Statistical Physics on Sparse Random Graphs: Mathematical Perspective

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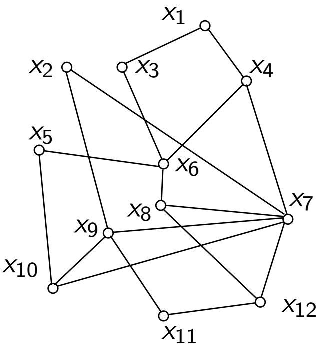
Stanford University

Northwestern, July 19, 2016

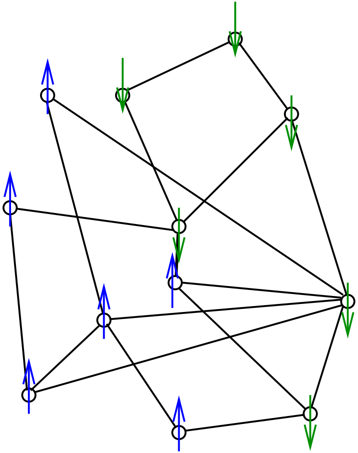
Factor model [DM10, Eqn. (1.4)]



$G = (V, E)$, $V = [n]$, $\underline{x} = (x_1, \dots, x_n)$, $x_i \in \mathcal{X}$ (finite set).

$$\mu(\underline{x}) = \frac{1}{Z_G} \prod_{(ij) \in E} \psi(x_i, x_j) \prod_{i \in V} \bar{\psi}(x_i) \,.$$

Ising models



Ising models [DM10, Eqn. (1.20)]

$G_n = (V_n \equiv [n], E_n)$, finite, non-directed graphs.

$x_i \in \{+1, -1\}$

$$\mu(\underline{x}) = \frac{1}{Z_n(\beta, B)} \exp\left\{\beta \sum_{(ij)\in E_n} x_i x_j + B \sum_{i=1}^{n} x_i\right\}$$

$\beta \geq 0$ Ferromagnetic, $\beta < 0$ Anti-ferromagnetic, $B \geq 0$ WLOG

Potts models [DMS13, Eqn. (1.3)]

$G_n = (V_n \equiv [n], E_n)$, finite, non-directed graphs.

$x_i \in [q] \equiv \{1, 2, \ldots, q\}$

$$\mu(\underline{x}) = \frac{1}{Z_n(\beta, B)} \prod_{(ij) \in E_n} e^{\beta \mathbf{1}\{x_i = x_j\}}$$

— distribution on spin configurations $\underline{x} \in [q]^V$.

Interaction strength parametrized by β;

May add *external field B* towards distinguished spin 1

Natural generalization of **Ising models** ($q = 2$);

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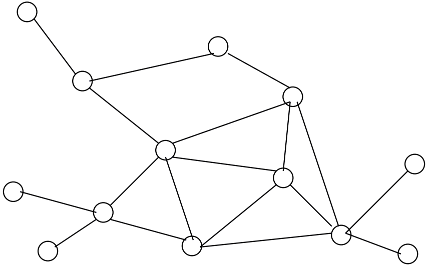

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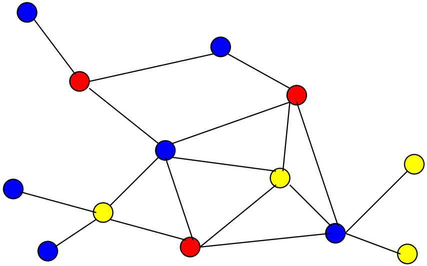

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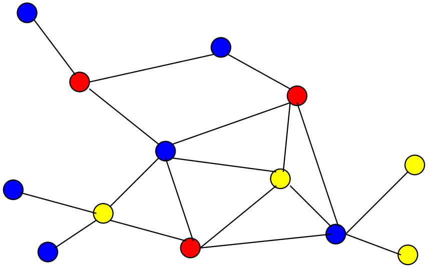
Uniform measure over proper colorings [DM10, Eqn. (1.22)]



$$\mu(\underline{x}) = \frac{1}{Z_G} \prod_{(i,j)\in E} \psi(x_i, x_j)\,, \qquad \psi(x,y) = \mathbb{I}(x \neq y)\,.$$

Z_G counts number of proper q-colorings of G.

AF Potts at $\beta = -\infty$, $B = 0$

The hard-core model [SS14]

$$\mu(\underline{x}) = \frac{1}{Z_G(\lambda)} \prod_{(ij)\in E} (1 - x_i x_j)$$

— supported on $\underline{x} \in \{0,1\}^V$ encoding **independent sets**

(no neighbors occupied)

Occupied vertices are weighted by a *fugacity* λ

$Z_G(1)$ counts independent sets of G;
$\lambda = \infty$ for largest indepednent set of G.

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Free energy density [DM10, Eqn. (2.1)]

Graph sequences $G_n = ([n], E_n)$

$$Z_n \equiv \sum_{\underline{x}} \prod_{(ij)\in E_n} \psi(x_i, x_j) \prod_{i=1}^{n} \bar{\psi}(x_i)$$

Find asymptotic **free energy density**

$$\phi \equiv \lim_{n\to\infty} n^{-1} \log Z_n \quad \text{(if limit exists);}$$

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Random graph models [DM10, Sec. 1.2.4]

Erdős-Renyi Random graph $G = (V, E) \sim G_{n,m}$

$|V| = n$ vertices

Uniform among graphs of $m = |E|$ edges

(or edges independently present with probability $p = m/\binom{n}{2}$)

Average degree $d = 2m/n$

Random regular graph $G = (V, E) \sim \mathcal{G}_{n,d}$

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Uniformly random among graphs with $|\partial i| = d \quad \forall i \in V$.

$d = O(1)$ for (uniformly) sparse graphs

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Locally tree-like graphs: regular limit

Consider graphs that are **sparse** ($|E_n| \asymp |V_n|$) and **locally tree-like** — *the local neighborhood of a uniformly random vertex converges in distribution to a (random) rooted tree*

the random d-regular graph $\mathscr{G}_{n,d}$ converges (locally) to the infinite d-regular tree $\mathbf{T}_d$, denoted $\mathscr{G}_{n,d} \rightsquigarrow \mathbf{T}_d$

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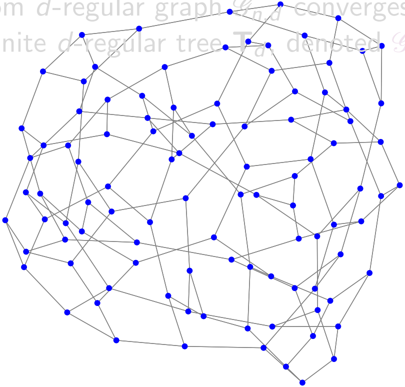

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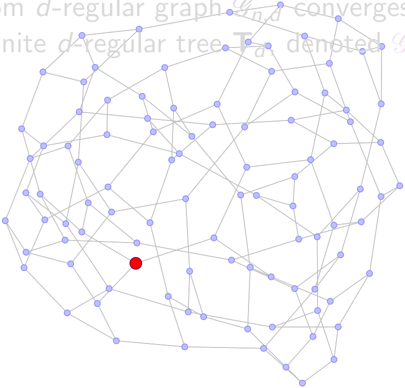

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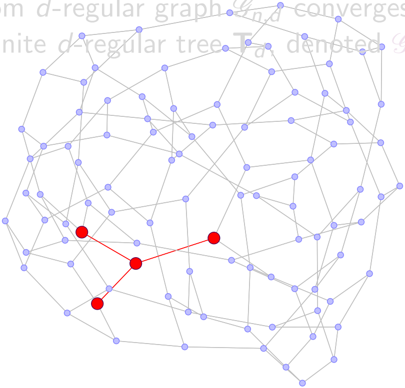

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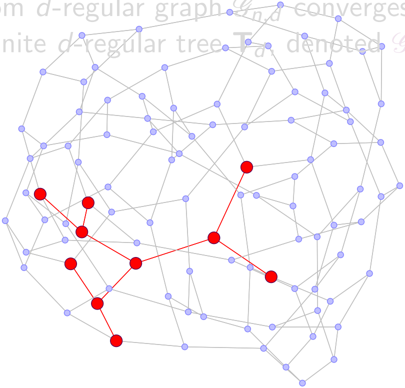

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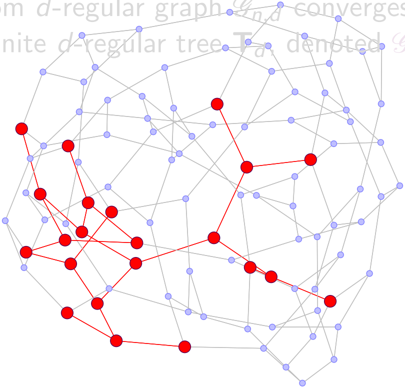

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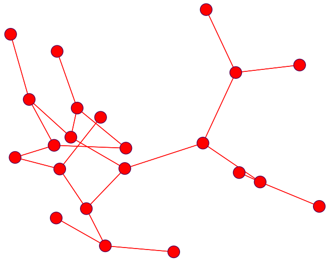

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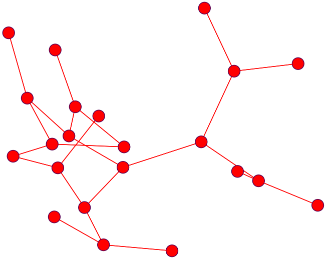

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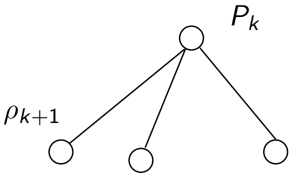

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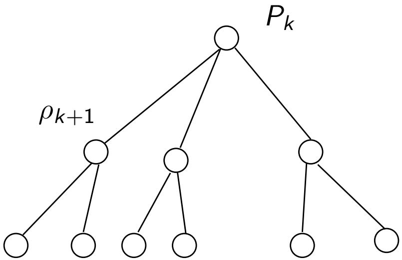

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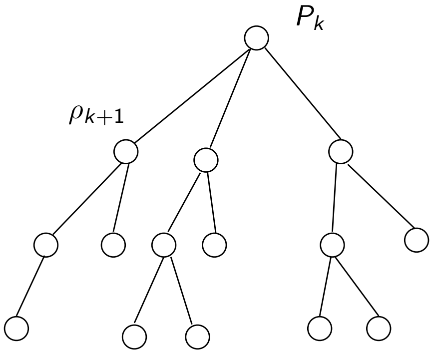

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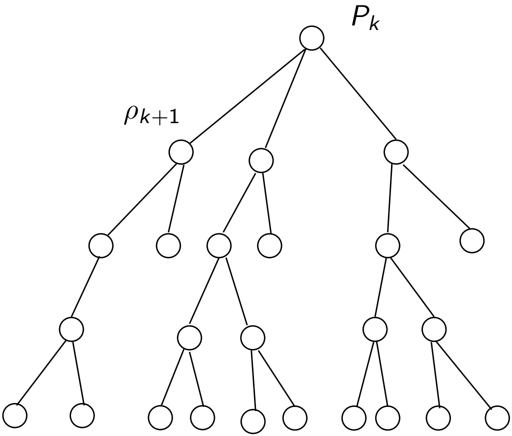

Uniformly sparse, locally tree-like graphs: GW limit

The *Unimodular* Galton-Watson tree:

$P \equiv \{P_k\}_{k \ge 0}$	Degree distribution, law of L (of finite mean $\overline{P}$)
$\rho \equiv \{\frac{k}{\overline{P}} P_k\}_{k \ge 0}$	Law of K (degree of uniform edge)
$\mathsf{T}(P, \rho, t)$	t-generations UGW tree (root degree P, else ρ)
$\mathsf{B}_i(t)$	Ball of radius t in G_n centered at node i

Definition (DM10, Dfn. 2.1)

$\{G_n\}$ *converges locally* to $\mathsf{T}(P, \rho, \infty)$ if for uniformly random $I_n \in [n]$ and fixed t, law of $\mathsf{B}_{I_n}(t)$ converges as $n \to \infty$ to $\mathsf{T}(P, \rho, t)$.
$\{G_n\}$ *uniformly sparse* if $\{|\partial I_n|\}$ is uniformly integrable.

see Benjamini–Schramm '01, Aldous–Lyons '07

Tree-like graphs: examples

Random regular graph $\mathscr{G}_{n,d} \rightsquigarrow$ regular tree $\mathbf{T}_d$

Sparse Erdős–Rényi graph $G_{n,nd/2} \rightsquigarrow \mathrm{T}(P,\rho,\infty)$
with $P = \rho$ Poisson(d) [DM10, Prop. 2.6]

Graphs of fixed degree distribution $P \rightsquigarrow \mathrm{T}(P,\rho,\infty)$
[DM10, Prop. 2.5]

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Tree-like graphs: motivation

Encompasses natural random graph ensembles such as sparse Erdős–Rényi graphs or random regular graphs;

Natural randomized computational problems also described by a locally tree-like constraint structure (random hard-core, q-coloring)

Trees are amenable to *exact analysis* (recursive equations) — physicists predict many exact asymptotic results for the setting $G_n \rightsquigarrow T$, based on comparisons between models on G_n and on T

The **Bethe replica symmetric** prediction for ϕ in graphs $G_n \rightsquigarrow T$: $\phi = \Phi^{\text{Bethe}}$ explicit which *depends only on the limit tree T*

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Mean field equations

$$\mu_{i\to j}(\,\cdot\,) \equiv \text{Marginal of } x_i \text{ on without edge } (i,j)$$
$$\equiv \text{replacing factor } \psi(x_i, x_j) \text{ by } 1$$

Belief propagation, Bethe-Peierls (BP) equations
(Replica Symmetric cavity method):

$$\mu_{i\to j}(x_i) \approx \frac{1}{z_{i\to j}} \bar{\psi}(x_i) \prod_{l\in\partial i\setminus j} \sum_{x_l} \psi(x_i, x_l)\mu_{l\to i}(x_l)$$

Exact if $G = T$ a tree;
approximate marginals of $\mu(\,\cdot\,)$ by measures on trees.

[DM10, Dfn. 3.4 & Prop. 3.7]

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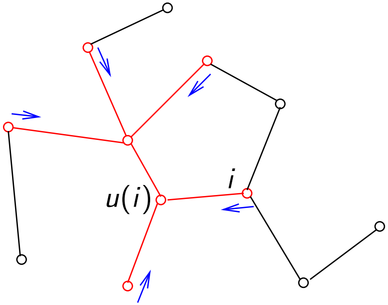
Bethe-Peierls approximation [DM10, Dfn. 3.10]



A tree $T = (U, E_U) \subseteq G$ with $\partial i \in U$ or $\partial i \cap U = \{u(i)\}$

$$\mu_U(\underline{x}_U) \approx \nu_U(\underline{x}_U) = \frac{1}{Z_U} \prod_{(i,j)\in E_U} \psi(x_i, x_j) \prod_{i \notin \partial U} \bar{\psi}(x_i) \prod_{i \in \partial U} h_{i\to u(i)}(x_i) \,.$$

Bethe Gibbs measures [DMS13, Eqn. (1.6)-(1.12)]

On $G_n \rightsquigarrow T$, $\Phi^{\text{Bethe}} = \Phi(\nu^\star)$, with Φ an explicit function, and $\nu^\star$ a certain infinite-volume Gibbs measure on T — represents proposed local limit "$\mu_n \rightsquigarrow \nu^\star$" of models μ_n on G_n.

The replica symmetric prediction assumes special structure for $\nu^\star$: marginal on any finite subgraph $U \subset T$ given by

$$\nu^\star(\underline{x}_U) \cong \binom{\text{weight of } \underline{x}_U}{\text{within } U} \times \prod_{v \in \partial U} h^\star(x_v)$$

Bethe Gibbs measure with entrance law $h^\star$

Consistency mandates fixed-point relations for $h^\star$ (BP equations) — Φ^{Bethe} is equivalently defined in terms of BP solutions $h^\star$

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Bethe free energy density for $\mathbf{T}_d$ [DMSS14, Dfn. 1.3-1.5]

If $G_n \rightsquigarrow \mathbf{T}_d$ — entrance law h in simplex $\mathcal{H}$ of prob. measures on $\mathcal{X}$, with $h^\star \in \mathcal{H}^\star$ set of fixed points of mapping $\mathrm{BP} : \mathcal{H} \to \mathcal{H}$:

$$(\mathrm{BP}h)(x) \equiv \frac{1}{z_h}\,\bar{\psi}(x)\Big(\sum_y \psi(x,y)h(y)\Big)^{d-1}, \quad x \in \mathcal{X}$$

(z_h the normalizing constant)

The Bethe functional

$$\boldsymbol{\Phi}(h) = \underbrace{\log\Big\{\sum_x \bar{\psi}(x)\Big(\sum_y \psi(x,y)h(y)\Big)^d\Big\}}_{\text{“vertex term” } \boldsymbol{\Phi}^{\mathrm{vx}}(h)} - \underbrace{\frac{d}{2}\log\Big\{\sum_{x,y}\psi(x,y)h(x)h(y)\Big\}}_{\text{“edge term” } \boldsymbol{\Phi}^{\mathrm{e}}(h)}.$$

$\boldsymbol{\Phi}^{\mathrm{Bethe}} \equiv \sup\{\boldsymbol{\Phi}(h) : h \in \mathcal{H}^\star\}$.

Interpolation in phase-space: Ferromagnetic Ising [DM10, Thm. 2.15]

For $\beta \geq 0$, $B > 0$ if $G_n \rightsquigarrow T$ then $\phi = \Phi^{\text{Bethe}}(h^+)$.

Proof outline.

0. $B = 0$ given by limit $B \downarrow 0$, so fix $B > 0$.

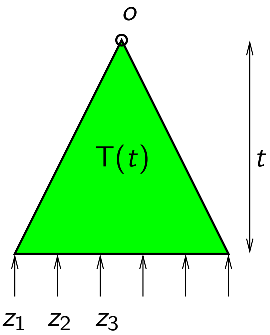
1. Reduce to expectations of local quantities

$$\frac{\mathrm{d}}{\mathrm{d}B} \log Z_n(\beta, B) = \sum_{i=1}^{n} \langle x_i \rangle_n$$

   ($\langle \cdot \rangle_n$ denote expectation under Ising on G_n).

2. Prove convergence of local expectations to tree values.

2. Convergence to tree values



T	infinite tree with max degree $d_{\max}$
T(t)	first t generations
$\mu^{t,z}(\cdot)$	Ising model on T(t) boundary condition z
$\mu_o^{t,z}$	root spin expectation

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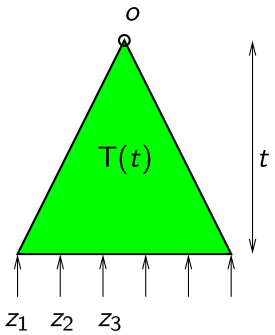

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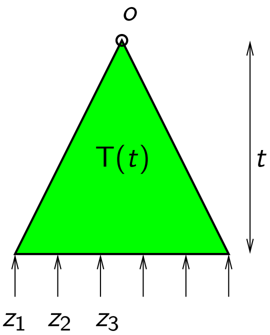

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Decorrelation

Uniform (Gibbs measure uniqueness)

$$|\mu_o^{t,z(1)} - \mu_o^{t,z(2)}| \leq |\mu_o^{t,+} - \mu_o^{t,-}| \to 0\,.$$

Easier

True only at high temperature ($\beta < \beta_c$)

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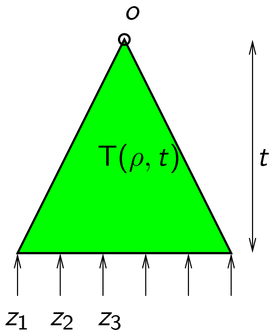
Why non-uniform control? Phase transition. . .

For $\beta > \beta_{\rm c} \equiv \mathrm{atanh}(1/\overline{\rho})$

$$\lim_{B\to 0+}\lim_{n\to\infty}\mathbb{E}\langle x_I\rangle_n = -\lim_{B\to 0-}\lim_{n\to\infty}\mathbb{E}\langle x_I\rangle_n > 0$$

. . . and its tree counterpart



$$z = (+1, +1, \ldots, +1) \quad \Rightarrow \quad \lim_{t\to\infty} \langle x_o \rangle_t > 0$$

$$z = (-1, -1, \ldots, -1) \quad \Rightarrow \quad \lim_{t\to\infty} \langle x_o \rangle_t < 0$$

Decorrelation

Non uniform

$$|\mu_o^{t,+} - \mu_o^{t,\mathrm{free}}| \to 0\,.$$

Any temperature

Trick

$$0 \leq \mu_o^{t,+} - \mu_o^{t,\mathrm{free}} \leq \epsilon\{\mu_o^{t,\mathrm{free}} - \mu_o^{t-1,\mathrm{free}}\} \to 0$$

($\mu_o^{t,\mathrm{free}}$ monotone by Griffiths Inequality [DM10, Prop. 2.11])

[Ising specific, but strategy extends to Potts, Hard-core models]

Bethe Gibbs measures: Non-uniqueness

A major challenge in verifying the Bethe prediction is the **non-uniqueness** of infinite-volume Gibbs measures

Ising already has non-uniqueness, but the correct measure can be isolated by monotonicity considerations

Situation is qualitatively different in $q \geq 3$ Potts models, where monotonicity considerations do not close the gap

If multiple Gibbs measures, physics prescription is to select *Bethe* Gibbs measure achieving highest value of Φ^{Bethe},
$\Phi^{\mathrm{Bethe}} = \sup\{\Phi(\nu^\star) : \nu^\star \text{ Bethe Gibbs measure}\}$

(restriction to Bethe Gibbs measures is an assumption)

— [DMSS14] prove such prediction holds

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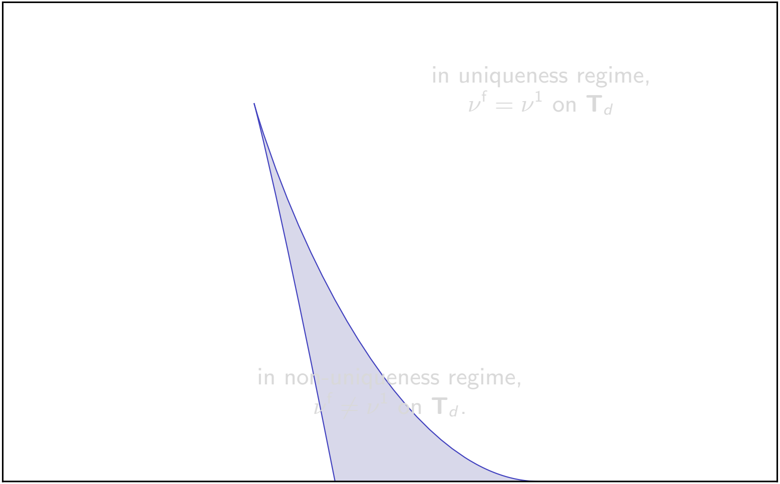
Potts: Non-uniqueness regime

Non-uniqueness regime for Potts on $\mathbf{T}_d$ $(d = 4, q = 30)$



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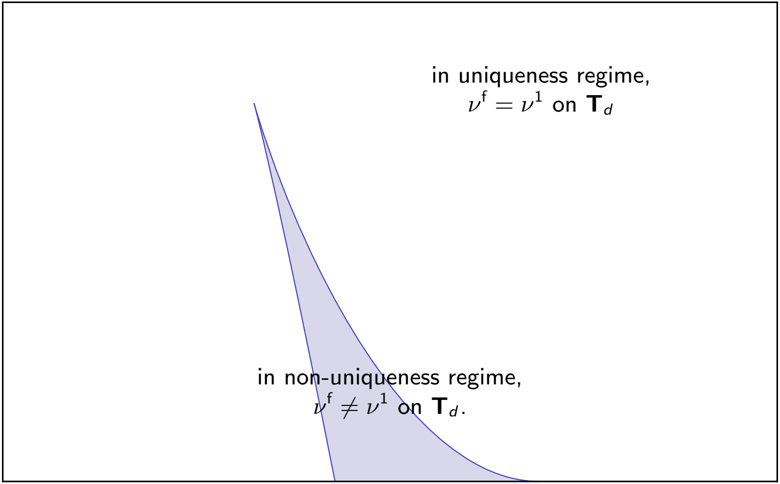

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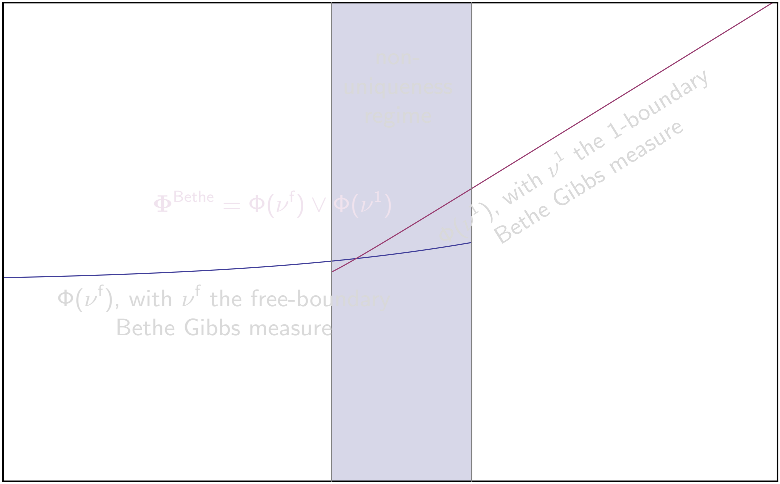
Potts: Bethe free energy

Potts Bethe prediction as function of β ($d = 4,\ q = 30,\ B = 0.05$)



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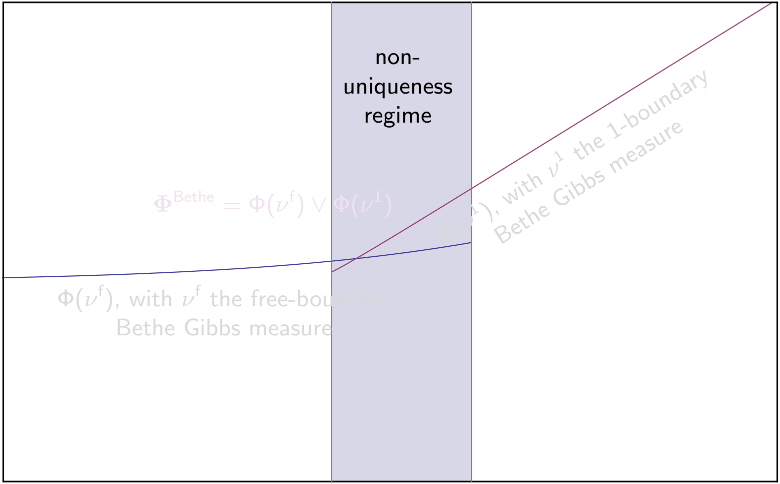

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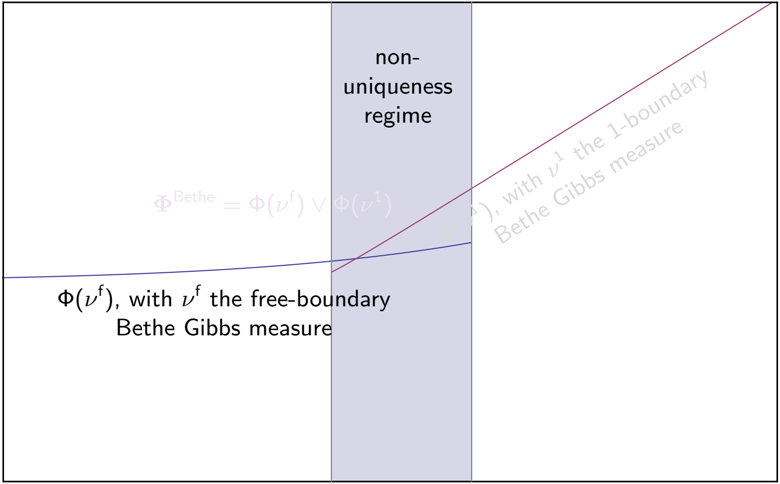

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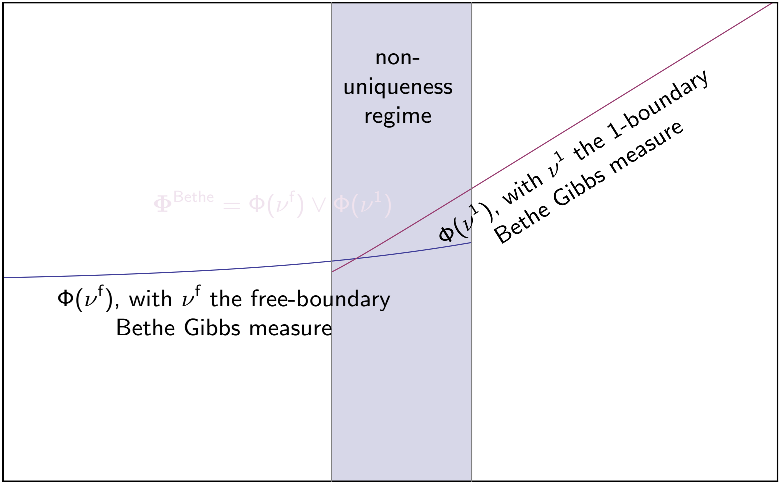

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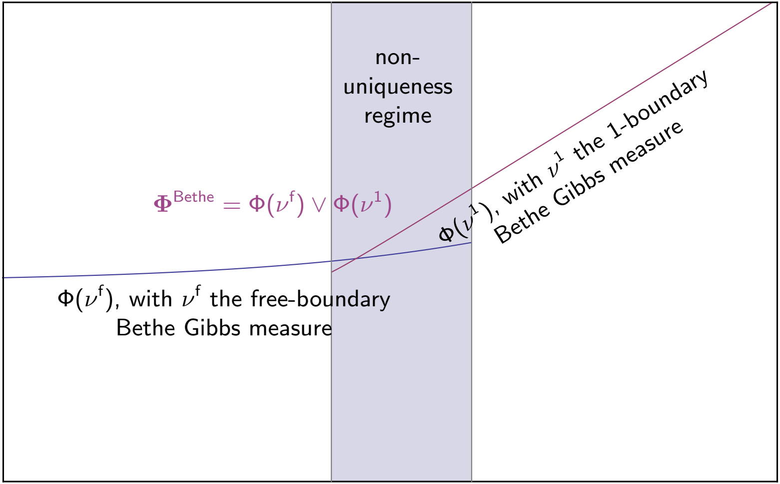

Free energy: other models

By interpolation in parameters and other means ...
mostly for $G_n \rightsquigarrow \mathbf{T}_d$ (regular tree) — simpler Bethe prediction.

Examples of $G_n \rightsquigarrow \mathbf{T}_d$ include
the (uniformly) random d-regular graph,
the (uniformly) random *bipartite* d-regular graph,
and any sequence G_n of d-regular graphs of divergent girth

THEOREM (Sly–Sun '14).
For the hard-core (independent set) model at any fugacity $\lambda > 0$, and for the anti-ferromagnetic Ising model at any $B \in \mathbb{R}$, the Bethe prediction holds on any bipartite sequence $G_n \rightsquigarrow \mathbf{T}_d$.
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