Decay of Correlations for the Hardcore Model in Random Regular Graphs

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joint work with

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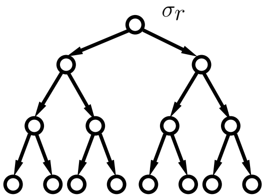
Information Flow on Trees



Broadcast Model

- Infinite d-ary tree T_d.
- Spins $[q]$, $\sigma_v \in [q]$ is spin at v.
- Choose $\sigma_r \sim \pi$.
- Independent Markov chain M on each edge of T.

Motivation

- Evolution of DNA, Noisy computation.
- Broadcast model gives rise to the **free Gibbs measure**.
- Random constraint satisfaction.

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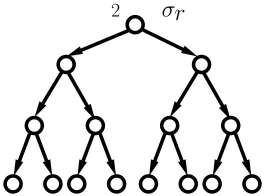

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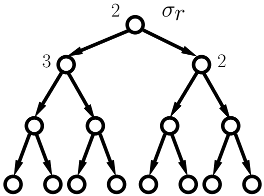

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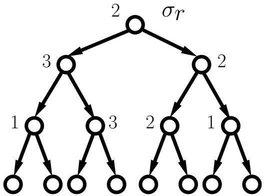

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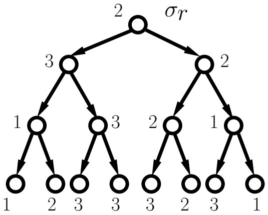

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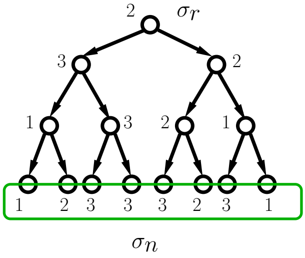

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What is reconstruction?

- Reconstruction is **solvable** if for some i, j

$$\limsup_n d_{tv}(\sigma_n^i, \sigma_n^j) > 0$$

- Mutual information between values at level n and root does not go to 0.

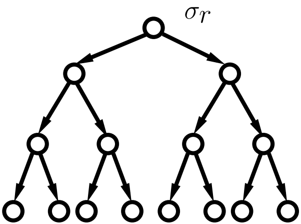
$$\mathbb{P}(\sigma_r = i | \sigma_n) \not\to \pi_i \quad \text{a.s.}$$

- Otherwise, **non-solvable** or **non-reconstruction**.

Related to...

- Can the phylogenetic tree be constructed efficiently from DNA?
- Extremality of the free Gibbs measure.
- Phase transitions in the solution space of random constraint satisfaction problems.

Symmetric channel



- Keep state w.p. λ, randomize w.p., $1-\lambda$.
- Second eigenvalue $\lambda(M) = \lambda$.
- $$M_{i,j} = \lambda \mathbb{1}_{i=j} + \frac{1-\lambda}{q}.$$
- Critical λ_c s.t.
  - Reconstruction solvable for $\lambda > \lambda_c$.
  - Non-solvable for $0 < \lambda < \lambda_c$.

Percolation bound

Non-solvable when $\lambda d \leq 1$. All components finite a.s.

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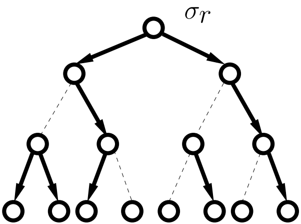

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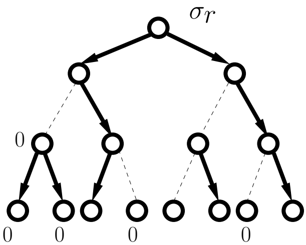

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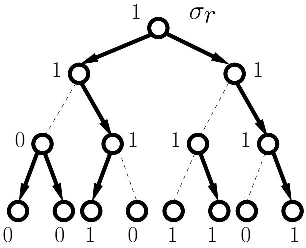

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Kesten-Stigum bound

Theorem (Kesten-Stigum '68)

For $\lambda^2 d > 1$, reconstruction is (census) solvable.

Theorem (Mossel-Peres '03)

The threshold for census reconstruction is $\lambda^2 d = 1$.

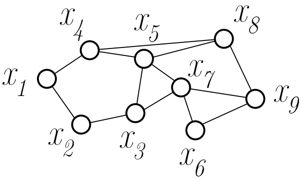
- *Kesten-Stigum CLT's imply a.s. the frequencies are π.*

Tightness of Kesten-Stigum

- [Bleher-Ruiz-Zagrebnov '95], [Ioffe '96], [Evans-Kenyon-Peres-Schulman '00] K-S bound tight for binary symmetric channel ($q = 2$, Ising model).
- [Mossel '01] When $q \geq 18$, K-S bound is **not** tight. Reconstruction beyond census.
- [Mezard-Montanari '06] Statistical physics predictions: for $\theta > 0$ (ferromagnetic)
  - K-S bound tight when $q \leq 4$.
  - K-S bound **not** tight for $q \geq 5$.

  [Sly '08] rigorously verified these predictions, for large enough d.
- [Brightwell-Winkler '04] Reconstruction beyond K-S in the "hardcore model" of lattice gases.

Graphical models



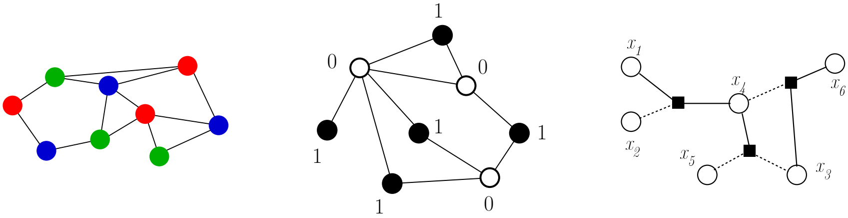
$$G = (V, E), \quad |V| = n, \quad \underline{x} = (x_1, \dots, x_n), \quad x_i \in [q], \quad \psi : [q] \times [q] \to \mathbb{R}.$$

$$\mu(\underline{x}) = \frac{1}{Z} \prod_{(x_i \sim x_j)} \psi(x_i, x_j)$$

- G has bounded degree.
- G is random.
- G has girth $g(n) \to \infty$, except for $o(n)$ vertices.

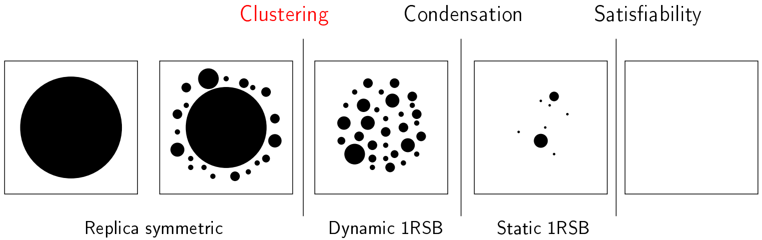
Random constraint satisfaction problems

A CSP is a set of variables subject to some constraints: decide whether there exists some variable assignment satisfying all constraints.



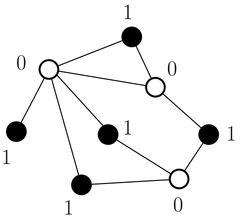
- Proper k-colorings of random graph (of average degree d).
- Independent sets (hardcore model) of a random graph.
- Random k-satisfiability.
- Using theory of spin glasses (random instances of satisfiability)...

Clustering of the solution space in random CSPs



- Replica symmetric (decay of correlations) vs. replica symmetry breaking (long range correlations).
- [MM '06] conjectured that 1 step replica symmetry breaking occurs exactly when the K-S bound is not tight.
- For $d < d_c(k)$ almost all solutions are contained in a giant cluster.
- For $d > d_c(k)$, there are exponentially many clusters. [Mezard-Mora-Zecchina '05], [Achlioptas-Coja-Oghlan '08]
- [Mezard-Montanari '06] conjectured that clustering threshold coincides with tree reconstruction. Shown asymptotically in [Montanari-Restrepo-Tetali '09].

Hardcore Model



- Probability distribution over independent sets (ISs), fugacity $\lambda > 0$,

$$\mathbb{P}(\sigma) = \frac{1}{Z}\lambda^{|\sigma|}\mathbb{1}(\sigma \text{ is an independent set})$$

- Partition function Z

$$Z = \sum_{\sigma \in I(G)} \lambda^{|\sigma|}$$

Reconstruction for hardcore measure on the tree T_d

- Broadcast process for the hardcore measure on T_d

$$M = \begin{pmatrix} p_{11} & p_{10} \\ p_{01} & p_{00} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\alpha}{1-\alpha} & \frac{1-2\alpha}{1-\alpha} \end{pmatrix}$$

- Density of independent sets $\alpha = \alpha(\lambda, d)$ where

$$\lambda = \frac{\alpha}{1-\alpha}\left(\frac{1-\alpha}{1-2\alpha}\right)^d .$$

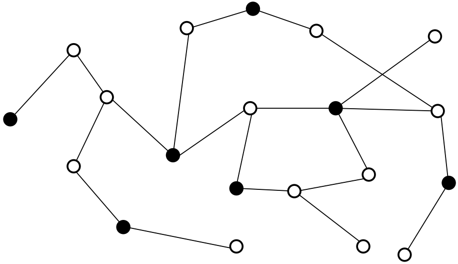
- Reconstruction solvable for λ if $\mathbb{P}(\sigma_r = 1|\sigma_n) \not\to \alpha(\lambda, d)$.
- [Brightwell-Winkler '04] Reconstruction solvability for

$$\alpha \geq \frac{\ln d + \ln\ln d + 1 + o(1)}{d}, \quad \lambda \geq (e + o(1))(\ln d)^2.$$

- [Martin '03] Non-solvable for $\lambda \leq e - 1$.
- [B-Sly-Tetali '10] Non-solvable for $\lambda \leq \frac{(\ln 2 - o_d(1))\ln(d)}{2\ln\ln d}$,

$$\alpha \leq \frac{\ln d + \ln\ln d - \ln\ln\ln d - \ln 2 + \ln\ln 2 - o(1)}{d}, .$$

Reconstruction in graphs



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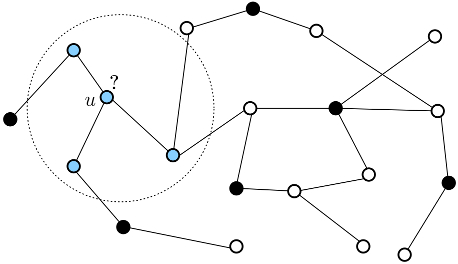

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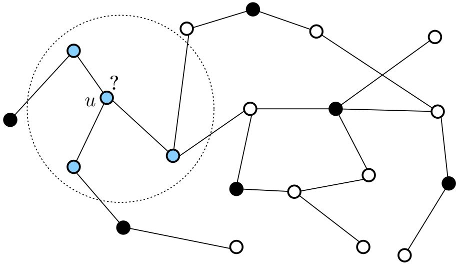


- $\{G_n\}$ a family of random graphs, $|V(G_n)| \to \infty$.
- $\{G_n\}$ has non-reconstruction if for uniform $u \in V(G_n)$.

$$\lim_{L\to\infty} \limsup_{n} \mathbb{E}\Big|\mathbb{P}\left(\sigma_u = 1 | \sigma(\partial B_u(L)), u\right) - \alpha(\lambda, d)\Big| = 0$$

Reconstruction in graphs through local weak convergence

Theorem (B-Sly-Tetali '16)

For $\lambda < \lambda_c$, and large enough d, the d-regular random graph has non-reconstruction iff the d-regular tree has non-reconstruction.

$$\mathbb{P}_{T_d}(\sigma_x = 1 | \sigma_{L_\ell}) \xrightarrow{\mathbb{P}} \alpha(\lambda, d) \text{ as } \ell \to \infty.$$

$$\Leftrightarrow$$

$$\lim_{\ell \to \infty} \limsup_n \mathbb{E} \left| \mathbb{P}_{G_n} \left(\sigma_u = 1 | \sigma(\partial B_u(\ell)) \right) - \alpha(\lambda, d) \right| = 0$$

where λ_c is the fugacity corresponding to

$$\alpha_c = \frac{2 \ln d - (3 + o(1) \ln \ln d)}{d}.$$

Convergence in probability locally

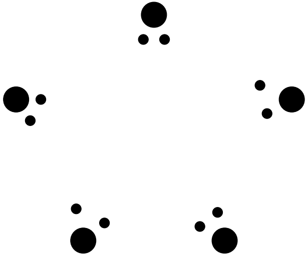
[Montanari-Mossel-Sly '12] Convergence in probability locally:

- For all r, $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}_{U_n}\left(d_{tv}(\mathbb{P}_{G_n}(\sigma(B_r(u) \in \cdot)), \mathbb{P}_{T_d}(\sigma(B_r(x)) \in \cdot))) > \varepsilon\right) = 0$$

- Idea: use a type of second moment argument to show convergence in probability locally of hardcore measure to tree measure.

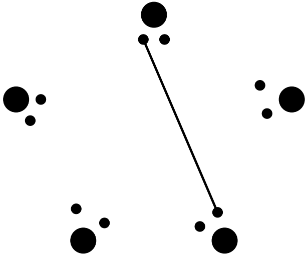
The configuration model



- Analyze the configuration model - d-regular multigraphs on n-vertices $\mathcal{C}(n, d)$.

$$\mathbb{P}(\text{Simple}) = (1 + o(1)) \exp\left(\frac{1 - d^2}{4}\right)$$

The configuration model



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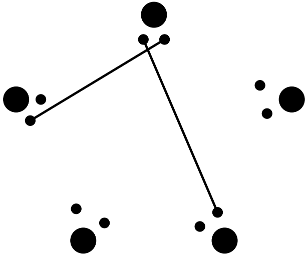

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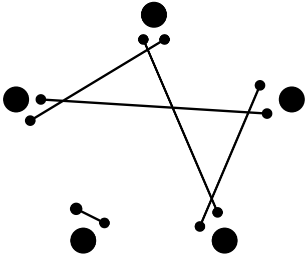

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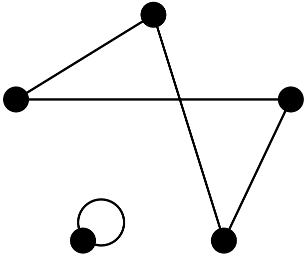

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The partition function of the hardcore model on $\mathcal{C}(n,d)$

$$Z_{G,\alpha} := \sum_{\sigma:|\sigma|=\alpha n} \lambda^{\alpha n}, \qquad Z_G := \sum_{\alpha} Z_{G,\alpha}.$$

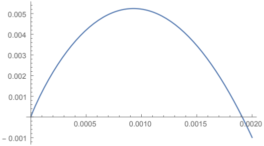


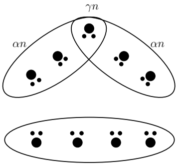
Figure: $\Phi(\alpha)$ vs. α.

$$\mathbb{E}\left(Z_{G,\alpha}\right) = \binom{n}{\alpha n}\lambda^{\alpha n}\prod_{i=0}^{\alpha n d-1}\frac{(1-\alpha)nd-i}{nd-1-2i} = \widetilde{\Theta}(1)\exp\left(n\Phi(\alpha)\right).$$

$\Phi(\alpha)$ maximized at α^* which solves $\lambda = \frac{\alpha}{1-\alpha}\left(\frac{1-\alpha}{1-2\alpha}\right)^d$.

$$\alpha^*(\lambda,d) = (1+o_d(1))\frac{\ln(\lambda d/\ln(\lambda d))}{d}.$$

The second moment of $Z_{G,\alpha}$



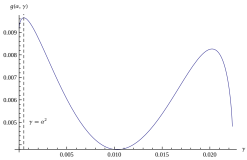
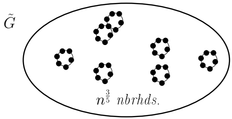


Figure: Projection $g(\alpha, \gamma)$ of f onto the maximizer $\overline{\varepsilon}(\alpha, \gamma)$.

Main facts/results: $G \sim \mathcal{C}(n, d)$

- $\mathbb{E}((Z_{G,\alpha})^2) = \sum_{\gamma,\varepsilon} \mathbb{E}(Z^{(2)}_{G,\alpha,\gamma,\varepsilon}) = \exp(nf(\alpha,\gamma,\varepsilon) + O(\ln(n)))$
- For any $0 \le \alpha \le \frac{1}{2}$, $2\Phi(\alpha) = f(\alpha, \alpha^2, \alpha(1-2\alpha))$.
- Let $\lambda < \lambda_c$. The function f is maximized at $(\alpha^*, (\alpha^*)^2, \alpha^*(1-2\alpha^*))$ and decays quadratically about this point.
- $\mathbb{E}((Z_G)^2) = \tilde{\Theta}(1)(\mathbb{E}(Z_G))^2$.

Relating graph and tree measures: punctured graph $\tilde{G}$



$\tilde{G}$: Choose $n^{3/5}$ vertices at random and delete their local neighborhood of radius r.

Study Z_G conditioned on the boundaries of local neighborhoods B.

Main results:

- Whp $O(n^{3/5})$ vertices of B have degree $d-1$ and $O(n^{1/5})$ have degree $d-2$.
- $\mathbb{E}(Z_{\tilde{G},\sigma})$ is proportional to a product measure on B.
- Second moment calculation for $\tilde{G}$. For each $\sigma \in \{0,1\}^B$

$$\mathbb{E}((Z_{G,\sigma})^2) = \exp\left(O(n^{1/5})\right)\mathbb{E}(Z_{G,\sigma})^2 \tag{1}$$

A set of bad boundaries $\mathcal{B}$

Local weak convergence: Let $G_n \sim \mathcal{C}(n,d)$. Then,

$$\frac{|\{i : \|\mathbb{P}_{G_n}(\partial B_r(s_i) \in \cdot), \mathbb{P}_{T_d}(\partial B_r(x) \in \cdot)\|_{tv} > \varepsilon\}|}{n^{3/5}} \xrightarrow{\mathbb{P}} 0$$

If not, there is a set $\mathcal{B}$ of boundary configurations on which graph and tree measure differ.

- By (1) and Azuma's ineq, $\mathcal{B}$ has large stationary prob. :

$$\mathbb{P}\left(\sum_{\sigma\in\mathcal{B}} Z_{G,\sigma} \geq \exp\left(-O(n^{\frac{1}{2}+\delta(\varepsilon)})\right)\mathbb{E}\left(Z_G\right)\right) \geq 1 - 2\exp\left(-n^{2\delta(\varepsilon)}\right).$$

- $\mathbb{E}(Z_G)$ and $\mathbb{E}\left(\sum_{\sigma\in\mathcal{B}} Z_{G,\sigma}\right)$ related by tree measure $\mathbb{P}_{T_d}(\mathcal{B})$.
- On these events, the part. fn. is much larger than expected

$$\sum_{\sigma\in\mathcal{B}} Z_{G,\sigma} \geq \exp\left(cn^{\frac{3}{5}}\right)\mathbb{E}\left(\sum_{\sigma\in\mathcal{B}} Z_{G,\sigma}\right).$$

Second moment implies $\mathcal{B}$ is unlikely

$$
\begin{aligned}
&\mathbb{P}\left(\sum_{\sigma\in\mathcal{B}} Z_{G,\sigma} \geq \exp\left(-O(n^{\frac{1}{2}+\delta})\right)\mathbb{E}\left(Z_G\right)\right) \\
&\leq \frac{\mathbb{E}\left(\sum_{\sigma\in\mathcal{B}} Z_{G,\sigma}\right)}{\exp\left(-O(n^{\frac{1}{2}+\delta})\right)\mathbb{E}\left(Z_G\right)} \qquad \text{(Markov's ineq.)} \\
&\leq \frac{\exp\left(-cn^{\frac{3}{5}}\right)}{\exp\left(-O(n^{\frac{1}{2}+\delta})\right)} \leq \exp\left(-cn^{\frac{3}{5}}\right)
\end{aligned}
$$

by choosing δ such that $\frac{3}{5} > \frac{1}{2} + \delta$.

Phase transitions for the hardcore measure

- [Barbier-Krzakala-Zdeborová-Zhang '13] Several earlier works showed the hardcore model undergoes a continuous "full replica symmetry breaking" transition for graphs of small degree.
- [Coja-Oghlan-Efthymiou '11] Clustering for hardcore measure in Erdös-Rényi graphs beyond $(1 + o_d(1)) \log(d)/d$. [Rahman-Virag '14, Gamarnik-Sudan '14] random regular graphs.
- [BKZZ13] use cavity methods to show that for degree $d > 16$, there is a discontinuous phase transition while for $d < 16$, it is continuous.
- [BKZZ13] "1 RSB approach": condensation transition exists and is at

$$\alpha_c \approx \frac{2(\log(d) + \log\log(d) + 1 - \log(2))}{d} + o(1/d).$$

Thank you for your attention!