}_\alpha(f)(x) = A^{-\frac{\alpha}{2}} f, \quad \left(f \in \mathrm{Dom}(\mathrm{A}^{-\frac{\alpha}{2}}) \cap \mathrm{L}^1(\mathrm{S})\right)$$

Theorem (N. Varopoulos 1985)

Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and set* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

$$||\mathcal{I}_\alpha(f)||_q \leq C_{p,n,\alpha} ||f||_p.$$

Can keep track of constants:

$$\|A^{-\frac{\alpha}{2}} f\|_{\frac{np}{n-p\alpha}} \leq \left(\frac{p}{p-1}\right)^{1-\alpha/n} \frac{2nC^{\alpha/n}}{\alpha(n-p\alpha)\Gamma(\alpha/2)} \|f\|_p.$$

Sobolev for any semigroup of dimesnion $n > 2$.

With $n > 2$, $p = 2$, $\alpha = 1$,

$$\|f\|_{\frac{2n}{n-2}} \leq 2^{1-1/n} \frac{2nC^{1/n}}{(n-2)\sqrt{\pi}} \sqrt{\mathcal{E}(f)}, \quad (\mathcal{E} \text{ Dirchlet form of semigroup}) \qquad (*)$$

where C constant in $p_t(x, y) \leq Ct^{-n/2}$.

For any semigroup of dimension $n > 2$, we have (as on $\mathbb{R}^d$)

S $\Rightarrow$ Log-S $\Rightarrow$ Heat(n) $\Rightarrow$ S

Let B_t Brownian motion in $\mathbb{R}^d$ starting at z, $\mathbb{P}_z$ and $\mathbb{E}_z$ be the probability and $\mathbb{E}$ the expectation of the BM stating with Lebesgue measure. Then
Fix $0 < a < \infty$. Consider the martingale

$$M(f)_t = T_{a-t}f(B_t), \qquad 0 \leq t \leq a,$$

$$\text{Itô: } \Rightarrow \qquad M(f)_t = T_a f(z) + \int_0^t \nabla_x T_{a-s} f(B_s) \cdot dB_s.$$

$$\text{Martingale Transform:} \quad \int_0^a (a-s)^{\alpha/2} \nabla_x T_{a-s} f(B_s) \cdot dB_s.$$

$$\begin{aligned}
& \mathbb{E}\left|\int_0^a (a-s)^{\alpha/2} \nabla_x T_{a-s} f(B_s) \cdot dB_s\right|^p \\
= & \int_{\mathbb{R}^d} \mathbb{E}_z \left|\int_0^a (a-s)^{\alpha/2} \nabla_x T_{a-s} f(B_s) \cdot dB_s\right|^p dz \\
\leq & c_p a^{p\alpha} \int_{\mathbb{R}^d} \mathbb{E}_z |f(B_a)|^p dz = c_p a^{p\alpha} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p_a(x-z)|f(x)|^p dx \right) dz \\
= & c_p a^{p\alpha} \int_{\mathbb{R}^d} |f(x)|^p dx < \infty
\end{aligned}$$

Theorem (D. Applebaum, R.B–2014)

For $f \in C_0^\infty$ *set*

$$\mathcal{S}^{a,\alpha} f(x) = \mathbb{E}\left(\int_0^a (a-s)^{\alpha/2} \nabla (T_{a-s} f)(B_s) \cdot dB_s \mid B_a = x\right).$$

Then

$$\begin{aligned}\mathcal{S}^{a,\alpha} f(x) &= -\int_0^a s^{\alpha/2} T_s(\Delta T_s f)(x) ds \\ &= \left(-\frac{1}{2} a^{\alpha/2} T_{2a} f(x) + \frac{\alpha}{4}\int_0^a s^{\alpha/2-1} T_{2s} f(x) ds\right)\end{aligned}$$

and as $a \to \infty$,

$$\lim_{a\to\infty} \mathcal{S}^{a,\alpha} f(x) \to \frac{\Gamma(\alpha/2)}{2} \mathcal{I}_\alpha(f)(x), \quad a.e.$$

Burkholder-Gundy: there is a C_p independent of a such that

$$\|\mathcal{S}^{a,\alpha} f(x)\|_q \leq \left\| \int_0^a (a-s)^{\alpha/2} \nabla (T_{a-s} f)(B_s) \cdot dB_s \right\|_q$$
$$\leq C_q \left\| \left(\int_0^a (a-s)^{\alpha} |\nabla (T_{a-s} f)(B_s)|^2 \, ds \right)^{1/2} \right\|_q$$

Lemma

$$\left(\int_0^a (a-s)^{\alpha} |\nabla (T_{a-s} f)(B_s)|^2 \, ds \right)^{1/2}$$
$$\leq C_{p,\alpha,d} \left(\sup_{0<s<2a} |(T_{(2a-s)}|f|)(B_s)| \right)^{p/q} ||f||_p^{\frac{q-p}{q}}$$

Raise to q, take $\mathbb{E}$ expectation and apply Doob to first term.

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Lemma

$$\left(\int_0^a (a-s)^{\alpha} |\nabla(T_{a-s}f)(B_s)|^2 \, ds \right)^{1/2}$$

$$\leq C_{p,\alpha,d} \left(\sup_{0<s<2a} |(T_{(2a-s)}|f|)(B_s)| \right)^{p/q} ||f||_p^{\frac{q-p}{q}}$$

Raise to q, take $\mathbb{E}$ expectation and apply Doob to first term.

A "Hedberg 1972 proof of Applebaum-Bañuelos 2014 Theorem"

Several weakness in above martingale proof (outside of best constant). Does not work for general symmetric Markovian semigroups

(i) Proof is "very" $\mathbb{R}^d$: Uses the gradient.

(ii) Proof uses a simple estimate on gradient on $\mathbb{R}^d$:

$$|\nabla_x T_t f(x)| \leq \frac{C_d}{\sqrt{t}} T_{2t}|f|(x)$$

which follows from the fact that for $p_t(x)$ Gausian on $\mathbb{R}^d$,

$$|\nabla_x p_t(x)| \leq 2^{\frac{d+4}{2}} \frac{1}{\sqrt{t}} p_{2t}(x).$$

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Proof can be extended to manifolds of non-negative Ricci curvature ...

Different martingale representation via Poisson (Cauchy) semigroup on $\mathbb{R}^d$

$$P_t f(x) = C_n \int_{R^d} \frac{f(y)}{(t^2 + |x-y|^2)^{n+1}/2} dy = \int_0^\infty T_s f(x) \eta_t(ds),$$

$\eta_t(ds) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds$. ("half-stable subordination")

Theorem (R.B. many years ago)

$$\mathcal{T}^{a,\alpha} f(x) = \mathbb{E}^a [\int_0^\tau Y_s^\alpha \frac{\partial u_f}{\partial y}(Z_s) dY_s | X_\tau = x],$$

$u_f(x,y) = P_y f(x)$ *the harmonic extension of* f.

$$\lim_{a\to\infty} \mathcal{T}^{a,\alpha} f = \frac{\Gamma(\alpha+2)}{2^{\alpha+2}} \mathcal{I}_\alpha(f)$$

$Z_s = (X_s, Y_s)$, X_s B.M. on $\mathbb{R}^d$, Y_s B.M. on $\mathbb{R}^+ = (0,\infty)$. $\tau=$ first time $Y_s = 0$

Theorem (Daesung Kim (2015))

Formula extend to symmetric Markovian semigroups and most importantly, it can be used to prove HLS. (Proof: Based on "Littlewood-Paley" theory.)

Given $T : L^p(\mu) \to L^q(\mu)$, with $\|Tf\|_q \le A\|f\|_p$,

$$\lambda^q \mu\{|Tf| > \lambda\} \le \int |Tf|^q d\mu \le A^q \|f\|_p^q$$

$$|||Tf|||_{weak(q)} = \sup_{\lambda>0} \lambda \left(\mu\{|Tf| > \lambda\}\right)^{\frac{1}{q}} \le A\|f\|_p$$

For cube $(0,1]^d \subset \mathbb{R}^d$, define.

$$||f||_{weak(q)} = \sup \left\{ \frac{1}{|E|^{1-1/q}} \int_E |f| dx \, : \, E \subset (0,1]^d, |E| > 0 \right\}.$$

Theorem (R.B., A. Osekowski-2015: version I_α on dyadic martingales on $\mathbb{R}^d$)

$$\|I_\alpha f\|_{weak(q)} \le K(\alpha, d, p)\|f\|_p, \quad 1/q = 1/p - \alpha/d$$

$$K(\alpha, d, p) = \frac{2^{d-\alpha} - 2^{-\alpha}}{2^{d-\alpha} - 1} \left(1 + \frac{(1 - 2^{-\alpha})^{p'}}{(1 - 2^{-d})^{p'-1}(2^{(d-p\alpha)(p'-1)} - 1)} \right)^{1/p'}$$

p' conjugate exponent of p. The inequality is sharp.

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Proof via Bellman functions. Not a "Hedberg 1972 proof of Bañuelos-Osekowski 2015 theorem"

Thank you!